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On $\theta(\Delta)-$open sets in grill topological spaces

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Abstract

The objective of this paper is to present and examine these concepts within the context of grill topological spaces, introducing novel categories of $\theta(\Delta)-$closed sets and $\theta(\Delta)-$continuous functions specific to grill topological spaces.

Keywords: : $\theta-$cluster point, $\theta-$closed sets, $\theta-$continuous function.
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1. INTRODUCTION

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. In the course of our discussion, the closure and interior of A will be represented by $Cl(A)$ and $Int(A)$, respectively. Fomin [2] pioneered the notion of $\theta-$continuous functions, while Velicko [5] introduced the concept of $\theta-$closed sets in topological spaces. As a reminder from [2], a function $f : (X, \mathcal{T}) \longrightarrow (Y, P)$ from a topological space $(X, \mathcal{T})$ to another topological space (Y, p) is termed a θ-continuous function at $x \in X$ if, for every open set V containing $f(x)$, there exists an open set U containing x such that $f(Cl(U)) \subseteq Cl(V)$. The function f is considered $\theta-$continuous if it is $\theta-$continuous at each point in X. Additionally, according to [5], a point $x \in X$ is designated as a $\theta-$cluster point of A if $Cl(U) \cap A \neq \phi$ for every open set U in X containing x. The set encompassing all $\theta-$cluster points of A is termed the $\theta-$cluster set of A and is denoted by $Cl^{\theta}(A)$. A subset A of a topological space is termed a $\theta-$closed set in X if $Cl^{\theta}(A) = A$. The complement of a $\theta-$closed set in X is referred to as a $\theta-$open set in X.

In this paper, we introduce the concept of a $\Delta-$open set along with pertinent results. Subsequently, we present and explore categories of $\theta(\Delta)-$closed sets. Finally, we introduce and investigate classes of $\theta(\Delta)-$continuous functions.

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2. Preliminaries

Theorem 2.1. *[7] For a topological space* $(X, \mathcal{T})$ *and* $A \subseteq X$*, the following hold:*

1. $Int(X - A) = X - Cl(A)$.
2. $Cl(X - A) = X - Int(A)$.

Theorem 2.2. *[7] Let* A *and* B *be two subset of a topological space* $(X, \mathcal{T})$*. If B is an open set in* X *then* $Cl(A) \cap B \subseteq Cl(A \cap B)$.

Theorem 2.3. *[5] Every* $\theta-$*closed set is closed set.*

Definition 2.4. *[1] Anon-empty collection* $\mathcal{G}$ *of subsets of a topological space*$(X, \mathcal{T})$ *is said to be a grill on* X *if* G *satisfies following conditions:*

1. $\phi \in \mathcal{G}$.
2. $A \in \mathcal{G}$ *and* $A \subseteq B \Longrightarrow B \in G.$;
3. $A, B \subseteq X$*and* $A \cup B \in \mathcal{G} \Longrightarrow A \in \mathcal{G}$ *or* $B \in \mathcal{G}$,

For a topological space X, the operator $\phi : P(X) \longrightarrow P(X)$ from the power set $P(X)$ of X to $P(X)$ was first defined [3], as $\phi(A) = \{x \in X : U \cap A \in \mathcal{G},$ for each open set containing $x\}$. The operator $\psi(A) : P(X) \longrightarrow P(X)$, is given by $\psi(A) = A \cup \phi(A)$ for $A \in P(X)$. This operator was also show in [4] to called a Kuratowski closure operator. So for a grill topological space $(X, \mathcal{T}, \mathcal{G})$, there exists an unique topological $\mathcal{T}_{\mathcal{G}}$ on X. This topological defined by

$$\mathcal{T}_{\mathcal{G}} = \{U \subseteq X : \psi(X - U) = X - U\}.$$

For any $A \subseteq X, \psi(A) =_{\mathcal{G}} Cl(A)$ such that $\mathcal{G}Cl(A)$ denotes the set of all $G-$closure points of A in topological space $(X, \mathcal{T}_{\mathcal{G}})$. The intersection of all closed subsets of $(X, \mathcal{T}, \mathcal{G})$ containing $A$is denoted by ${}_{\mathcal{G}}Cl(A)$ and the interior set of $A$is defined as the union of all open subsets of $(X, \mathcal{T}, \mathcal{G})$ contained in A and is denoted by ${}_{\mathcal{G}}Int(A)$.

Theorem 2.5. *[4] Let* $(X, \mathcal{T}, \mathcal{G})$ *be a grill topological space and* $A, B \subseteq X$*, the following properties hold:*

1. $A \subseteq B$ *implies that* $\phi(A) \subseteqq \phi(B)$.
2. $\phi(A \cup B) = \phi(A) \cup \phi(B)$.
3. $\phi(\phi(A)) \subseteqq \phi(A) = Cl(\phi(\phi A)) \subseteq Cl(A)$.
4. *if* $U \in \mathcal{T}$*then* $U \cap \phi(A) \subseteq \phi(U \cap A)$.

