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On Some Properties of the Degenerate Hyperbolic Functions

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Abstract

In this paper, we establish some limit properties of the degenerate hyperbolic functions. Using analytical methods, we obtain some monotonic properties and other properties in the form of inequalities.

Keywords: Degenerate hyperbolic sine, Degenerate hyperbolic cosine, Degenerate hyperbolic tangent, limits, inequality.
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1. Introduction

In [1], the degenerate hyperbolic functions were introduced and defined as

$$\cosh_\lambda(t) = \frac{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}{2}, \tag{1.1}$$

$$\sinh_\lambda(t) = \frac{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}{2}, \tag{1.2}$$

$$\tanh_\lambda(t) = \frac{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}, \tag{1.3}$$

for $\lambda \in (0, \infty)$ and $t \in \mathbb{R}$.

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The reciprocals of the degenerate hyperbolic cosine, sine and tangent functions are defined as (see[1])

$$\operatorname{sech}_{\lambda}(t) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}, \tag{1.4}$$

$$\operatorname{csch}_{\lambda}(t) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}, \tag{1.5}$$

$$\coth_{\lambda}(t) = \frac{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}, \tag{1.6}$$

for $\lambda \in (0, \infty)$ and $t \in \mathbb{R}$. Figures 1,2,3,4,5 and 6 below, are the plots of the degenerate hyperbolic cosine, sine, tangent, secant, cosecant and cotangent functions respectively. The introduction of the degenerate hyperbolic functions was motivated by the degenerate exponential function, which is well known in the literature (see [2],[3],[4],[5], [6], [7]), and the recent interest of many reseachers in establishing degenerate versions of some special functions(see [8], [9]). In [10], the hyperbolic secant function has been applied to handle noise in data processing. Dalloo et.al in [11], proposed a cost effective and low latency design in computing the exponential and the hyperbolic functions. Also serving as a motivation, is the wide range application of the hyperbolic functions in engineering and other fields.

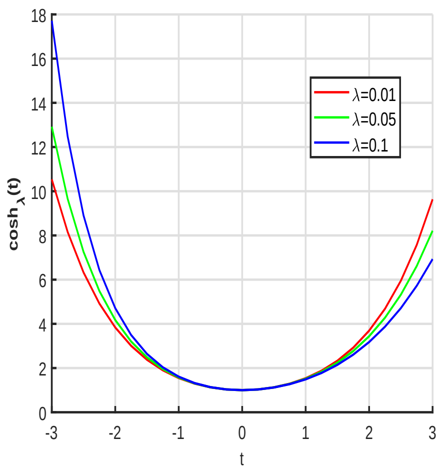

Figure 1: Plot of $\cosh_{\lambda}(t)$

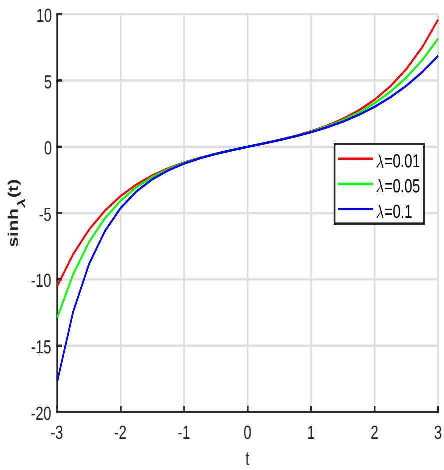

Figure 2: Plot of $\sinh_{\lambda}(t)$

In [12], several identities on degenerate hyperbolic functions were obtained, which originated from Volkenborn and the fermionic p-adic integrals on $\mathbb{Z}_p$.

In this paper, we establish some limit and monotonic properties involving the degenerate hyperbolic functions. We also obtain the degenerate hyperbolic generalizations of some properties satisfied by the ordinary hyperbolic functions (see [13], [14], [15]) .

