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Decomposable positive map from $\mathbb{M}_3(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{M}_2(\mathbb{C}))$

C. A. WINDA [a,*], N. B. OKELO[b], P. M. OMOKE [c]

[a,b,c] Department of Pure and Applied Mathematics,
Jaramogi Oginga Odinga University of Science and Technology, Kenya

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Abstract

In most literature, the decomposition of positive maps from $\mathbb{M}_3$ to $\mathbb{M}_2$ are discussed where the matrix elements are complex numbers. In this paper we construct a positive maps $\phi_{(\mu,c_1,c_2)}$ from $\mathbb{M}_3(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{M}_2(\mathbb{C}))$. The Choi matrices for complete positivity and complete copositivity are visualized as tensor matrix $\mathbb{M}_3 \otimes \mathbb{M}_2$ with $\mathbb{M}_2(\mathbb{C})$ as the entry elements. The construction allows us to describe decomposability on positive semidefinite matrices.

Keywords: Positive maps, 2-positivity, Choi matrix, completely positivity, decomposable maps.
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1. Introduction

Positive linear maps on C^*-algebras, particularly those of finite dimensions have been very important in quantum information theory and quantum channels. Stinespring [6] initiated the concept of completely positive maps with his representation(or dilation) theorem. Arveson in [1] and [2] found the application of completely positive maps in operator theory and further developed extensively in operator algebra and mathematical physics. Woronowicz [11], Theorem 3.1.6 showed that every positive linear map ϕ form $\mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_m(\mathbb{C})$ is decomposable if and only if $m \leqslant 3$. In [7], [8] and [9] Theorem 1, Størmer gives conditions for decomposability of positive maps; For $\mathcal{A}$ be a C^*-algebra and linear map ϕ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $\mathbb{M}_n(\mathcal{A})^+$. Choi [4] gave the first example of indecomposable map, for a 3-dimension case.

Yang, Leung and Tang [12] showed that every 2-positive linear map from $\mathbb{M}_3(\mathbb{C})$ to $\mathbb{M}_3(\mathbb{C})$ is decomposable. Though we are motivated by the question in [12] that enquire if there exist indecomposable 2-positive maps from $\mathbb{M}_3(\mathbb{C})$ to $\mathbb{M}_4(\mathbb{C})$), we show there is a decomposable positive map from $\mathbb{M}_3(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{M}_2(\mathbb{C}))$.

*Corresponding author: windac758@gmail.com

In most literature, the authors have studied case $\mathbb{M}_3$ to $\mathbb{M}_2$ where the matrix elements are complex numbers. In this paper we construct a positive maps $\phi_{(\mu,c_1,c_2)}$ from $\mathbb{M}_3(\mathbb{C})$ to $\mathbb{M}_2$ where the matrix elements of $\mathbb{M}_2$ is a 2×2 positive matrix $\mathbb{M}_2(\mathbb{C})$. We find conditions on the triplet μ, c_1, c_2 for which the map is positive, completely positive, 2-positive and decomposable.

A matrix $X \in \mathbb{M}_n(\mathbb{C})$ is called positive semi-definite if it is hermitian and all its eigenvalues are positive. It is denoted as $X \geqslant 0$. The set of all positive semi-definite matrices in $\mathbb{M}_n(\mathbb{C})$ is denoted by $\mathbb{M}_n(\mathbb{C})^+$. Let the identity map on and the transpose map on $\mathbb{M}_n(\mathbb{C})^+$ be denoted by $\mathcal{I}_n$ and τ_n respectively. A linear map ϕ is from $\mathbb{M}_n(\mathbb{C})$ to $\mathbb{M}_m(\mathbb{C})$ is called positive if $\phi(\mathbb{M}_n(\mathbb{C}))^+ \subseteq \mathbb{M}_m(\mathbb{C})^+$. A map ϕ from $\mathbb{M}_n(\mathbb{C})$ to $\mathbb{M}_m(\mathbb{C})$ is $k-$*positive* if $\mathcal{I}\otimes\phi : \mathbb{M}_k\otimes\mathbb{M}_n \longrightarrow \mathbb{M}_k\otimes\mathbb{M}_m$ is positive. On the other hand, ϕ from $\mathbb{M}_n(\mathbb{C})$ to $\mathbb{M}_m(\mathbb{C})$ is $k-$*copositive* if the map $\tau_n\otimes\phi : \mathbb{M}_k\otimes\mathbb{M}_n \longrightarrow \mathbb{M}_k\otimes\mathbb{M}_n$ is positive. The Choi result in [3] affirms that the positive map ϕ is completely positive if and only if it's Choi matrix is positive semidefinite.