Theorem 2.6. *[6] If* A *is a subset of a grill topological space* $(X, \mathcal{T}, \mathcal{G})$ *and* U*is an open set in*

$$(X, \mathcal{T})$$

then $U \cap \psi(A) \subseteq \psi(U \cap A)$.

3. Δ−open sets

Definition 3.1. *A subset* A *of a grill topological space* $(X, \mathcal{T}, \mathcal{G})$ *is said to be* Δ−*open set if* $A \subseteq Cl[\mathcal{G}Int(\psi(A))]$. *The complement of* $\mathcal{G}^S$−*open set is said to be* Δ−*closed set. For a grill topological space* $(X, \mathcal{T}, \mathcal{G})$, *the set of all* $\mathcal{G}^S$−*open sets in* X *denoted by* $\Delta O(X, \mathcal{T})$ *and the set of all* $\mathcal{G}^S$−*closed sets in* X *denoted by* $\Delta C(X, \mathcal{T})$.

Example 3.2. *In a grill topological space* $(X, \mathcal{T}, \mathcal{G})$, *where* $X = \{a, b, c\}, T = \{\phi, X, a, b\}$*and* $G = \{b, a, b, c, b\}, X, \Delta O(X, \mathcal{T}) = \phi, X, b, a, b, c, b$ *and* $\Delta C(X, \mathcal{T}) = \{\phi, X, a, c, c, a\}$.

Theorem 3.3. *A subset* A *of a grill topological space* $(X, \mathcal{T}, \mathcal{G})$ *is* Δ−*closed set if and only if* $Int[\psi(_{\mathcal{G}}Int(A))] \subseteq A$.

Proof. A is a Δ−closed set in X if and only if $X - A$ is a Δ−open set in X if and only if

$$(X - A) \subseteq Cl[_{\mathcal{G}}Int(\psi(X - A))]$$

. if and only if by using Theorem 2.1,

$$\begin{aligned}(X - A) \subseteq Cl[\mathcal{G}Int(\psi(X - A))] &= Cl[_{\mathcal{G}}Int(_{\mathcal{G}}Cl(X - A))] \\ &= Cl[_{\mathcal{G}}Int(X -_{\mathcal{G}} Int(A))] = Cl[X -_{\mathcal{G}} Cl(\mathcal{G}Int(A))] \\ &= X - Int[_{\mathcal{G}}Cl(_{\mathcal{G}}Int(A))] = X - Int[\psi(_{\mathcal{G}}Int(A))].\end{aligned}$$

if and only if $Int[\psi(\mathcal{G}Int(A))] \subseteq A$. □

Theorem 3.4. *Let* $(X, \mathcal{T}, \mathcal{G})$ *be a grill topological space. If* A_λ*is* Δ−*open set for each* $\lambda \in \Delta$ *then* $\cup_{\lambda\in I}A_\lambda$ *is* Δ−*open set, where I is an index set.*

Proof. Since $A_\lambda$is Δ−open set for each $\lambda \in I$, then $A_\lambda \subseteq Cl[\mathcal{G}Int(\psi(A_\lambda))]$ for each $\lambda \in \Delta$. Then by

$$\begin{aligned}\cup_{\lambda\in\Delta}A_\lambda &\subseteq \cup_{\lambda\in I}Cl[\mathcal{G}Int(\psi(A_\lambda))] \\ &\subseteq Cl[\cup_{\lambda\in I}\mathcal{G}Int(\psi(A_\lambda))] \subseteq Cl[\mathcal{G}Int(\cup_{\lambda\in I}\psi(A_\lambda))] \\ &\subseteq Cl[\mathcal{G}Int(\cup_{\lambda\in I}(A_\lambda \cup \phi(A_\lambda))] \subseteq Cl[\mathcal{G}Int((\cup_(\lambda \in I)A_\lambda) \cup (\cup_{\lambda\in I}\phi(A_\lambda))] \\ &\subseteq Cl[\mathcal{G}Int(\cup_{\lambda\in I}A_\lambda \cup \phi(\cup_{\lambda\in I}A_\lambda))] = Cl[\mathcal{G}Int(\psi(\cup_{\lambda\in I}A_\lambda))]\end{aligned}$$

Hence $\cup_{\lambda\in I}A_\lambda$ is Δ−open set. □

Theorem 3.5. *Let* $(X, \mathcal{T}, \mathcal{G})$ *be a grill topological space. If* U *is an open set in* $(X, \mathcal{T})$ *and A is* $\mathcal{G}^S$−*open set then* $U \cap A$ *is* $\mathcal{G}^S$−*open set.*

Proof. Since A is Δ−open set then $A \subseteq Cl[_{\mathcal{G}}Int(\psi(A))]$. Then by Theorem 2.6 and 2.2,

$$\begin{aligned}U \cap A \subseteq U \cap Cl[_{\mathcal{G}}Int(\psi(A))] &\subseteq Cl[U \cap_{\mathcal{G}} Int(\psi(A))] \\ &= Cl[_{\mathcal{G}}Int(U) \cap_{\mathcal{G}} Int(\psi(A))] \\ &= Cl[_{\mathcal{G}}Int(U \cap \psi(A))] \subseteq Cl[_{\mathcal{G}}Int(\psi(U \cap A))]\end{aligned}$$

