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Common N-tupled Fixed Point Result in Rectangular Metric Spaces and their Applications to a System of N-Integral Equations

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Abstract

This manuscript establishes N-tupled fixed point result in rectangular b-metric. Using the obtained result, we also give an existence and uniqueness theorem for a non-linear N-integral equations class. Also, give an example of the validity of our result.

Keywords: Rectangular b-metric space, N-tupled coincidence point, Contractive mappings, W-compatible mapping pairs, Common N-tupled fixed point.
2010 MSC: Primary 47H10, secondary 54H25.

1. Introduction and Preliminaries

Branciari [6] introduced the concept of rectangular metric space as follows:

Definition 1.1. [6]. Let Δ be a non-empty set, and let $d : \Delta \times \Delta \to [0, \infty)$ be a mapping such that for all $\gamma_1, \vartheta_1 \in \Delta$, the following conditions holds:

1. $d(\gamma_1, \vartheta_1) = 0$ if and only if $\gamma_1 = \vartheta_1$;
2. $d(\gamma_1, \vartheta_1) = d(\vartheta_1, \gamma_1)$;
3. $d(\gamma_1, \eta_1) \leqslant d(\gamma_1, u_1) + d(u_1, v_1) + d(v_1, \eta_1)$ $\forall$ distinct points $u_1, v_1 \in \Delta \setminus \{\gamma_1, \eta_1\}$.

Then (Δ, d) is called a rectangular or generalized metric space. After that, fixed point results in rectangular metric spaces have been studied by many authors (see e.g [1, 3, 5, 7, 9, 10, 13, 15, 19, 23, 24, 25, 26, 27]). On the other hand, the concept of b-metric space was introduced by Bakhtin [4] 1989 and Czerwik [8] 1993, which is defined as:

Definition 1.2. [8]. Let Δ be a non-empty set and $s \geqslant 1$ be a given real number. A function $d : \Delta \times \Delta \to [0, \infty)$ is a b-metric on Δ, if for all $\gamma_1, \vartheta_1, \eta_1 \in \Delta$, the following conditions hold:

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1. $d(\gamma_1, \vartheta_1) = 0$ if and only if $\gamma_1 = \vartheta_1$;
2. $d(\gamma_1, \vartheta_1) = d(\vartheta_1, \gamma_1)$;
3. $d(\gamma_1, \eta_1) \leqslant s[d(\gamma_1, \vartheta_1) + d(\vartheta_1, \eta_1)]$.

the pair (Δ, d, s) is called a b-metric space, and the number s is called the coefficient of (Δ, d). After that, fixed point results in rectangular metric spaces have been studied by many authors (see e.g [2, 11, 12, 16, 20, 29, 30, 31, 32]). Recently, George et al. [16] and Roshan et al. [22] introduced the notion of rectangular b-metric space as follows:

Definition 1.3. [16, 22]. Let Δ be a nonempty set, $s \geqslant 1$ be a given real number and let $d : \Delta \times \Delta \to [0, \infty)$ be a mapping such that for all $\gamma_1, \vartheta_1 \in \Delta$, the following conditions hold:

1. $d(\gamma_1, \vartheta_1) = 0$ if and only if $\gamma_1 = \vartheta_1$;
2. $d(\gamma_1, \vartheta_1) = d(\vartheta_1, \gamma_1)$;
3. $d(\gamma_1, \eta_1) \leqslant s[d(\gamma_1, u_1) + d(u_1, v_1) + d(v_1, \eta_1)]$ $\forall$ distinct points $u_1, v_1 \in \Delta \setminus \{\gamma_1, \eta_1\}$.