2. Positivity

Let $X \in \mathbb{M}_n(\mathbb{C})$ be a positive semidefinite matrix denoted by $X = [x_i\bar{x}_j]$, where $x_i = (x_1,\ldots,x_n)^T \in \mathbb{C}^n$ is a column vector and $\bar{x}_j$ is the transpose conjugate(row vector) of x_i. We denote the diagonal entries $x_n\bar{x}_n$ by α_n.

Let $X \in \mathbb{M}_3$ be a positive semidefinite matrix with complex entries. Let $0 < \mu \leqslant 1$, $c_1, c_2 > 0$ and $r \in \mathbb{N}$. Then we define the family of positive maps $\phi_{(\mu,c_1,c_2)}$ as follows:

$$\phi_{(\mu,c_1,c_2)} : \mathbb{M}_3(\mathbb{C})^+ \longrightarrow \mathbb{M}_2(\mathbb{M}_2)\mathbb{C})^+.$$

$$X \mapsto \left(\begin{array}{cc|cc} P_1^{\mu} & -c_1x_1\bar{x}_2 & 0 & -\mu x_1\bar{x}_3 \\ -c_1x_2\bar{x}_1 & P_2^{\mu} & -c_2x_2\bar{x}_3 & 0 \\ \hline 0 & -c_2x_3\bar{x}_2 & P_3^{\mu} & 0 \\ -\mu x_3\bar{x}_1 & 0 & 0 & P_4^{\mu} \end{array}\right), \tag{2.1}$$

where

$$\begin{aligned} P_1^{\mu} &= \mu^{-r}(\alpha_1 + c_1\alpha_2\mu^r + c_2\alpha_3\mu^r) \\ P_2^{\mu} &= \mu^{-r}(\alpha_2 + c_1\alpha_3\mu^r + c_2\alpha_1\mu^r) \\ P_3^{\mu} &= \mu^{-r}(\alpha_1 + \alpha_2 + \alpha_3) \\ P_4^{\mu} &= \mu^{-r}(\alpha_3 + c_1\alpha_1\mu^r + c_2\alpha_2\mu^r) \end{aligned}$$

The matrix $\phi_{(\mu,c_1,c_2)}(X)$ is visualized as a 2×2 block matrix in $\mathbb{M}_2(\mathbb{M}_2(\mathbb{C}))$.

The linear map ϕ is uniquely determined by the polynomial function;

$$F(z,x) := \nu\phi(x_i\bar{x}_j)\nu^T$$

as a biquadratic function in $x := (x_1, x_2, x_3)$ and $\nu := (\nu_1, \nu_2, \nu_3, \nu_4)$. The map ϕ is positive if and only if the biquadratic form $F(z,x)$ is a sum of squares (positive semi-definite).

We characterize of the positivity of the map ϕ for $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4$ and $t \in \mathbb{C}$.

Lemma 2.1. *Let* $0 < \mu < 1$ *and* $c_1, c_2 \geqslant 0$. *Then the function*

$$\begin{aligned} F(\nu_1,\nu_2,\nu_3,\nu_4,t) &= \mu^{-r}(1+c_1\mu^r+c_2|t|\mu^r)\nu_1^2+\mu^{-r}(1+c_1|t|\mu^r+c_2\mu^r)\nu_2^2+\mu^{-r}(2+|t|)\nu_3^2 \\ &\quad +\mu^{-r}(|t|+c_1\mu^r+c_2\mu^r)\nu_4^2-2c_1\nu_1\nu_2-2c_2\Re(t)\nu_2\nu_3-2\mu\Re(t)\nu_1\nu_4 \end{aligned}$$

is positive semidefinite for every $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{R}$ *and* $t \in \mathbb{C}$ *if and only if the following two conditions are satisfied:*