Hence $U \cap A$ is Δ−open set. □

For a grill topolo Δ gical space$(X, \mathcal{T}, \mathcal{G})$ and a subset A of X, the$\Delta-$closure set of A is defined as the intersection of all $\Delta-$closed sets containing A and is denoted by $^s_{\mathcal{G}}CI(A)$. The $\Delta-$interior set of A is defined as the union of all $\Delta-$open sets of X contained in A and is denoted by $^s_{\mathcal{G}}Int(A)$, It is clear that $^s_{\mathcal{G}}CI(A)$ is a $\mathcal{G}^S-$closed subset of X and $^s_{\mathcal{G}}CI(A)$ is a $\Delta-$open subset of X.

For a subset $A \subseteq X$ of a grill topological space$(X, \mathcal{T}, \mathcal{G})$, it is clear from the definition of $^s_{\mathcal{G}}CI(A)$ and $^s_{\mathcal{G}}CI(A)$ that$A \subseteq {}^s_{\mathcal{G}}CI(A)CI(A)$ and $^s_{\mathcal{G}}CI(A) \subseteq A$.

Theorem 3.6. *Theorem 3.6. For a subset* $A \subseteq X$ *of grill topological space*$(X, \mathcal{T}, \mathcal{G}), {}^s_{\mathcal{G}}CI(A) = A$ *if and only if A is a* $\Delta-$*closed set.*

Proof. Let $^s_{\mathcal{G}}CI(A) = A$. Then from definition of $^s_{\mathcal{G}}CI(A)$ and Theorem **??** ,$^s_{\mathcal{G}}CI(A)$ is a $\mathcal{G}^S-$ closed set and so A is a $\Delta-$closed set. Conversely, we have $A \subseteq {}^s_{\mathcal{G}}CI(A)$. Since A is a $\mathcal{G}^S-$closed set, then it is clear from the definition of$^s_{\mathcal{G}}CI(A), {}^s_{\mathcal{G}}CI(A) \subseteq A$. Hence $A = {}^s_{\mathcal{G}}CI(A)$. □

Theorem 3.7. *For a subset* $A \subseteq X$ *of grill topological space*$(X, \mathcal{T}, \mathcal{G}),^s_{\mathcal{G}} CI(A) = A$ *if and only if A is a* $\Delta\mathcal{G}^S-$*open set.*

Proof. Similar to the proof of Theorem 3.6. □

Theorem 3.8. *For a subset* $A \subseteq X$ *of grill topological space*$(X, \mathcal{T}, \mathcal{G}), x \in {}^s_{\mathcal{G}}CI(A)$ *if and only if for all* $\Delta-$*open set* U *containing* $x, U \cap A \neq \phi$.

Proof. Let$x \in {}^s_{\mathcal{G}}CI(A)$ and U be a $\mathcal{G}^S-$open set containing x. If $U \cap A = \phi$then $A \subseteq X - U$. Since X – U is a $\Delta-$closed set containing A, then $^s_{\mathcal{G}}CI(A) \subseteq X\check{}U$and so $x \in {}^s_{\mathcal{G}}CI(A) \subseteq X - U$. Hence this is a contradiction, because $x \in U.ThereforeU \cap A \neq \phi$. Conversely, let $x \notin {}^s_{\mathcal{G}}CI(A)$. Then$X - {}^s_{\mathcal{G}}CI(A)$ is a $\Delta-$open set containing x. Hence by hypothesis, $[X - {}^s_{\mathcal{G}}CI(A)] \cap A \neq \phi$. But this is a contradiction, because $X - {}^s_{\mathcal{G}}CI(A) \subseteq X - A$. □

Theorem 3.9. *For a subset*$A \subseteq X$ *of grill topological space*$(X, \mathcal{T}, \mathcal{G})$, $x \in {}^s_{\mathcal{G}}CI(A)$ *if and only if there is* Δ*-open set U such that* $x \in U \subseteq A$.

Proof. Let $x \in {}^s_{\mathcal{G}}CI(A)$ and take $U =^s_{\mathcal{G}} (CI(A)$. Then by definition of $^s_{\mathcal{G}}CI(A)$ we get that U is a Δ-open set and $x \in U \subseteq A$.

Conversely, let there is $\Delta-$open set U such that $x \in U \subseteq A$. Then $x \in U \subseteqq {}^s_{\mathcal{G}}Int(A)$. □

Theorem 3.10. *For a subsets*$A, B \subseteq X$ *of grill topological space* $(X, \mathcal{T}, \mathcal{G})$, *the following hold:*

1. *AifA* $\subseteq B$ *then* $^s_{\mathcal{G}}CI(A) \subseteq^s_{\mathcal{G}} CI(B)$.
2. $^s_{\mathcal{G}}CI(A) \cup^s_{\mathcal{G}} CI(B) \subseteq^s_{\mathcal{G}} CI(A \cup B)$.
3. $^s_{\mathcal{G}}CI(A \cap B) \subseteq^s_{\mathcal{G}} CI(A) \cap^s_{\mathcal{G}} CI(B)$.
4. $^s_{\mathcal{G}}CI(A) \subseteq Cl(A)$.