Then (Δ, d, s) is called a rectangular b-metric space or a generalized b-metric space and the number s is called the coefficient of (Δ, d). For detail see [14, 18, 21, 28]

Definition 1.4. [17] Let $\Delta \neq \emptyset$. $\Theta : \Delta^n \to \Delta$, $(\eta^1, \eta^2 \cdots \eta^n) \in \Delta^n$, is said to be fixed point of Θ if

$$\eta^1 = \Theta(\eta^1, \eta^2, \eta^3 \cdots \eta^n), \eta^2 = \Theta(\eta^2, \eta^3, \eta^4 \cdots \eta^1) \cdots \eta^n = \Theta(\eta^n, \eta^2, \eta^3 \cdots \eta^{n-1}).$$

Definition 1.5. [17] Let $\Delta \neq \emptyset$. $\Theta : \Delta^n \to \Delta$ and $\theta : \Delta \to \Delta$, $(\eta^1, \eta^2 \cdots \eta^n) \in \Delta^n$, is said to be N-tupled coincidence point of θ and Θ if

$$\theta(\eta^1) = \Theta(\eta^1, \eta^2, \eta^3 \cdots \eta^n), \theta(\eta^2) = \Theta(\eta^2, \eta^3, \eta^4 \cdots \eta^1) \cdots \theta(\eta^n) = \Theta(\eta^n, \eta^2, \eta^3 \cdots \eta^{n-1}).$$

In the concerned work, we prove some common N-tupled fixed point theorems for mappings in a rectangular b-metric. We also discuss the existence and uniqueness of solutions for a class of nonlinear N-integral equations by using the established result.

2. Main Results

Definition 2.1. Let $\Delta \neq \emptyset$. A mapping $\Theta : \Delta^n \to \Delta$ and $\theta : \Delta \to \Delta$, are said to w-compatible if $\theta(\Theta(\eta^1, \eta^2, \eta^3 \cdots \eta^n)) \subseteq \Theta(\theta\eta^1, \theta\eta^2, \theta\eta^3 \cdots \theta\eta^n)$ whenever

$$\eta^1 = \Theta(\eta^1, \eta^2, \eta^3 \cdots \eta^n), \eta^2 = \Theta(\eta^2, \eta^3, \eta^4 \cdots \eta^1) \cdots \eta^n = \Theta(\eta^n, \eta^2, \eta^3 \cdots \eta^{n-1}).$$

Theorem 2.2. *Assume that d_1 and d_2 be two rectangular b-metrices on Δ with coefficient $s \geqslant 1$ $(d_2(\zeta, \xi) \leqslant d_1(\zeta, \xi))$. Furthermore, $\Theta : \Delta^n \to \Delta$ and $\theta : \Delta \to \Delta$ be two mappings. Suppose that there exist λ_1, λ_2 and $\lambda_3 \in [0, 1)$ with $0 \leqslant \lambda_1 + \lambda_2 + \lambda_3 < 1$ and $0 \leqslant s\lambda_3 < 1$ such that the below condition holds*

$$\begin{aligned} d_1(\Theta(\zeta^1, \zeta^2, \zeta^3, .., \zeta^n), \Theta(\eta^1, \eta^2, \eta^3, \cdots, \eta^n)) + d_1(\Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^1)), \Theta(\eta^2, \eta^3, \eta^4, \cdots, \eta^n)) + \cdots \\ + d_1(\Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1}), \Theta(\eta^n, \eta^1, \eta^2, \cdots, \eta^{n-1})) \\ \leqslant \lambda_1 \left[d_2(\theta\zeta^1, \theta\eta^1) + d_2(\theta\zeta^2, \theta\eta^2) + \cdots + d_2(\theta\zeta^n, \theta\eta^n) \right] \\ + \lambda_2 \left[d_2(\theta(\zeta^1), \Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n)) + d_2(\theta(\zeta^2), \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^1)) + \cdots \right. \\ \left. + d_2(\theta(\zeta^n), \Theta(\zeta^n, \zeta^2, \zeta^3, \cdots, \zeta^{n-1})) \right] + \lambda_3 \left[d_2(\theta(\eta^1), \Theta(\eta^1, \eta^2, \eta^3, \cdots, \eta^n)) \right. \\ \left. + d_2(\theta(\eta^2), \Theta(\eta^2, \eta^3, \eta^4, \cdots, \eta^1)) + \cdots + d_2(\theta(\eta^n), \Theta(\eta^n, \eta^2, \eta^3, \cdots, \eta^{n-1})) \right], \end{aligned} \quad (2.1)$$

for all $(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n), (\eta^1, \eta^2, \eta^3, \cdots, \eta^n) \in \Delta^n$. *If* $\Theta(\Delta^n) \subseteq \theta(\Delta)$, $\theta(\Delta)$ *is complete, then* θ *and* Θ *have a* N*-tupled coincidence point* $(\zeta^1, \zeta^2, \cdots, \zeta^n) \in \Delta^n$*, satisfying that*

$$\theta(\zeta^1) = \Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n),\ \theta(\zeta^2) = \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^1), \cdots, \theta(\zeta^n) = \Theta(\zeta^n, \zeta^2, \zeta^3, \cdots, \zeta^{n-1}).$$