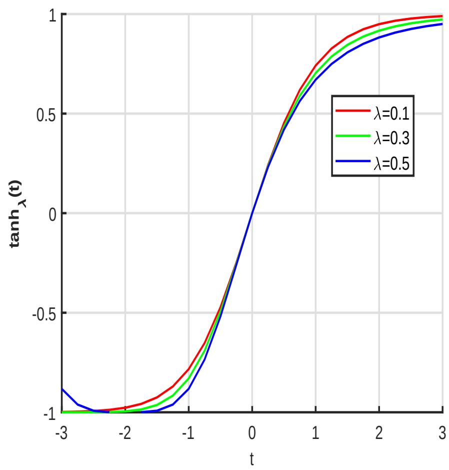

Figure 3: Plot of $\tanh_\lambda(t)$

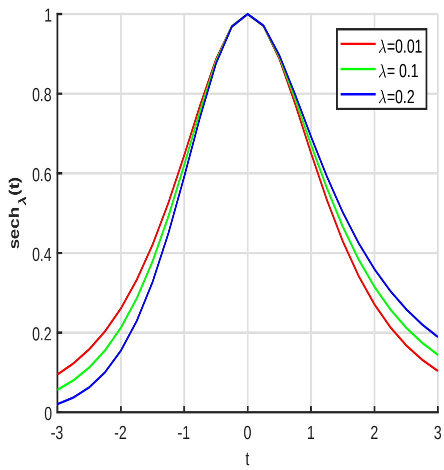

Figure 4: Plot of $\text{sech}_\lambda(t)$

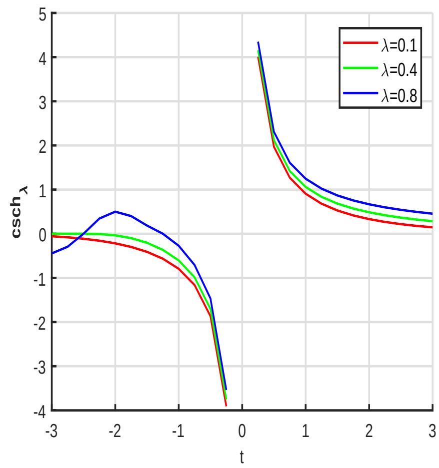

Figure 5: Plot of $\text{csch}_\lambda(t)$

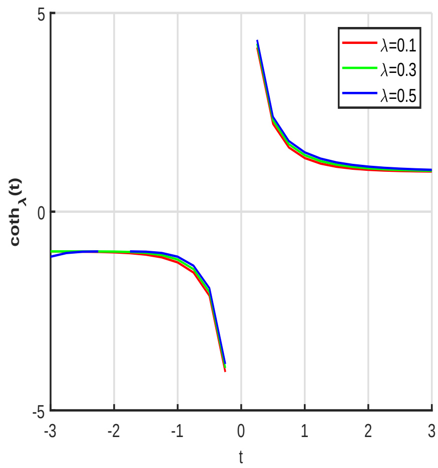

Figure 6: Plot of $\coth_\lambda(t)$

2. Results

Proposition 2.1. *The derivatives of the degenerate hyperbolic functions satisfy:*

$$\left(\cosh_\lambda(t)\right)' = \frac{\sinh_\lambda(t)}{(1+\lambda t)}, \tag{2.1}$$

$$\left(\sinh_\lambda(t)\right)' = \frac{\cosh_\lambda(t)}{(1+\lambda t)}, \tag{2.2}$$

$$\left(\tanh_\lambda(t)\right)' = \frac{\operatorname{sech}^2_\lambda(t)}{(1+\lambda t)}, \tag{2.3}$$

$$\left(\coth_\lambda(t)\right)' = -\frac{\operatorname{csch}^2_\lambda(t)}{(1+\lambda t)}, \tag{2.4}$$

$$\left(\operatorname{sech}_\lambda(t)\right)' = -\frac{\operatorname{sech}_\lambda(t)\tanh_\lambda(t)}{(1+\lambda t)}, \tag{2.5}$$

$$\left(\operatorname{csch}_\lambda(t)\right)' = -\frac{\operatorname{csch}_\lambda(t)\coth_\lambda(t)}{(1+\lambda t)}, \tag{2.6}$$

for all $t \in (-\infty, \infty)$ *and* $\lambda \in (0, \infty)$.