Proof. 1. Let $x \in^s_{\mathcal{G}} CI(A)$. Then by Theorem 3.8, for all $\Delta-$open sets U containing $x, U \cap A \neq \phi$. Since $A \subseteq B$, then $U \cap B \neq \phi$. Hence $x \in^s_{\mathcal{G}} CI(B)$. That is, $^s_{\mathcal{G}}CI(A) \subseteq^s_{\mathcal{G}} CI(B)$.

2. It is clear from the Part (1).
3. It is clear from the Part (1).
4. It is clear from Theorem 3.8 and from every open set U is $\Delta-$open set.
   Similar for the proof of the last theorem, we can proof the following theorem:

□

Theorem 3.11. *For a subsets* $A, B \subseteq X$ *of grill topological space* $(X, \mathcal{T}, \mathcal{G})$*, the following hold:*

1. *if* $A \subseteq B$ *then* ${}^s_{\mathcal{G}}\mathrm{Int}(A) \subseteq^s_{\mathcal{G}} \mathrm{Int}(B)$.
2. ${}^s_{\mathcal{G}}\mathrm{Int}(A) \cup^s_{\mathcal{G}} \mathrm{Int}(B) \subseteq^s_{\mathcal{G}} \mathrm{Int}(A \cup B)$.
3. ${}^s_{\mathcal{G}}\mathrm{Int}(A \cap B) \subset^s_{\mathcal{G}} \mathrm{Int}(A) \cap^s_{\mathcal{G}} \mathrm{Int}(B)$
4. $\mathrm{Int}(A) \subseteq^s_{\mathcal{G}} \mathrm{Int}(A)$.

Theorem 3.12. *For a subset* $A \subseteq X$ *of grill topological space* $(X, \mathcal{T}, \mathcal{G})$*, the following hold:*

1. ${}^s_{\mathcal{G}}\mathrm{Int}(X - A) = X -^s_{\mathcal{G}} \mathrm{CI}(A)$.
2. ${}^s_{\mathcal{G}}\mathrm{CI}(X - A) = X -^s_{\mathcal{G}} \mathrm{Int}(A)$.

Proof. 1. Since $A \subseteq^s_{\mathcal{G}} \mathrm{CI}(A)$ then$X -^s_{\mathcal{G}} \mathrm{CI}(A) \subseteq X - A$. Since $X -^s_{\mathcal{G}} \mathrm{CI}(A)$ is a $\Delta-$ open set in $(X, \mathcal{T}, \mathcal{G})$ then

$$X -^s_{\mathcal{G}} \mathrm{CI}(A) =^s_{\mathcal{G}} \mathrm{Int}[X -^s_{\mathcal{G}} \mathrm{CI}(A)] \subseteq^s_{\mathcal{G}} \mathrm{Int}(X - A).$$

For the other side, let $x \in^s_{\mathcal{G}} \mathrm{Int}(X - A)$. Then there is $\Delta-$open set U such that $x \in U \subseteq X - A$. Then$X - U$ is $\Delta-$closed set containing A and $x \notin X - U$. Hence $x \in^s_{\mathcal{G}} \mathrm{CI}(A)$, that is,$x \in X -^s_{\mathcal{G}} \mathrm{CI}(A)$.

2. Since ${}^s_{\mathcal{G}}\mathrm{Int}(A) \subseteq A$ then$X - A \subseteq X -^s_{\mathcal{G}} \mathrm{Int}(A)$. Since$X -^s_{\mathcal{G}} \mathrm{Int}(A)$ is a $\Delta-$closed set in $(X, \mathcal{T}, \mathcal{G})$, then

$${}^s_{\mathcal{G}}\mathrm{CI}(X - A) \subseteq^s_{\mathcal{G}} \mathrm{CI}[X -^s_{\mathcal{G}} \mathrm{Int}(A)] = X -^s_{\mathcal{G}} \mathrm{Int}(A).$$

For the other side, let $x \in^s_{\mathcal{G}} \mathrm{Int}(X - A)$. Then there is $\Delta-$open set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\Delta-$closed set containing A and $x \notin X - U$. Hence $x \notin^s_{\mathcal{G}} \mathrm{CI}(A)$, that is,$x \in X -^s_{\mathcal{G}} \mathrm{CI}(A)$. □

Theorem 3.13. *For a subset* $A \subseteq X$ *of grill topological space* $(X, \mathcal{T}, \mathcal{G})$*, the following hold:*

1. *if* G *is an open set of* X *then* ${}^s_{\mathcal{G}}\mathrm{CI}(A) \cap G \subseteq^s_{\mathcal{G}} \mathrm{CI}(A \cap G)$.
2. *if* G *is a closed set of Xthen* ${}^s_{\mathcal{G}}\mathrm{Int}(A \cup G) \subseteq^s_{\mathcal{G}} \mathrm{Int}(A) \cup G$.