Moreover, if θ *and* Θ *are w-compatible, then* θ *and* Θ *have a unique common* N*-tupled fixed point of the form* $(\eta, \eta, \cdots, \eta)$*, which satisfies that* $\eta = \theta\eta \in \Theta(\eta, \eta, \cdots, \eta)$.

Proof. Let $\zeta_0^1, \zeta_0^2, \zeta_0^3, \cdots, \zeta_0^n \in \Delta$ be arbitrary. Since $\Theta(\Delta^n) \subseteq \theta(\Delta)$ so there exist $\zeta_1^1, \zeta_1^2, \zeta_1^3, \cdots, \zeta_1^n \in \Delta$ such that $\theta\zeta_1^1 = \Theta(\zeta_0^1, \zeta_0^2, \zeta_0^3, \cdots, \zeta_0^n)$, $\theta\zeta_1^2 = \Theta(\zeta_0^2, \zeta_0^3, \zeta_0^4, \cdots, \zeta_0^1), \cdots, \theta\zeta_1^n = \Theta(\zeta_0^n, \zeta_0^1, \zeta_0^2, \cdots, \zeta_0^{n-1})$. By a similar way, there exist $\zeta_2^1, \zeta_2^2, \zeta_2^3, \cdots, \zeta_2^n \in \Delta$. Since $\Theta(\Delta^n) \subseteq \theta(\Delta)$, there exist $\zeta_2^1, \zeta_2^2, \zeta_2^3, \cdots, \zeta_2^n \in \Delta$ such that $\theta\zeta_2^1 = \Theta(\zeta_1^1, \zeta_1^2, \zeta_1^3, \cdots, \zeta_1^n))$, $\theta\zeta_2^2 = \Theta(\zeta_1^2, \zeta_1^3, \zeta_1^4, \cdots, \zeta_1^1)$, $\cdots\ \theta\zeta_2^n = \Theta(\zeta_1^n, \zeta_1^1, \zeta_1^2, \cdots, \zeta_1^{n-1})$. Repeating the procedure, we can construct n sequences $\{\zeta_n^1\}, \{\zeta_n^2\}\cdots$ and $\{\zeta_n^n\}$ we have $\theta\zeta_{n+1}^1 = \Theta(\zeta_n^1, \zeta_n^2, \zeta_n^3, \cdots, \zeta_n^n)$, $\theta\zeta_{n+1}^2 = \Theta(\zeta_n^2, \zeta_n^3, \zeta_n^4, \cdots, \zeta_n^1)\cdots$ and, $\theta\zeta_{n+1}^n = \Theta(\zeta_n^n, \zeta_n^1, \zeta_n^2, \cdots, \zeta_n^{n-1})$ for $n \geqslant 0$,

without loss of generality, we can assume that $\theta\zeta_n^1 \neq \theta\zeta_{n+1}^1$, $\theta\zeta_n^2 \neq \theta\zeta_{n+1}^2, \cdots, \theta\zeta_n^n \neq \theta\zeta_{n+1}^n$, for all $n \geqslant 0$. By taking $(\zeta^1, \zeta^2, \cdots, \zeta^n) = (\zeta_n^1, \zeta_n^2, \cdots, \zeta_n^n)$ and $(u^1, u^2, \cdots, u^n) = (\zeta_{n+1}^1, \zeta_{n+1}^2, \cdots, \zeta_{n+1}^n)$ in (2.1), we obtain