Proof. We have

$$\left(\cosh_\lambda(t)\right)' = \frac{(1+\lambda t)^{\frac{1}{\lambda}-1} - (1+\lambda t)^{-\frac{1}{\lambda}-1}}{2} = \frac{\sinh_\lambda(t)}{(1+\lambda t)},$$

and this gives the result (2.1). Next,

$$\left(\sinh_\lambda(t)\right)' = \frac{(1+\lambda t)^{\frac{1}{\lambda}-1} + (1+\lambda t)^{-\frac{1}{\lambda}-1}}{2} = \frac{\cosh_\lambda(t)}{(1+\lambda t)},$$

which gives the results (2.2). Next,

$$\left(\tanh_\lambda(t)\right)' = \frac{\left[\begin{array}{l} \left[(1+\lambda t)^{\frac{1}{\lambda}-1} + (1+\lambda t)^{-\frac{1}{\lambda}-1}\right]\left[(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}\right] \\ -\left[(1+\lambda t)^{\frac{1}{\lambda}-1} - (1+\lambda t)^{-\frac{1}{\lambda}-1}\right]\left[(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}\right] \end{array}\right]}{\left[(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}\right]^2} \tag{2.7}$$

$$
\begin{aligned}
&= \frac{\left[\begin{array}{c} (1+\lambda t)^{\frac{2}{\lambda}-1} + 2(1+\lambda t)^{-1} + (1+\lambda t)^{-\frac{2}{\lambda}-1} \\ -\left[(1+\lambda t)^{\frac{2}{\lambda}-1} - 2(1+\lambda t)^{-1} + (1+\lambda t)^{-\frac{2}{\lambda}-1}\right] \end{array}\right]}{\left[(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}\right]^2} \\
&= \frac{4(1+\lambda t)^{-1}}{\left[(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}\right]^2} \\
&= \frac{(1+\lambda t)^{-1}}{\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2} \\
&= \frac{1}{(1+\lambda t)\cosh_\lambda^2(t)} \\
&= \frac{\operatorname{sech}_\lambda^2(t)}{(1+\lambda t)}
\end{aligned} \tag{2.8}
$$

and this gives the desired results (2.3). Next,

$$
\begin{aligned}
(\coth_\lambda(t))' &= \frac{\left[\begin{array}{c} \left[(1+\lambda t)^{\frac{1}{\lambda}-1} - (1+\lambda t)^{-\frac{1}{\lambda}-1}\right]\left[(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}\right] \\ -\left[(1+\lambda t)^{\frac{1}{\lambda}-1} + (1+\lambda t)^{-\frac{1}{\lambda}-1}\right]\left[(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}\right] \end{array}\right]}{\left[(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}\right]^2} \\
&= \frac{\left[\begin{array}{c} (1+\lambda t)^{\frac{2}{\lambda}-1} - 2(1+\lambda t)^{-1} + (1+\lambda t)^{-\frac{2}{\lambda}-1} \\ -\left[(1+\lambda t)^{\frac{2}{\lambda}-1} + 2(1+\lambda t)^{-1} + (1+\lambda t)^{-\frac{2}{\lambda}-1}\right] \end{array}\right]}{\left[(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}\right]^2} \\
&= -\frac{4(1+\lambda t)^{-1}}{\left[(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}\right]^2} \\
&= -\frac{(1+\lambda t)^{-1}}{\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2} \\
&= -\frac{1}{(1+\lambda t)\sinh_\lambda^2(t)} \\
&= -\frac{\operatorname{csch}_\lambda^2(t)}{(1+\lambda t)}
\end{aligned} \tag{2.9}
$$