Proof. 1. Let $x \in^s_{\mathcal{G}} \mathrm{CI}(A) \cap G$. Then $x \in^s_{\mathcal{G}} \mathrm{CI}(A)$ and $x \in \mathcal{G}$. Let V be any $\Delta-$open set in $(X, \mathcal{T}, \mathcal{G})$ containing x. By Theorem 2.1,**??**, $V \cap G$ is $\Delta-$open set containing x. Since $x \in^s_{\mathcal{G}} \mathrm{CI}(A)$ then by Theorem 3.8, $(V \cap G) \cap A \neq \phi$. This implies,$V \cap (G \cap A) \neq \phi$. Hence by Theorem 3.8, $x \in^s_{\mathcal{G}} \mathrm{CI}(A \cap G)$. That is, ${}^s_{\mathcal{G}}\mathrm{CI}(A) \cap G \subseteq {}^s_{\mathcal{G}}\mathrm{CI}(A \cap G)$.

2. Since $\mathcal{G}$ is a closed set X then by The part (1) and Theorem 3.12,

$$\begin{aligned}
X - [{}^s_{\mathcal{G}}\mathrm{Int}(A) \cup \mathcal{G}] &= [X -^s_{\mathcal{G}} \mathrm{Int}(A)] \cap [X - G] = [{}^s_{\mathcal{G}}\mathrm{CI}(X - A)] \cap [X - G] \\
&\subseteq {}^s_{\mathcal{G}}\mathrm{CI}[(X - A) \cap (X - G)] \subseteq^s_{\mathcal{G}} \mathrm{CI}(X - A) \cap^s_{\mathcal{G}} \mathrm{CI}(X - G) \\
&= {}^s_{\mathcal{G}}\mathrm{CI}(X - A) \cap (X - G) \\
&= (X -^s_{\mathcal{G}} \mathrm{Int}(A)) \cap (X - G) = X - ({}^s_{\mathcal{G}}\mathrm{Int}(A) \cup \mathcal{G}).
\end{aligned}$$

Hence ${}^s_{\mathcal{G}}\mathrm{Int}(A \cup \mathcal{G}) \subseteq \mathrm{Int}(A) \cup \mathcal{G}$.

□

4. $\Theta(\Delta)$−closed set

Let $(X, \mathcal{T}, \mathcal{G})$ be grill topological space and $A \subseteq X$. A point $x \in X$ is called $\Theta(\Delta)$−cluster point of A if ${}^{s}_{\mathcal{G}}CI(U) \cap A \neq \phi$ for every Δ−open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x. The set of all $\Theta(\Delta)$−cluster points of A is called the$\Theta(\Delta)$−cluster set of Aand denoted by ${}^{s}_{\mathcal{G}}Cl^{\theta}(A)$.

Definition 4.1. *A subset A of grill topological space* $(X, \mathcal{T}, \mathcal{G})$ *is called* $\Theta(\Delta)$*−closed set in* $(X, \mathcal{T}, \mathcal{G})$ *if* $\Theta(\Delta) - \mathcal{G}^{s}Cl^{\theta}(A) = A$. *The complement of*$\Theta(\Delta)$*−closed set in* $(X, \mathcal{T}, \mathcal{G})$ *is called*$\Theta(\Delta)$*−open set in* $(X, \mathcal{T}, \mathcal{G})$.

Theorem 4.2. *Every* θ*−closed set in a space* (X, T) *is*$\Theta(\Delta)$*−closed set in grill topological space* $(X, \mathcal{T}, \mathcal{G})$.

Proof. Let A be a θ−closed set in a space (X, T), that is, $Cl^{\theta}(A) = A$. It is clear that $A \subseteq^{s}_{\mathcal{G}} Cl^{\theta}(A)$. We prove that${}^{s}_{\mathcal{G}}Cl^{\theta}(A) \subseteq A$. Let $x \in^{s}_{\mathcal{G}} Cl^{\theta}(A)$. Then $U \cap A \neq \phi$ for every Δ−open set U of $(X, \mathcal{T}, \mathcal{G})$ containing x, since $U \subseteq^{s}_{\mathcal{G}} CI(U)$, then${}^{s}_{\mathcal{G}}CI(U) \cap A \neq \phi$ for every Δ−open setU in $(X, \mathcal{T}, \mathcal{G})$ containing x. Since ${}^{s}_{\mathcal{G}}CI(U) \subseteq Cl(U)$ then $Cl(U) \cap A \neq \phi$ for every Δ−open set $U \in (X, \mathcal{T}, \mathcal{G})$ containing x. Then $x \in Cl^{\theta}(A) = A$. Hence ${}^{s}_{\mathcal{G}}Cl^{\theta}(A) \subseteq A$. That is, A is a$\Theta(\Delta)$−closed set in grill topological space $(X, \mathcal{T}, \mathcal{G})$.

The converse of the last theorem need not be true.