$$\begin{aligned}
&d_1(\theta\zeta_{n+1}^1, \theta\zeta_{n+2}^1) + d_1(\theta\zeta_{n+1}^2, \theta\zeta_{n+2}^2) + \cdots + d_1(\theta\zeta_{n+1}^n, \theta\zeta_{n+2}^n)\\
&= d_1(\Theta(\zeta_n^1, \zeta_n^2, \zeta_n^3, \cdots, \zeta_n^n), \Theta(\zeta_{n+1}^1, \zeta_{n+1}^2, \zeta_{n+1}^3, \cdots, \zeta_{n+1}^n))\\
&+ d_1(\Theta(\zeta_n^2, \zeta_n^3, \zeta_n^4, \cdots, \zeta_n^1), \Theta(\zeta_{n+1}^2, \zeta_{n+1}^3, \zeta_{n+1}^4, \cdots, \zeta_{n+1}^1))\\
&+ \cdots + d_1(\Theta(\zeta_n^n, \zeta_n^1, \zeta_n^2, \cdots, \zeta_n^{n-1}), \Theta(\zeta_{n+1}^n, \zeta_{n+1}^1, \zeta_{n+1}^2, \cdots, \zeta_{n+1}^{n-1}))\\
&\leqslant \lambda_1\big[d_2(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_2(\theta\zeta_n^2, \theta\zeta_{n+1}^2) +\\
&\cdots + d_2(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big] + \lambda_2\big[d_2(\theta(\zeta_n^1), \Theta(\zeta_n^1, \zeta_n^2, \zeta_n^3, \cdots, \zeta_n^n)) + d_2(\theta(\zeta_n^2), \Theta(\zeta_n^2, \zeta_n^3, \zeta_n^4, \cdots, \zeta_n^1)) +\\
&\cdots + d_2(\theta(\zeta_n^n), \Theta(\zeta_n^n, \zeta_n^1, \zeta_n^2, \cdots, \zeta_n^{n-1}))\big] + \lambda_3\big[d_2(\theta(\zeta_{n+1}^1), \Theta(\zeta_{n+1}^1, \zeta_{n+1}^2, \zeta_{n+1}^3, \cdots, \zeta_{n+1}^n))\\
&+ d_2(\theta(\zeta_{n+1}^2), \Theta(\zeta_{n+1}^2, \zeta_{n+1}^3, \zeta_{n+1}^4, \cdots, \zeta_{n+1}^1))\\
&+ \cdots + d_2(\theta(\zeta_{n+1}^n), \Theta(\zeta_{n+1}^n, \zeta_{n+1}^1, \zeta_{n+1}^2, \cdots, \zeta_{n+1}^{n-1}))\big]
\end{aligned}$$

$$\begin{aligned}
&= \lambda_1\big[d_2(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_2(\theta\zeta_n^2, \theta\zeta_{n+1}^2) + \cdots + d_2(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big] + \lambda_2\big[d_2(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_2(\theta\zeta_n^2, \theta\zeta_{n+1}^2)\\
&+ \cdots + d_2(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big]\\
&+ \lambda_3\big[d_2(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_2(\theta\zeta_n^2, \theta\zeta_{n+1}^2) + \cdots + d_2(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big] \quad (2.2)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \lambda_1\big[d_1(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_1(\theta\zeta_n^2, \theta\zeta_{n+1}^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big] + \lambda_2\big[d_1(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_1(\theta\zeta_n^2, \theta\zeta_{n+1}^2)\\
&+ \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big]\\
&+ \lambda_3\big[d_1(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_1(\theta\zeta_n^2, \theta\zeta_{n+1}^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big] \quad (2.3)
\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}
&d_1(\theta\zeta_{n+1}^1, \theta\zeta_{n+2}^1) + d_1(\theta\zeta_{n+1}^2, \theta\zeta_{n+2}^2) + \cdots + d_1(\theta\zeta_{n+1}^n, \theta\zeta_{n+2}^n)\\
&\leqslant \frac{\lambda_1 + \lambda_2}{1 - \lambda_3}\big[d(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_1(\theta\zeta_n^2, \theta\zeta_{n+1}^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\big]. \quad (2.4)
\end{aligned}$$