$$\mu^{-r} > c_1. \tag{2.2}$$

$$\mu^{-r} > c_2. \tag{2.3}$$

Proof. If $\nu_1 = 0$. Then,

$$\begin{aligned} &F(0,\nu_2,\nu_3,\nu_4,t) \\ &= \mu^{-r}(1+c_1|t|\mu^r+c_2\mu^r)\nu_2^2+\mu^{-r}(2+|t|)\nu_3^2+\mu^{-r}(|t|+c_1\mu^r+c_2\mu^r)\nu_4^2-2c_2\Re(t)\nu_2\nu_3 \\ &= (c_1|t|+c_2)\nu_2^2+2\mu^{-r}\nu_3^2+\mu^{-r}(|t|+c_1\mu^r+c_2\mu^r)\nu_4^2+(\mu^{-r}\nu_2^2-2\nu_2\nu_3c_2\Re(t)+\mu^{-r}|t|\nu_3^2) \\ &= (c_1|t|+c_2)\nu_2^2+\mu^{-r}(|t|+c_1\mu^r+c_2\mu^r)\nu_4^2+\mu^{-r}(\nu_2-\mu^r c_2\Re(t)\nu_3)^2 \\ &+ (2\mu^{-r}+\mu^{-r}|t|-\mu^r c_2^2\Re(t)^2)\nu_3^2. \end{aligned}$$

$F(0,\nu_2,\nu_3,\nu_4,t)$ is positive when the coefficient of ν_3^2 satisfy the inequality,

$$\mu^{-2r}(2+|t|)-c_2^2\Re^2(t)^2 \geqslant 0. \tag{2.4}$$

Letting $t = x+iy$. We have that,

$$\begin{aligned} \mu^{-r}(2+|t|)-\mu^r c_2^2\Re^2(t)^2 &= 2\mu^{-2r}+(\mu^{-2r}(|x|^2+|y|^2)-x^2c_2^2) \\ &= 2\mu^{-2r}+\mu^{-2r}|y|^2+|x|^2(\mu^{-2r}-c_2^2) \end{aligned}$$

is positive whenever $\mu^{-r} \geqslant c_2$ hold.

If $\nu_2 = 0$. Then ,

$$\begin{aligned} &F(\nu_1,0,\nu_3,\nu_4,t) \\ &= \mu^{-r}(1+c_1\mu^r+c_2|t|\mu^r)\nu_1^2+\mu^{-r}(2+|t|)\nu_3^2+\mu^{-r}(|t|+c_1\mu^r+c_2\mu^r)\nu_4^2-2\mu\Re(t)\nu_1\nu_4 \\ &= \mu^{-r}(c_1\mu^r+c_2|t|\mu^r)\nu_1^2+\mu^{-r}(2+|t|)\nu_3^2+\mu^{-r}(c_1\mu^r+c_2\mu^r)\nu_4^2+(\mu^{-r}\nu_1^2-2\nu_1\nu_4\mu\Re(t)+\mu^{-r}|t|\nu_4^2) \\ &= \mu^{-r}(c_1\mu^r+c_2|t|\mu^r)\nu_1^2+\mu^{-r}(2+|t|)\nu_3^2+\mu^{-r}(c_1\mu^r+c_2\mu^r)\nu_4^2 \\ &+ \mu^{-r}(\nu_1-\mu^{1+r}\Re(t)\nu_4)^2+\mu^{-r}(|t|-\mu^{2+2r}\Re(t)^2)\nu_4^2 \\ &\geqslant 0. \end{aligned}$$

If $\nu_3 = 0$. Then,

$$\begin{aligned} &F(\nu_1,\nu_2,0,\nu_4,t) \\ &= \mu^{-r}(1+c_1\mu^r+c_2|t|\mu^r)\nu_1^2+\mu^{-r}(1+c_1|t|\mu^r+c_2\mu^r)\nu_2^2+\mu^{-r}(|t|+c_1\mu^r+c_2\mu^r)\nu_4^2 \\ &- 2c_1\nu_1\nu_2-2\mu\Re(t)\nu_1\nu_4 \\ &= c_2|t|\nu_1^2+\mu^{-r}(1+c_2\mu^r)\nu_2^2+(c_1+c_2)\nu_4^2+(\mu^{-r}\nu_1^2-2\nu_1\nu_4\mu\Re(t)+\mu^{-r}|t|\nu_4^2)+c_1(\nu_1^2-2\nu_1\nu_2+|t|\nu_2^2) \\ &= c_2|t|\nu_1^2+\mu^{-r}(1+c_2\mu^r)\nu_2^2+(c_1+c_2)\nu_4^2+\mu^{-r}(\nu_1-\mu^{1+r}\Re(t)\nu_4)^2 \\ &\quad +\mu^{-r}(|t|-\mu^{2+2r}\Re(t)^2)\nu_4^2+c_1(\nu_1-\nu_2)^2+c_1(|t|^2-1)\nu_2^2 \end{aligned}$$