Example 4.3. *In a grill topological space* $(X, \mathcal{T}, \mathcal{G})$, *where* $X = \{a, b, c\}, \mathcal{T} = \{\phi, X, \{a, b\}\}$ *and* $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$, *the set*$\{a\}$ *is a* $\Theta(\Delta)$*−closed set in* $(X, \mathcal{T}, \mathcal{G})$ *but it is not* θ*−closed set in (X, T).*

Theorem 4.4. *Every*$\Theta(\Delta)$*−closed set is* Δ*−closed set.*

Proof. Let $(X, \mathcal{T}, \mathcal{G})$ be a grill topological space and A be a $\Theta(\Delta)$−closed set, that is,${}^{s}_{\mathcal{G}}Cl^{\theta}(A) = A$. It is clear thatA $\subseteq^{s}_{\mathcal{G}} CI(A)$. We prove that ${}^{s}_{\mathcal{G}}CI(A) \subseteq A$. Let $x \in^{s}_{\mathcal{G}} CI(A)$. Then $U \cap A \neq \phi$ for every Δ−open set U in $(X, \mathcal{T}, \mathcal{G})$ containing x. Since $U \subseteq^{s}_{\mathcal{G}} CI(U)$ then ${}^{s}_{\mathcal{G}}CI(U) \cap A \neq \phi$ for every Δ−open set $U \in (X, \mathcal{T}, \mathcal{G})$ containing x. Then $x \in^{s}_{\mathcal{G}} Cl^{\theta}(A) = A$. Hence ${}^{s}_{\mathcal{G}}CI(A) \subseteq A$. That is, A is a Δ−closed set in grill topological space $(X, \mathcal{T}, \mathcal{G})$. □

The converse of the last theorem need not be true.

Example 4.5. *Let* $(X, \mathcal{T}, \mathcal{G})$ *be a grill topological space.* $X = \{a, b, c\}, \mathcal{T} = \{\phi, X, \{a, b\}\}$ *and* $\mathcal{G} = P(X) - \{\phi\}$. *The set* $\{b\}$ *is a* Δ*−closed set in* $(X, \mathcal{T}, \mathcal{G})$ *but it is not* $\Theta(\Delta)$*−closed set, where* $P(X)$ *is the power of* X.

Theorem 4.6. *For every* Δ*−open set G in grill topological space* $(X, \mathcal{T}, \mathcal{G})$,${}^{s}_{\mathcal{G}}Cl^{\theta}(G) =^{s}_{\mathcal{G}} CI(G)$.

Proof. Let $x \in^{s}_{\mathcal{G}} CI(G)$. Then for every Δ−open set U in $(X, \mathcal{T}, \mathcal{G})$ containing $x, U \cap G \neq \phi$.SinceU $\subseteq^{s}_{\mathcal{G}} CI(U)$then${}^{s}_{\mathcal{G}}CI(U) \cap G \neq \phi$. Hence $x \in^{s}_{\mathcal{G}} Cl^{\theta}(U)$. That is,${}^{s}_{\mathcal{G}}CI(G) \subseteq^{s}_{\mathcal{G}} Cl^{\theta}(G)$. For the other side, let $x \in^{s}_{\mathcal{G}} Cl^{\theta}(G)$. Then for every Δ−open set Uin $(X, \mathcal{T}, \mathcal{G})$ containing x,${}^{s}_{\mathcal{G}} CI(U) \cap G \neq \phi$. Since G is Δ−open set U in $(X, \mathcal{T}, \mathcal{G})$. Then by Theorem 3.13, ${}^{s}_{\mathcal{G}}CI(U) \cap G =^{s}_{\mathcal{G}} CI(U \cap G)$. Then ${}^{s}_{\mathcal{G}}CI(U \cap G) \neq \phi$. Hence $U \cap G \neq \phi$. That is, $x \in^{s}_{\mathcal{G}} CI(U)$. Hence ${}^{s}_{\mathcal{G}}Cl^{\theta}(G) \subseteq^{s}_{\mathcal{G}} CI(G)$.

□

From Theorems 4.2 and 4.4 we have the following relation for $\Theta(\Delta)-$-closed set with the other known sets.

Theorem 4.7. *A subset* U *is* $\Theta(\Delta)-$*open set in grill topological space* $(X,\mathcal{T},\mathcal{G})$ *if and only if for each* $x \in U$ *there is* $\Delta-$*open set* V *in* $(X,\mathcal{T},\mathcal{G})$ *containing* x *such that* $^s_{\mathcal{G}}CI(V) \subseteq U$.

Proof. Suppose that U is $\Theta(\Delta)-$open set in $(X,\mathcal{T},\mathcal{G})$ and $x \in U$. Then $x \notin X - U =^s_{\mathcal{G}} Cl^{\theta}(X-U)$. Then there is$\Delta-$open set V in $(X,\mathcal{T},\mathcal{G})$ containing x such that$^s_{\mathcal{G}}CI(V) \cap (X - U) = \phi$. That is,$^s_{\mathcal{G}}CI(V) \subseteq U$.