Taking $\lambda = \frac{\lambda_1+\lambda_2}{1-\lambda_3}$ by the condition $0 \leqslant \lambda_1+\lambda_2+\lambda_3 < 1$, then we have $0 \leqslant \lambda < 1$. (2.4) implies that

$$\begin{aligned} &d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2}) + d_1(\theta\zeta^2_{n+1}, \theta\zeta^2_{n+2}) + \cdots + d_1(\theta\zeta^n_{n+1}, \theta\zeta^n_{n+2}) \\ &\qquad \leqslant \lambda\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^2_n, \theta\zeta^2_{n+1}) + \cdots + d_1(\theta\zeta^n_n, \theta\zeta^n_{n+1})\big]. \end{aligned} \tag{2.5}$$

By taking $\delta_n = d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^2_n, \theta\zeta^2_{n+1}) + \cdots + d_1(\theta\zeta^n_n, \theta\zeta^n_{n+1})$. Repeating the above inequality (2.5) $n+1$ times, we obtain,

$$\delta_{n+1} \leqslant \lambda\delta_n \leqslant \lambda^2\delta_{n-1} \leqslant \cdots \leqslant \lambda^{n+1}\delta_0. \tag{2.6}$$

As $(\zeta^1, \zeta^2, \cdots, \zeta^n) = (\zeta^1_n, \zeta^2_n, \cdots, \zeta^n_n)$ and $(u^1, u^2, \cdots, u^n) = (\zeta^1_{n+2}, \zeta^2_{n+2}, \cdots, \zeta^n_{n+2})$ in (2.1), also with (2.6), we get

$$\begin{aligned} &d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+3}) + d_1(\theta\zeta^2_{n+1}, \theta\zeta^2_{n+3}) + \cdots + d_1(\theta\zeta^n_{n+1}, \theta\zeta^n_{n+3}) \\ &= d_1(\Theta(\zeta^1_n, \zeta^2_n, \zeta^3_n, \cdots, \zeta^n_n), \Theta(\zeta^1_{n+2}, \zeta^2_{n+2}, \zeta^3_{n+2}, \cdots, \zeta^n_{n+2})) \\ &+ d_1(\Theta(\zeta^2_n, \zeta^3_n, \zeta^4_n, \cdots, \zeta^1_n), \Theta(\zeta^2_{n+2}, \zeta^3_{n+2}, \zeta^4_{n+2}, \cdots, \zeta^1_{n+2})) + \cdots \\ &+ d_1(\Theta(\zeta^n_n, \zeta^1_n, \zeta^2_n, \cdots, \zeta^{n-1}_n), \Theta(\zeta^n_{n+2}, \zeta^1_{n+2}, \zeta^2_{n+2}, \cdots, \zeta^{n-1}_{n+2})) \\ &\leqslant \lambda_1\big[d_2(\theta\zeta^1_n, \theta\zeta^1_{n+2}) + d_2(\theta\zeta^2_n, \theta\zeta^2_{n+2}) + \cdots + d_2(\theta\zeta^n_n, \theta\zeta^n_{n+2})\big] \\ &+ \lambda_2\big[d_2(\theta(\zeta^1_n), \Theta(\zeta^1_n, \zeta^2_n, \zeta^3_n, \cdots, \zeta^n_n)) + d_2(\theta(\zeta^2_n), \Theta(\zeta^2_n, \zeta^3_n, \zeta^4_n, \cdots, \zeta^1_n)) + \cdots \\ &+ d_2(\theta(\zeta^n_n), \Theta(\zeta^n_n, \zeta^2_n, \zeta^3_n, \cdots, \zeta^{n-1}_n))\big] \\ &+ \lambda_3\big[d_2(\theta(\zeta^1_{n+2}), \Theta(\zeta^1_{n+2}, \zeta^2_{n+2}, \zeta^3_{n+2}, \cdots, \zeta^n_{n+2})) \\ &+ d_2(\theta(\zeta^2_{n+2}), \Theta(\zeta^2_{n+2}, \zeta^3_{n+2}, \zeta^4_{n+2}, \cdots, \zeta^1_{n+2})) + \cdots + d_2(\theta(\zeta^n_{n+2}), \Theta(\zeta^n_{n+2}, \zeta^1_{n+2}, \zeta^2_{n+2}, \cdots, \zeta^{n-1}_{n+2}))\big] \end{aligned}$$