$F(\nu_1,\nu_2,0,\nu_4,t)$ is positive whenever $\mu^{-r}-c_1 \geqslant 0$ hold. That is, the coefficients of ν_2^2 is such that,

$$\mu^{-r}+c_2+c_1(|t|-1) = (\mu^{-r}-c_1)+c_2+c_1|t| \geqslant 0. \tag{2.5}$$

If $v_4 = 0$. Then,

$$
\begin{aligned}
&F(v_1, v_2, v_3, 0, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 - 2c_1v_1v_2 - 2c_2\Re(t)v_2v_3\\
&= \mu^{-r}(1 + c_2|t|\mu^r)v_1^2 + c_2v_2^2 + 2\mu^{-r}v_3^2 + c_1(v_1^2 - 2v_1v_2 + |t|v_2^2) + (\mu^{-r}v_2^2 - 2c_2\Re(t)v_2v_3 + \mu^{-r}|t|v_3^2)\\
&= \mu^{-r}(1 + c_2|t|\mu^r)v_1^2 + c_2v_2^2 + 2\mu^{-r}v_3^2 + c_1(v_1 - v_2)^2 + c_1(|t|^2 - 1)v_2^2\\
&+ \mu^{-r}(v_2 - \mu^r c_2\Re(t)v_3)^2 + (\mu^{-r}|t| - \mu^r c_2^2\Re(t)^2)v_3^2\\
&\geqslant 0
\end{aligned}
$$

whenever the inequalities (2.4) and (2.5) hold.
Now let $v_i \neq 0, i = 1, 2, 3, 4$ and assume that there exist $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$ such that $v_1 \neq 0$ and $F(v_1, v_2, v_3, v_4, t) < 0$. Since $0 < \mu < 1$ and $c_1, c_2 \geqslant 0$. Then,

$$
\begin{aligned}
&F(v_1, v_2, v_3, v_4, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2\\
&\quad -2c_1v_1v_2 - 2c_2\Re(t)v_2v_3 - 2\mu\Re(t)v_1v_4\\
&= \mu^{-r}v_1^2 + \mu^{-r}v_2^2 + 2\mu^{-r}v_3^2 + (c_1 + c_2)\mu^{-r}v_4^2 + c_1(v_1 - v_2)^2 + c_1(|t|^2 - 1)v_2^2 + \mu^{-r}(v_2 - \mu^r c_2\Re(t)v_3)^2\\
&\quad +(\mu^{-r}|t| - \mu^r c_2^2\Re(t)^2)v_3^2 + \mu^{-r}(v_1 - \mu^{1+r}\Re(t)v_4)^2 + (\mu^{-r}|t| - \mu^{2+2r}\Re(t)^2)v_4^2\\
&< 0
\end{aligned}
$$

is a contradiction when the inequalities (2.4) and (2.5) hold . Thus $F(v_1, v_2, v_3, v_4, t) \geqslant 0$ for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$. □

Proposition 2.2. *The linear map $\phi_{(\mu, c_1, c_2)}$ is positive provided Lemma 2.1 is satisfied. are satisfied.*

Proof. We need to show that,

$$
\phi\left(\begin{pmatrix} q \\ s \\ t \end{pmatrix} \begin{pmatrix} \bar{q} & \bar{s} & \bar{t} \end{pmatrix}\right) \in \mathbb{M}_4^+
$$

for every $q, s, t \in \mathbb{C}$.
Thai is.