Conversely, suppose that U is not $\Theta(\Delta)-$open set. Then $X-U$ is not $\Theta(\Delta)-$closed set. That is, there is $x \in^s_{\mathcal{G}} Cl^{\theta}(X-U)$ and $x \notin X-U$. Since $x \in U$ then by the hypothesis, there is $\Delta-$open set$V$in $(X,\mathcal{T},\mathcal{G})$ containing x such that $^s_{\mathcal{G}}CI(V) \subseteq U$. This implies, $^s_{\mathcal{G}}CI(V) \cap (X - U) = \phi$ and this contradiction since $x \in^s_{\mathcal{G}} Cl^{\theta}(X-U)$. Hence U is $\theta - \mathcal{G}^S-$open set. □

5. $\Theta(\Delta)-$continuous function

Definition 5.1. *A function* $f:(X,\mathcal{T},\mathcal{G}) \longrightarrow (Y,P)$ *of a grill topological space* $(X,\mathcal{T},\mathcal{G})$ *into a space* (Y,P) *is called* $\Theta(\Delta)-$*continuous function if for each* $x \in X$ *and each open set* V *in* (Y,P) *containing* $f(x)$, *there exists* $\Delta-$*open set* U *in* $(X,\mathcal{T},\mathcal{G})$ *containing* x *such that* $f(^s_{\mathcal{G}}CI(U)) \subseteq_P Cl(V)$.

Theorem 5.2. *A function* $f:(X,\mathcal{T},\mathcal{G}) \longrightarrow (Y,P)$ *is* $\Theta(\Delta)-$*continuous if and only if* $^s_{\mathcal{G}}Cl^{\theta}(f^{-1}(V)) \subseteq f^{-1}(PCl(V))$ *for every open set* V *in* (Y,P).

Proof. Suppose that f is $\Theta(\Delta)-$continuous. Let V be any open set in of (Y,P). Let $x \notin f^{-1}(PCl(V))$. Then $f(x) \notin_P Cl(V)$. Then $f(x) \in Y - PCl(V)$. Since $Y - PCl(V)$ is open set in (Y,P) containing x and f is $\Theta(\Delta)-$continuous then there exists $\Delta-$open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that $f(\mathcal{G}^sCI(U)) \subseteq_P Cl(Y -_P Cl(V))$. This implies,

$$f(\mathcal{G}^sCI(U)) \subseteq_P Cl(Y - PCl(V)) = Y -_p Int(PCl(V)).$$

Hence $f(\mathcal{G}^sCI(U)) \cap pInt(PCl(V)) = \phi$. Since $V =_p Int(V) \subseteq_p Int(PCl(V))$ then $f(\mathcal{G}^sCI(U)) \cap V = \phi$ and so $\mathcal{G}^sCI(U) \cap f^{-1}(V) = \phi$. Since U is $\Delta-$open set in $(X,\mathcal{T},\mathcal{G})$ containing x then $x \notin^s_{\mathcal{G}} Cl^{\theta}(f^{-1}(V))$. Hence $^s_{\mathcal{G}}Cl^{\theta}(f^{-1}(V)) \subseteq f^{-1}(_PCl(V))$.
Conversely, Let $x \in X$ be any point in X and V be any open set (Y,P) containing $f(x)$. Since $V \cap (Y -_P Cl(V)) = \phi$ then $f(x) \notin_P Cl(Y -_P Cl(V))$. This implies, $x \notin f^{-1}[PCl(Y - PCl(V))]$. Since $Y -_P Cl(V)$ is an open set in (Y,P) then by the hypothesis,

$$^s_{\mathcal{G}}Cl^{\theta}[f^{-1}(Y -_P Cl(V))] \subseteq f^{-1}[_PCl(Y - PCl(V))].$$

Then $x \notin^s_{\mathcal{G}} Cl^{\theta}[f^{-1}(Y - PCl(V))]$. Hence there is $\Delta-$open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that $\mathcal{G}^sCI(U) \cap f^{-1}(Y -_P Cl(V)) = \phi$. This implies, $f(\mathcal{G}^sCI(U)) \subseteq_P Cl(V)$. Hence f is $\Theta(\Delta)-$continuous. □

Theorem 5.3. *A function* $f:(X,\mathcal{T},\mathcal{G}) \longrightarrow (Y,P)$ *is* $\Theta(\Delta)-$*continuous if and only if*

$$^s_{\mathcal{G}}Cl^{\theta}[X - f^{-1}(_PCl(V))] \subseteq X - f^{-1}(V)$$

for every open set V *in* (Y,P).