$$\begin{aligned} &\leqslant \lambda_1\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+2}) + d_1(\theta\zeta^2_n, \theta\zeta^2_{n+2}) + \cdots + d_1(\theta\zeta^n_n, \theta\zeta^n_{n+2})\big] \\ &\quad + \lambda_2\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^2_n, \theta\zeta^2_{n+1}) + \cdots. + d_1(\theta\zeta^n_n, \theta\zeta^n_{n+1})\big] \\ &\quad + \lambda_3\big[d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+3}) + d_1(\theta\zeta^2_{n+2}, \theta\zeta^2_{n+3}) + \cdots + d_1(\theta\zeta^n_{n+2}, \theta\zeta^n_{n+3})\big] \end{aligned} \tag{2.7}$$

$$\textit{Put} \quad \delta^\star_n = d_1(\theta\zeta^1_n, \theta\zeta^1_{n+2}) + d_1(\theta\zeta^2_n, \theta\zeta^2_{n+2}) + \cdots + d_1(\theta\zeta^n_n, \theta\zeta^n_{n+2}).$$

From $\lambda = \frac{\lambda_1+\lambda_2}{1-\lambda_3} \in [0,1)$ we have $\lambda_1+\lambda_2+\lambda_3\lambda^2 \leqslant \lambda_1+\lambda_2+\lambda_3\lambda$. Consequently, by the use of (2.6) and (2.7), we have

$$\delta^\star_{n+1} \leqslant \lambda_1\delta^\star_n + (\lambda_2+\lambda_3\lambda^2)\delta_n \leqslant (\lambda_1+\lambda_2+\lambda_3\lambda^2)\max\{\delta_n, \delta^\star_n\} \leqslant \lambda\max\{\delta_n, \delta^\star_n\}. \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$\begin{aligned} &\delta^\star_{n+1} \leqslant \lambda\max\{\delta_n, \delta^\star_n\} \leqslant \lambda\max\{\lambda\delta_{n-1}, \lambda\max\{\delta_{n-1}, \delta^\star_{n-1}\}\} \\ &\qquad = \lambda^2\max\{\delta_{n-1}, \delta^\star_{n-1}\} \leqslant \lambda^3\max\{\delta_{n-2}, \delta^\star_{n-2}\} \leqslant \cdots \leqslant \lambda^{n+1}\max\{\delta_0, \delta^\star_0\}. \end{aligned} \tag{2.9}$$

Next, we show that $\{\theta\zeta^1_n\}$, $\{\theta\zeta^2_n\}$,$\cdots$,$\{\theta\zeta^n_n\}$ are Cauchy sequences in $\theta(\Delta)$. For this, we consider $d_1(\zeta_n, \zeta_{n+p})$ in two cases.

Case 1. p is an odd number, assume that $p = 2m+1$.

$$\begin{aligned}
d_1(\theta\zeta^1_n, \theta\zeta^1_{n+p}) &= d_1(\theta\zeta^1_n, \theta\zeta^1_{n+2m+1}) \\
&\leqslant s\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2}) + d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+2m+1})\big] \\
&\leqslant s\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2})\big] \\
&+ s^2\big[d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+3}) + d_1(\theta\zeta^1_{n+3}, \theta\zeta^1_{n+4}) + d_1(\theta\zeta^1_{n+4}, \theta\zeta^1_{n+2m+1})\big] \\
&\leqslant s\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2})\big] \\
&+ s^2\big[d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+3}) + d_1(\theta\zeta^1_{n+3}, \theta\zeta^1_{n+4})\big] \\
&+ s^3\big[d_1(\theta\zeta^1_{n+4}, \theta\zeta^1_{n+5}) + d_1(\theta\zeta^1_{n+5}, \theta\zeta^1_{n+6}) + d_1(\theta\zeta^1_{n+6}, \theta\zeta^1_{n+2m+1})\big] \leqslant \cdots \\
&\leqslant s\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2})\big] \\
&+ s^2\big[d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+3}) + d_1(\theta\zeta^1_{n+3}, \theta\zeta^1_{n+4})\big] + s^3\big[d_1(\theta\zeta^1_{n+4}, \theta\zeta^1_{n+5}) + d_1(\theta\zeta^1_{n+5}, \theta\zeta^1_{n+6})\big] + \cdots \\
&+ s^m\big[d_1(\theta\zeta^1_{n+2m-2}, \theta\zeta^1_{n+2m-1}) + d_1(\theta\zeta^1_{n+2m-1}, \theta\zeta^1_{n+2m}) + d_1(\theta\zeta^1_{n+2m}, \theta\zeta^1_{n+2m+1})\big].
\end{aligned}$$