$$
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}^T
\left(\begin{array}{cc|cc}
P_1^\mu & -c_1q\bar{s} & 0 & -\mu q\bar{t} \\
-c_1s\bar{q} & P_2^\mu & -c_2s\bar{t} & 0 \\
\hline
0 & -c_2t\bar{s} & P_3^\mu & 0 \\
-\mu t\bar{q} & 0 & 0 & P_4^\mu
\end{array}\right)
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \geqslant 0 \tag{2.6}
$$

where,

$$
\begin{aligned}
P_1^\mu &= \mu^{-r}(|q|^2 + c_1|s|^2\mu^r + c_2|t|\mu^r)\\
P_2^\mu &= \mu^{-r}(|s|^2 + c_1|t|\mu^r + c_2|q|^2\mu^r)\\
P_3^\mu &= \mu^{-r}(|q|^2 + |s|^2 + |t|)\\
P_4^\mu &= \mu^{-r}(|t| + c_1|q|^2\mu^r + c_2|s|^2\mu^r)
\end{aligned}
$$

for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $q, s, t \in \mathbb{C}$.
Taking $q = s = 0$.

$$
F(v_1, v_2, v_3, v_4, t) = c_1\mu^{-r}|t|v_1^2 + c_1|t|v_2^2 + \mu^{-r}|t|v_3^2 + \mu^{-r}|t|v_4^2 \geqslant 0.
$$

If $q = 0$. Since $0 < \mu \leqslant 1$, by inequality (2.4),

$$\begin{aligned} F(v_1, v_2, v_3, v_4, t) \\ &= (c_1 + c_2|t|)v_1^2 + \mu^{-r}(1 + c_1|t|)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + (\mu^{-r}|t| + c_2)v_4^2 - 2c_2\mathfrak{R}(t)v_2v_3 \\ &\geqslant 0. \end{aligned}$$

If $s = 0$,

$$\begin{aligned} F(v_1, v_2, v_3, v_4, t) \\ &= \mu^{-r}(1 + c_2|t|)v_1^2 + (c_1|t| + c_2)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r)v_4^2 - 2\mu\mathfrak{R}(t)v_1v_4 \\ &= c_2|t|v_1^2 + (c_1|t| + c_2)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + c_1v_4^2 + \mu^{-r}(v_1 - \mu^{1+r}\mathfrak{R}(t)v_4)^2 \\ &+ (\mu^{-r}|t| - \mu^{2+r}\mathfrak{R}(t)^2)v_4^2 \\ &\geqslant 0. \end{aligned}$$

If q and s are not equal to zero. Assume that $q = s = 1$. Then, by Lemma 2.1

$$z^T\phi\left(\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} \begin{pmatrix} 1 & 1 & \bar{t} \end{pmatrix}\right) z \geqslant 0$$

since the polynomial,

$$\begin{aligned} F(v_1, v_2, v_3, v_4, t) &= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 \\ &\quad + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_1v_1v_2 - 2c_2\mathfrak{R}(t)v_2v_3 - 2\mu\mathfrak{R}(t)v_1v_4 \\ &= \mu^{-r}v_1^2 + \mu^{-r}v_2^2 + c_1(v_1 - v_2)^2 + c_1(|t|^2 - 1)v_2^2 + (2\mu^{-r} + \mu^{-r}|t| \\ &- \mu^r c_2^2\mathfrak{R}(t)^2)v_3^2 + \mu^{-r}(v_2 - \mu^r c_2\mathfrak{R}(t)v_3)^2 \\ &+ \mu^{-r}(v_1 - \mu^{1+r}\mathfrak{R}(t)v_4)^2 + (\mu^{-r}(c_1 + c_2 + |t|) - \mu^{2+2r}\mathfrak{R}(t)^2)v_4^2 \\ &\geqslant 0 \end{aligned}$$

for every $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $t \in \mathbb{C}$. □

3. Complete (co)positivity

The structure of the Choi matrix $C_{\phi_{(\mu,c_1,c_2)}} \in \mathbb{M}_3(\mathbb{M}_2(\mathbb{M}_2))$ is visualized as a block matrix whose entries are 2×2 matrices within the 6×6 matrix.