This yields the desired results (2.4). Next,

$$\begin{aligned}
\left(\operatorname{sech}_{\lambda}(t)\right)' &= \frac{-2\left[(1+\lambda t)^{\frac{1}{\lambda}-1}-(1+\lambda t)^{-\frac{1}{\lambda}-1}\right]}{\left[(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}} \\
&= \left[\frac{-2}{(1+\lambda t)\left[(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}\right]}\right]\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}\right] \\
&= -\frac{\operatorname{sech}_{\lambda}(t)\tanh_{\lambda}(t)}{(1+\lambda t)},
\end{aligned}$$

which gives the results (2.5). Finaly, we have

$$\begin{aligned}
\left(\operatorname{csch}_{\lambda}(t)\right)' &= \frac{-2\left[(1+\lambda t)^{\frac{1}{\lambda}-1}+(1+\lambda t)^{-\frac{1}{\lambda}-1}\right]}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}} \\
&= \left[\frac{-2}{(1+\lambda t)\left[(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}\right]}\right]\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}\right] \\
&= -\frac{\operatorname{csch}_{\lambda}(t)\coth_{\lambda}(t)}{(1+\lambda t)},
\end{aligned}$$

which gives the desired results (2.6). This concludes the proof. □

Proposition 2.2. *For* $t \in (-\infty, \infty), \lambda \in (0, \infty)$ *and* $n \in \mathbb{N}$*, the limit of the* nth *derivative of the degenerate sine function as* $\lambda \to 0$*, is given as*

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)+(-1)^{n+1}(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right] \\
&=\frac{1}{2}\left[e^{t}+(-1)^{n+1}e^{-t}\right]. \qquad (2.10)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\left(\sinh_{\lambda}(t)\right)' &= \frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-1}+(1+\lambda t)^{-\frac{1}{\lambda}-1}\right], \\
\left(\sinh_{\lambda}(t)\right)'' &= \frac{1}{2}\left[(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2}-(1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2}\right], \\
\left(\sinh_{\lambda}(t)\right)''' &= \frac{1}{2}\left[(1-2\lambda)(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-3}+(1+2\lambda)(1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-3}\right], \\
\left(\sinh_{\lambda}(t)\right)^{(4)} &= \frac{1}{2}\left[(1-3\lambda)(1-2\lambda)(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-4}-(1+3\lambda)(1+2\lambda)(1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-4}\right].
\end{aligned}$$

Continuing the process n number of times, we have

$$\left(\sinh_{\lambda}(t)\right)^{(n)} = \frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)+(-1)^{n+1}(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right].$$

From the properties of limits, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)+(-1)^{n+1}(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right]\\
&=\frac{1}{2}\left[\lim_{\lambda\to 0}\ (1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)+(-1)^{n+1}\lim_{\lambda\to 0}(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right]. \quad (2.11)
\end{aligned}$$

From equation (2.11), let

$$y=\lim_{\lambda\to 0}\ \left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)\right] \tag{2.12}$$

and

$$z=\lim_{\lambda\to 0}\ \left[(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right]. \tag{2.13}$$

Taking the natural logarithm on both sides of (2.12), we have

$$\begin{aligned}
\ln y &= \lim_{\lambda\to 0}\ln\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)\right]\\
&=\lim_{\lambda\to 0}\left(\frac{1}{\lambda}-n\right)\ln(1+\lambda t)+\lim_{\lambda\to 0}\ln\prod_{k=0}^{n-1}(1-\lambda k)\\
&=\lim_{\lambda\to 0}\frac{(1-\lambda n)\ln(1+\lambda t)}{\lambda}+\sum_{k=0}^{n-1}\lim_{\lambda\to 0}\ln(1-\lambda k)\\
&=\lim_{\lambda\to 0}\frac{(1-\lambda n)\ln(1+\lambda t)}{\lambda}+\sum_{k=0}^{n-1}\ln(1)\\
&=\lim_{\lambda\to 0}\left[-n\ln(1+\lambda t)+\frac{(1-\lambda n)\,t}{1+\lambda t}\right]\\
&=\lim_{\lambda\to 0}-n\ln(1+\lambda t)+\lim_{\lambda\to 0}+\frac{t-\lambda nt}{1+\lambda t}\\
&=t.
\end{aligned}$$