Proof. Suppose that f is $\Theta(\Delta)-$continuous. Let V be any open set in of (Y,P). Let $x \in X - f^{-1}(V)$. Then $f(x) \in V$. Since f is $\Theta(\Delta)-$continuous then there exists $\Theta(\Delta)-$open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that $f(\mathcal{G}^s CI(U)) \subseteq_P Cl(V)$. This implies, ${}^s_{\mathcal{G}}CI(U) \subseteq f^{-1}(PCl(V))$. Then $\mathcal{G}^s CI(U) \cap [X - f^{-1}({}_P Cl(V))] = \phi$. Since U is a $\Delta-$open set in $(X,\mathcal{T},\mathcal{G})$ containing x then $x \in \mathcal{G}^s Cl^\theta[X - f^{-1}(PCl(V))]$. Hence

$$\mathcal{G}^s Cl^\theta[X - f^{-1}({}_P Cl(V))] \subseteq X - f^{-1}(V).$$

Conversely, let $x \in X$ be any point in X and V be any open set in (Y,P) containing $f(x)$. Then $x \in f^{-1}(V)$, that is, $x \in X - f^{-1}(V)$. Then by the hypothesis, $x \in \mathcal{G}^s Cl^\theta[X - f^{-1}({}_P Cl(V))]$. That is, $\Delta-$open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that

$$\mathcal{G}^s CI(U) \cap [X - f^{-1}(PCl(V))] = \phi.$$

This implies,$\mathcal{G}^s CI(U) \subseteq f^{-1}({}_P Cl(V))$ and so $f(\mathcal{G}^s CI(U)) \subseteq_P Cl(V)$. Hence f is $\Theta(\Delta)-$continuous. □

Theorem 5.4. *For a function*$f : (X,\mathcal{T},\mathcal{G}) \longrightarrow (Y,P)$, *the following properties are equivalent:*

1. f *is* $\Theta(\Delta)-$*continuous.*
2. ${}^s_{\mathcal{G}}Cl^\theta(f^{-1}(B)) \subseteq f^{-1}(\mathcal{G}^s Cl^\theta(B))$*for every subset* $B \subseteq Y$.
3. $f(\mathcal{G}^s Cl^\theta(A)) \subseteq_p Cl^\theta(f(A))$ *for every subset* $A \subseteq X$.

Proof. $(1) \Rightarrow (2)$: Let Bbe any subset of Y. Suppose that $x \in f^{-1}(pCl^\theta(B))$. Then $f(x) \in_p Cl^\theta(B)$. Then there is an open set V in Y containing $f(x)$ such that ${}_P Cl(V) \cap B = \phi$. Since f is $\Theta(\Delta)-$continuous then there exists $\Delta-$open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that $f(\mathcal{G}^s CI(U)) \subseteq_P Cl(V)$. Then we have $f(\mathcal{G}^s CI(U)) \cap B = \phi$. This implies, ${}^s_{\mathcal{G}}CI(U) \cap f^{-1}(B) = \phi$. Hence $x \in \mathcal{G}^s Cl^\theta(f^{-1}(B))$. That is,

$$\mathcal{G}^s Cl^\theta(f^{-1}(B)) \subseteq f^{-1}(pCl^\theta(B)).$$

$(2) \Rightarrow (1)$: Let $x \in X$ be any point in X and V be any open set in (Y,P) containing $f(x)$. Since ${}_P Cl(V) \cap (Y -_P Cl(V)) = \phi$ then $f(x) \in_p Cl^\theta(Y - PCl(V))$. This implies,$x \in f^{-1}[pCl^\theta(Y - PCl(V))]$. Since ${}_p Cl^\theta(Y - PCl(V)) \subseteq Y$ then by the hypothesis,

$$\mathcal{G}^s Cl^\theta[f^{-1}({}_p Cl^\theta(Y - PCl(V)))] \subseteq f^{-1}[pCl^\theta({}_p Cl^\theta(Y - PCl(V)))] = f^{-1}[{}_p Cl^\theta(Y -_P Cl(V))].$$

Then $x \in \mathcal{G}^s Cl^\theta[f^{-1}({}_p Cl^\theta(Y -_P Cl(V)))]$. Hence there is $\Delta-$open set U in $(X,\mathcal{T},\mathcal{G})$ containing x such that ${}^s_{\mathcal{G}}CI(U) \cap f^{-1}[pCl^\theta(Y -_P Cl(V))] = \phi$. This implies, $f(\mathcal{G}^s CI(U)) \subseteq_P Cl(V)$. Hence f is $\Theta(\Delta)-$continuous.
$(2) \Rightarrow (3)$: Let A be any subset of X. Since $f(A) \subseteq Y$ then by the hypothesis,

$${}^s_{\mathcal{G}}Cl^\theta(A) \subseteq \mathcal{G}^s Cl^\theta[f^{-1}(f(A))] \subseteq f^{-1}[pCl^\theta(f(A))].$$

This implies, $f(\mathcal{G}^s Cl^\theta(A)) \subseteq pCl^\theta(f(A))$.
$(3) \Rightarrow (2)$: Let B be any subset of Y. Since $f^{-1}(B) \subseteq X$ then by the hypothesis,

$$f[\mathcal{G}^s Cl^\theta(f^{-1}(B))] \subseteq pCl^\theta[f(f^{-1}(B))] \subseteq pCl^\theta(B).$$

This implies, ${}^s_{\mathcal{G}}Cl^\theta(f^{-1}(B)) \subseteq f^{-1}(pCl^\theta(B))$. □

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