Proposition 3.1. *Let $\phi_{((\mu,c_1,c_2)}$ be a map given by (2.1). The following conditions are equivalent:*

(i) *$\phi_{(\mu,c_1,c_2)}$ is completely positive,*

(ii) *$\phi_{(\mu,c_1,c_2)}$ is 2-positive and,*

(iii) *$\mu^{-r} \geqslant c_1$ and $\mu^{-2r} \geqslant c_1^2 + c_2^2$.*

Proof. $(ii) \Rightarrow (iii)$.
Assume $\phi_{(\mu,c_1,c_2)}$ is 2-positive. Then

$$\mathcal{I}_2 \otimes \phi_{(\mu,c_1,c_2)}(P) = \left(\begin{array}{cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & -\mu \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & -c_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \mu^{-r} & \cdot \\
-\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \in \mathbb{M}_2(\mathbb{M}_2(\mathbb{M}_2)) \tag{3.1}$$

is positive semidefinite, where dots replace zeros. Since $\phi_{(\mu,c_1,c_2)}$ is 2-positive, the above matrix is positive definite. Therefore,

$$\begin{vmatrix}
\mu^{-r} & -c_1 & 0 & -\mu \\
-c_1 & \mu^{-r} & -c_2 & 0 \\
0 & c_2 & \mu^{-r} & 0 \\
-\mu & 0 & 0 & \mu^{-r}
\end{vmatrix} \geqslant 0 \tag{3.2}$$

and $\mu^{-r} \geqslant c_1$ and $\mu^{-2r} \geqslant c_1^2 + c_2^2$.
$(iii) \Rightarrow (i)$.
The Choi matrix of $\phi_{(\mu,c_1,c_2)}$ is of the form,

$$C_{\phi_{(\mu,c_1,c_2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
-\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \in \mathbb{M}_3(\mathbb{M}_2(\mathbb{M}_2)) \tag{3.3}$$

Since (iii) is satisfied, the inequality (3.2) holds, and consequently $C_{\phi_{(\mu,c_1,c_2)}}$ is positive definite. Hence, complete positivity of $\phi_{(\mu,c_1,c_2)}$ follows. □

Remark 3.2. The transposition in this case imply the Partial Positive transpose of the Choi matrix $C_{\phi_{(\mu,c_1,c_2)}} \in \mathbb{M}_3(\mathbb{M}_2)$. The transposition is operated with respect to a 2×2 matrix as the elements of $\mathbb{M}_3 \otimes \mathbb{M}_2$ matrix. This leads to the Partial Positive transpose Choi matrix $C^{\Gamma}_{\phi_{(\mu,c_1,c_2)}} \in \mathbb{M}_3(\mathbb{M}_2)$ with the structure. By Γ we denote partial transpose.

Proposition 3.3. *Let $\phi_{((\mu,c_1,c_2)}$ be a map given by (2.1). The following conditions are equivalent:*

(i) $\phi_{(\mu,c_1,c_2)}$ *is completely copositive,*

(ii) $\phi_{(\mu,c_1,c_2)}$ *is 2-copositive and,*

(iii) $\mu^{-r} \geqslant c_1$ *and* $c_1\mu^{-r} \geqslant c_2^2$

Proof. $(\mathrm{ii}) \Rightarrow (\mathrm{iii})$.
Assume $\phi_{(\mu,c_1,c_2)}$ is 2-copositive. Then

$$\tau_2 \otimes \phi_{(\mu,c_1,c_2)}(P) = \left(\begin{array}{cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & -\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & -c_2 \\
-c_1 & \cdot & -\mu & \cdot & \cdot & \mu^{-r} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \mu^{-r}
\end{array}\right) \in \mathbb{M}_2(\mathbb{M}_2(\mathbb{M}_2)) \tag{3.4}$$

is positive semidefinite with the minors positive when conditions in (iii) hold.

$(\mathrm{iii}) \Rightarrow (\mathrm{i})$
The choi matrix,

$$C^{\Gamma}_{\phi_{(\mu,c_1,c_2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & -c_2 & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1. & \cdot & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \tag{3.5}$$

in $\mathbb{M}_3(\mathbb{M}_2(\mathbb{M}_2))$.
Since (iii) is satisfied, by calculation of the minor, $C^{\Gamma}_{\phi_{(\mu,c_1,c_2)}}$ is positive semidefinite when $\mu^{-r} \geqslant c_1$ hold. Hence, complete copositivity follows.

□

Example 3.4. When $r = 3, \mu = \frac{1}{2}, c_1 = 1$ and $c_2 = 2$. Then,

$$C_{\phi_{(\frac{1}{2},1,2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
8 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & -2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 8 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{array}\right)$$

with eigenvalues

$$\{10.2477, 8.4449, 8., 8., 7.5551, 5.75232, 2., 2., 2., 1., 1., 1.\}$$

and

$$C^{\Gamma}_{\phi_{(\frac{1}{2},1,2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
8 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{array}\right)$$

with eigenvalues

$$\{9., 8.03553, 8., 8., 8., 7., 4., 2., 1., 1., 0.964466, 0.\}$$

.