This implies,

$$y=e^{t}.$$

Also, taking the natural logarithm on both sides of (2.13), we have

$$\begin{aligned}
\ln z &= \lim_{\lambda\to 0}\ln\left[(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right] \\
&= \lim_{\lambda\to 0}\left(-\frac{1}{\lambda}-n\right)\ln(1+\lambda t)+\lim_{\lambda\to 0}\ln\prod_{k=0}^{n-1}(1+\lambda k) \\
&= \lim_{\lambda\to 0}\frac{-(1+\lambda n)\ln(1+\lambda t)}{\lambda}+\sum_{k=0}^{n-1}\lim_{\lambda\to 0}\ln(1+\lambda k) \\
&= \lim_{\lambda\to 0}\frac{-(1+\lambda n)\ln(1+\lambda t)}{\lambda}+\sum_{k=0}^{n-1}\ln(1) \\
&= \lim_{\lambda\to 0}\left[-n\ln(1+\lambda t)-\frac{(1+\lambda n)t}{1+\lambda t}\right] \\
&= \lim_{\lambda\to 0}-n\ln(1+\lambda t)-\lim_{\lambda\to 0}+\frac{t+\lambda nt}{1+\lambda t} \\
&= -t.
\end{aligned}$$

Thus,

$$z = e^{-t}.$$

From equation (2.11), we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)+(-1)^{n+1}(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right] \\
&= \frac{1}{2}\left[y+(-1)^{n+1}z\right] \\
&= \frac{1}{2}\left[e^{t}+(-1)^{n+1}e^{-t}\right].
\end{aligned}$$

This completes the proof. □

Proposition 2.3. *The limit of the* nth *derivative of the degenerate cosine function as* $\lambda \to 0$*, is given as*

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k)+(-1)^{n+1}(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right] \\
&= \frac{1}{2}\left[e^{t}+(-1)^{n}e^{-t}\right],
\end{aligned} \tag{2.14}$$

for $t \in (-\infty,\infty)\,, \lambda \in (0,\infty)$ *and* $n \in \mathbb{N}$.

Proof. We have

$$\begin{aligned}(\cosh_\lambda(t))' =& \frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-1} - (1+\lambda t)^{-\frac{1}{\lambda}-1}\right],\\ (\cosh_\lambda(t))'' =& \frac{1}{2}\left[(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2} + (1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2}\right],\\ (\cosh_\lambda(t))''' =& \frac{1}{2}\left[(1-2\lambda)(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-3} - (1+2\lambda)(1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-3}\right],\\ (\cosh_\lambda(t))^{(4)} =& \frac{1}{2}\left[(1-3\lambda)(1-2\lambda)(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-4} + (1+3\lambda)(1+2\lambda)(1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-4}\right].\end{aligned}$$

Continuing the process n number of times, we have

$$(\cosh_\lambda(t))^{(n)} = \frac{1}{2}\left[(1+\lambda t)^{\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1-\lambda k) + (-1)^n(1+\lambda t)^{-\frac{1}{\lambda}-n}\prod_{k=0}^{n-1}(1+\lambda k)\right].$$

Now, replacing $(-1)^{n+1}$ in the proof of proposition (2.2) with $(-1)^n$, we obtain the desired results. □

Proposition 2.4. *For all* $t \in (-\infty, \infty)$ *and* $\lambda \in (0, \infty)$, *the degenerate hyperbolic functions satisfy the following identities*

$$\cosh_\lambda(t) + \sinh_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \tag{2.15}$$

$$\cosh_\lambda(t) - \sinh_\lambda(t) = (1+\lambda t)^{-\frac{1}{\lambda}}, \tag{2.16}$$

$$(\cosh_\lambda(t))'' + (\sinh_\lambda(t))'' = (1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2}, \tag{2.17}$$

$$(\cosh_\lambda(t))'' - (\sinh_\lambda(t))'' = (1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2}, \tag{2.18}$$

$$\cosh^2_\lambda(t) + \sinh^2_\lambda(t) = \cosh_\lambda(2t), \tag{2.19}$$

$$\cosh^2_\lambda(t) - \sinh^2_\lambda(t) = 1, \tag{2.20}$$

$$1 - \tanh^2_\lambda(t) = \text{sech}^2_\lambda(t), \tag{2.21}$$

$$\coth^2_\lambda(t) - 1 = \text{csch}^2_\lambda(t). \tag{2.22}$$

Proof. To begin with, we have

$$\begin{aligned}\cosh_\lambda(t) + \sinh_\lambda(t) =& \frac{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}{2} + \frac{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\\ =& (1+\lambda t)^{\frac{1}{\lambda}},\end{aligned}$$

which gives the result (2.15). Next,

$$\begin{aligned}\cosh_\lambda(t) - \sinh_\lambda(t) =& \frac{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}{2} - \left[\frac{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]\\ =& (1+\lambda t)^{-\frac{1}{\lambda}},\end{aligned}$$

which gives the equation (2.16). Next,

$$
\begin{aligned}
\left(\cosh_\lambda(t)\right)'' + \left(\sinh_\lambda(t)\right)'' =& \frac{(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2} + (1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2}}{2} \\
&+ \frac{(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2} - (1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2}}{2} \\
=& (1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2},
\end{aligned}
$$

which gives the result (2.17). Next,

$$
\begin{aligned}
\left(\cosh_\lambda(t)\right)'' - \left(\sinh_\lambda(t)\right)'' =& \frac{(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2} + (1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2}}{2} \\
&- \left[\frac{(1-\lambda)(1+\lambda t)^{\frac{1}{\lambda}-2} - (1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2}}{2}\right] \\
=& (1+\lambda)(1+\lambda t)^{-\frac{1}{\lambda}-2},
\end{aligned}
$$

which gives the result (2.18). Next,

$$
\begin{aligned}
\cosh_\lambda^2(t) + \sinh_\lambda^2(t) =& \left[\frac{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2 + \left[\frac{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2 \\
=& \frac{(1+\lambda t)^{\frac{2}{\lambda}} + (1+\lambda t)^{-\frac{2}{\lambda}} + 2}{4} + \frac{(1+\lambda t)^{\frac{2}{\lambda}} + (1+\lambda t)^{-\frac{2}{\lambda}} - 2}{4} \\
=& \frac{(1+\lambda t)^{\frac{2}{\lambda}} + (1+\lambda t)^{-\frac{2}{\lambda}}}{2} \\
=& \cosh_\lambda(2t),
\end{aligned}
$$

which gives the result (2.19). Next,

$$
\begin{aligned}
\cosh_\lambda^2(t) - \sinh_\lambda^2(t) =& \left[\frac{(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2 - \left[\frac{(1+\lambda t)^{\frac{1}{\lambda}} - (1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2 \\
=& \frac{(1+\lambda t)^{\frac{2}{\lambda}} + (1+\lambda t)^{-\frac{2}{\lambda}} + 2}{4} - \left[\frac{(1+\lambda t)^{\frac{2}{\lambda}} + (1+\lambda t)^{-\frac{2}{\lambda}} - 2}{4}\right] \\
=& 1,
\end{aligned}
$$