4. Decomposability of $\phi_{(\mu,c_1,c_2)}$

A positive linear map is decomposable if it is the sum of a completely positive linear map and a completely copositive linear map. The result of Choi [3] shows that a positive linear map ϕ from $\mathbb{M}_n$ to $\mathbb{M}_m$ is decomposable if and only if there exist $n \times m$ matrices v_i and W_j such that,

$$\phi(X) = V_i X V_i^* + W_j X^T W_j^*$$

for every X in $\mathbb{M}_n$, where T is the transpose of X.

Proposition 4.1. *The linear map $\phi_{(\mu,c_1,c_2)}$ is decomposable.*

Proof. Let $\eta,\xi \in (0,1)$ and $a_i, b_i \in \mathbb{R}^+$ for $i = 1,2$ such that $\eta^{-r} + \xi^{-r} = \mu^{-r}$ and $a_i + b_i = c_i$. We show that there exist 2-positive map $\phi_{(\eta,a_1,a_2)}$ and 2-copositive map $\phi_{(\xi,b_1,b_2)}$ whose sum is $\phi_{(\mu,c_1,c_2)}$. Let $C_{\phi_{(\mu,c_1,c_2)}}$ be

$$\left(\begin{array}{cc|cc|cc|cc|cc|cc}
\beta & \cdot & \cdot & \cdot & \cdot & -\beta_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -q\mu \\
\cdot & \beta_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -(1-q)\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & \beta_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \beta_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\beta_1 & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & -a_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_2 & -b_2 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -b_2 & \beta_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -(1-q)\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_1. & \cdot & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & -a_2 & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\
-q\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta
\end{array}\right) \tag{4.1}$$

(where $\beta = \eta^{-r} + \xi^{-r}$, $\beta_1 = a_1 + b_1$, $\beta_2 = a_2 + b_2$) in $\mathbb{M}_3(\mathbb{M}_2(\mathbb{M}_2)\mathbb{C})$ give be the sum of;

$$C_{\phi_{(\eta,a_1,a_2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
\eta^{-r} & \cdot & \cdot & \cdot & \cdot & -a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -q\mu \\
\cdot & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-a_1 & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & -a_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & -a_2 & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot \\
-q\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r}
\end{array}\right)$$

and

$$C_{\phi_{(\xi,b_1,b_2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
\xi^{-r} & \cdot & \cdot & \cdot & \cdot & -b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \xi^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -(1-q)\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-b_1 & \cdot & \cdot & \cdot & \cdot & \xi^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & -b_2 & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -b_2 & b_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -(1-q)\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1. & \cdot & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi^{-r}
\end{array}\right)$$

When $q = 1$. Then, from the Choi matrices $C_{\phi_{(\eta,a_1,a_2)}}$ and $C_{\phi_{(\xi,b_1,b_2)}}$ the linear maps $\phi(\eta, a_1, a_2)$ is completely positive and $\phi(\xi, b_1, b_2)$ is completely copositive. On the other hand, when $q = 0$. Then $\phi(\eta, a_1, a))$ is completely copositive and $\phi(\xi, b_1, b_2)$ is completely positive. Hence, $\phi(\mu, c_1, c_2)$ is decomposable. □

Note that the decomposition of these maps is not unique.

5. Conclusion

It is known that every positive linear map ϕ from $\mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_m(\mathbb{C})$ is decomposable if and only if $m \leqslant 3$. The map $\phi_{(\mu,c_1,c_2)}$ from $\mathbb{M}_3(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{M}_2(\mathbb{C}))$ is also decomposable with 2×2 matrices as the entry elements of the Choi matrix in $\mathbb{M}_3(\mathbb{M}_2)(\mathbb{C})$. However, a look at the example by Woronowicz [11] and Tang' [10] of a map from $\mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_4(\mathbb{C})$ when approached as a map from $\mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{M}_2(\mathbb{C}))$ fails to be decomposable with 2×2 matrices as the elements of it's Choi matrix.

Declaration of competing interest

There is no competing interest.

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