which gives the result (2.20). Next,

$$
\begin{aligned}
1-\tanh_{\lambda}^{2}(t) =& 1-\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}\right]^{2}\\
=& 1-\left[\frac{(1+\lambda t)^{\frac{2}{\lambda}}+(1+\lambda t)^{-\frac{2}{\lambda}}-2}{\left[(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}}\right]\\
=& \frac{(1+\lambda t)^{\frac{2}{\lambda}}+(1+\lambda t)^{-\frac{2}{\lambda}}+2-(1+\lambda t)^{\frac{2}{\lambda}}-(1+\lambda t)^{-\frac{2}{\lambda}}+2}{\left[(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}}\\
=& \frac{4}{\left[(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}}\\
=& \frac{1}{\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^{2}}\\
=& \frac{1}{\cosh_{\lambda}^{2}(t)}\\
=& \operatorname{sech}_{\lambda}^{2}(t),
\end{aligned}
$$

and this yields the desired results (2.21). Finally, we have

$$
\begin{aligned}
\coth_{\lambda}^{2}(t)-1 =& \left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}\right]^{2}-1\\
=& \left[\frac{(1+\lambda t)^{\frac{2}{\lambda}}+(1+\lambda t)^{-\frac{2}{\lambda}}+2}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}}\right]-1\\
=& \frac{(1+\lambda t)^{\frac{2}{\lambda}}+(1+\lambda t)^{-\frac{2}{\lambda}}+2-(1+\lambda t)^{\frac{2}{\lambda}}-(1+\lambda t)^{-\frac{2}{\lambda}}+2}{\left[(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}}\\
=& \frac{4}{\left[(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}\right]^{2}}\\
=& \frac{1}{\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^{2}}\\
=& \frac{1}{\sinh_{\lambda}^{2}(t)}\\
=& \operatorname{csch}_{\lambda}^{2}(t).
\end{aligned}
$$

This gives the result (2.22). □

The results that follow, were inspired by the works of [13], [14] and [15].

Lemma 2.5. *For all* $t \in (0, \infty)$ *and* $\lambda \in (0, \infty)$*, the inequality*

$$0 < \tanh_\lambda(t) < 1 \tag{2.23}$$

is valid.

Proof. Recall from (2.8) that

$$\begin{aligned}
(\tanh_\lambda(t))' &= \frac{(1+\lambda t)^{-1}}{\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2} \\
&= \frac{1}{(1+\lambda t)\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]^2} \\
&= \frac{1}{\left\{(1+\lambda t)^{\frac{1}{2}}\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}{2}\right]\right\}^2} \\
&= \frac{1}{\left[\frac{(1+\lambda t)^{\frac{1}{\lambda}+\frac{1}{2}}+(1+\lambda t)^{-\frac{1}{\lambda}+\frac{1}{2}}}{2}\right]^2} > 0,
\end{aligned}$$

which implies that $\tanh_\lambda(t)$ is increasing. Next, we have

$$\begin{aligned}
\lim_{t\to 0}\tanh_\lambda(t) &= \lim_{t\to 0}\frac{(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}} \\
&= \lim_{t\to 0}\frac{(1+\lambda t)^{\frac{2}{\lambda}}-1}{(1+\lambda t)^{\frac{2}{\lambda}}+1} \\
&= 0, \\
\lim_{t\to\infty}\tanh_\lambda(t) &= \lim_{t\to\infty}\frac{1-(1+\lambda t)^{-\frac{2}{\lambda}}}{1+(1+\lambda t)^{-\frac{2}{\lambda}}} = 1.
\end{aligned}$$

Thus for $t \in (0, \infty)$, we have

$$0 = \lim_{t\to 0}\tanh_\lambda(t) < \tanh_\lambda(t) < \lim_{t\to\infty}\tanh_\lambda(t) = 1.$$

This completes the proof. □

Theorem 2.6. *The inequality*

$$0 < (1+\lambda u)\ln\frac{\cosh_\lambda(t)}{\cosh_\lambda(r)} < t - r \tag{2.24}$$

is satisfied for all $\lambda, r, t \in (0, \infty)$ *such that* $r < t$ *and* $u \in (r, t)$.

Proof. Let $f(x) = \ln \cosh_\lambda(x)$ on the interval (r,t). By the mean value theorem, there exist a $u \in (r,t)$ such that

$$\begin{aligned} f'(u) &= \frac{\sinh_\lambda(u)}{(1+\lambda u)\cosh_\lambda(u)} \\ &= \frac{\tanh_\lambda(u)}{(1+\lambda u)} \\ &= \frac{\ln\cosh_\lambda(t) - \ln\cosh_\lambda(r)}{t-r}. \end{aligned}$$

This implies

$$\tanh_\lambda(u) = \frac{(1+\lambda u)}{t-r} \ln \frac{\cosh_\lambda(t)}{\cosh_\lambda(r)}.$$

Then by Lemma 2.5, we have

$$0 < \frac{(1+\lambda u)}{t-r} \ln \frac{\cosh_\lambda(t)}{\cosh_\lambda(r)} < 1,$$

which gives the desired results (2.24). □

Theorem 2.7. *For* $\lambda, t \in (0,\infty)$, *the inequality*

$$\frac{\ln(1+\lambda t)}{\lambda} < \sinh_\lambda(t)\cosh_\lambda(t) \tag{2.25}$$

holds.

Proof. Let

$$\theta(t) = \frac{\ln(1+\lambda t)}{\lambda} - \sinh_\lambda(t)\cosh_\lambda(t)$$

We have

$$\begin{aligned} \theta'(t) &= \frac{1}{1+\lambda t} - \left(\frac{\cosh_\lambda(t)\cosh_\lambda(t)}{1+\lambda t} + \frac{\sinh_\lambda(t)\sinh_\lambda(t)}{1+\lambda t} \right) \\ &= \frac{1}{1+\lambda t} - \frac{\cosh_\lambda^2(t)}{1+\lambda t} - \frac{\sinh_\lambda^2(t)}{1+\lambda t} \\ &= \frac{1-\cosh_\lambda^2(t) - \sinh_\lambda^2(t)}{1+\lambda t} \\ &= \frac{1-\left[\sinh_\lambda^2(t)+1\right] - \sinh_\lambda^2(t)}{1+\lambda t} \\ &= \frac{1-\sinh_\lambda^2(t) - 1 - \sinh_\lambda^2(t)}{1+\lambda t} \\ &= \frac{-2\sinh_\lambda^2(t)}{1+\lambda t} \\ &< 0. \end{aligned}$$

Therefore, $\theta(t)$ is decreasing and consequently $\theta(t) < \lim_{t\to 0^+} \theta(t) = 0$. This completes the proof. □

Theorem 2.8. *The inequality*

$$\frac{\sinh_\lambda(t)}{t} < \cosh_\lambda(t) \tag{2.26}$$

is valid for $t, \lambda \in (0, \infty)$.

Proof. Let

$$\xi(t) = \sinh_\lambda(t) - t\cosh_\lambda(t).$$

Then,

$$\begin{aligned}
\xi'(t) &= \frac{\cosh_\lambda(t)}{1+\lambda t} - \cosh_\lambda(t) - \frac{t\sinh_\lambda(t)}{1+\lambda t} \\
&= \frac{\cosh_\lambda(t) - (1+\lambda t)\cosh_\lambda(t) - t\sinh_\lambda(t)}{1+\lambda t} \\
&= \frac{[1-(1+\lambda t)]\cosh_\lambda(t) - t\sinh_\lambda(t)}{1+\lambda t} \\
&= -\frac{\lambda t\cosh_\lambda(t) + t\sinh_\lambda(t)}{1+\lambda t} \\
&< 0.
\end{aligned}$$

This implies that $\xi(t)$ is decreasing. Thus, $\xi(t) < \lim_{t\to 0^+} \xi(t) = 0$, which completes the proof. □

3. Conclusion

We have obtained some limit and monotonic properties as well as inequalities involving the degenerate hyperbolic functions. These established properties are useful in many fields in mathematics.

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Conflicts of interest

On behalf of all authors, the corresponding author certifies that there is no conflict of interest with the publication of this study.

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