That is

$$\begin{aligned}
d_1(\theta\zeta^1_n, \theta\zeta^1_{n+p}) &= d_1(\theta\zeta^1_n, \theta\zeta^1_{n+2m+1}) \\
&\leqslant s\big[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2})\big] \\
&+ s^2\big[d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+3}) + d_1(\theta\zeta^1_{n+3}, \theta\zeta^1_{n+4})\big] \\
&+ s^3\big[d_1(\theta\zeta^1_{n+4}, \theta\zeta^1_{n+5}) + d_1(\theta\zeta^1_{n+5}, \theta\zeta^1_{n+6})\big] + \cdots \\
&+ s^m\big[d_1(\theta\zeta^1_{n+2m-2}, \theta\zeta^1_{n+2m-1}) + d_1(\theta\zeta^1_{n+2m-1}, \theta\zeta^1_{n+2m})\big] + s^m d_1(\theta\zeta^1_{n+2m}, \theta\zeta^1_{n+2m+1}).
\end{aligned} \tag{2.10}$$

We can similarly prove the following result

$$\begin{aligned}
d_1(\theta\zeta^2_n, \theta\zeta^2_{n+p}) &= d_1(\theta\zeta^2_n, \theta\zeta^2_{n+2m+1}) \\
&\leqslant s\big[d_1(\theta\zeta^2_n, \theta\zeta^2_{n+1}) + d_1(\theta\zeta^2_{n+1}, \theta\zeta^2_{n+2})\big] + s^2\big[d_1(\theta\zeta^2_{n+2}, \theta\zeta^2_{n+3}) \\
&+ d_1(\theta\zeta^2_{n+3}, \theta\zeta^2_{n+4})\big] + s^3\big[d_1(\theta\zeta^2_{n+4}, \theta\zeta^2_{n+5}) + d_1(\theta\zeta^2_{n+5}, \theta\zeta^2_{n+6})\big] + \cdots \\
&+ s^m\big[d_1(\theta\zeta^2_{n+2m-2}, \theta\zeta^2_{n+2m-1}) \\
&+ d_1(\theta\zeta^2_{n+2m-1}, \theta\zeta^2_{n+2m})\big] + s^m d_1(\theta\zeta^2_{n+2m}, \theta\zeta^2_{n+2m+1}),
\end{aligned} \tag{2.11}$$

$$\cdots$$

and

$$\begin{aligned}
d_1(\theta\zeta^n_n, \theta\zeta^n_{n+p}) &= d_1(\theta\zeta^n_n, \theta\zeta^n_{n+2m+1}) \\
&\leqslant s\big[d_1(\theta\zeta^n_n, \theta\zeta^n_{n+1}) + d_1(\theta\zeta^n_{n+1}, \theta\zeta^n_{n+2})\big] + s^2\big[d_1(\theta\zeta^n_{n+2}, \theta\zeta^n_{n+3}) + d_1(\theta\zeta^n_{n+3}, \theta\zeta^n_{n+4})\big] \\
&+ s^3\big[d_1(\theta\zeta^n_{n+4}, \theta\zeta^n_{n+5}) + d_1(\theta\zeta^n_{n+5}, \theta\zeta^n_{n+6})\big] + \cdots \\
&+ s^m\big[d_1(\theta\zeta^n_{n+2m-2}, \theta\zeta^n_{n+2m-1}) + d_1(\theta\zeta^n_{n+2m-1}, \theta\zeta^n_{n+2m})\big] + s^m d_1(\theta\zeta^n_{n+2m}, \theta\zeta^n_{n+2m+1}).
\end{aligned} \tag{2.12}$$

Combining (2.8), (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
& d_1(\theta\zeta^1_n, \theta\zeta^1_{n+p}) + d_1(\theta\zeta^2_n, \theta\zeta^2_{n+p}) + \cdots + d_1(\theta\zeta^n_n, \theta\zeta^n_{n+p}) \\
&= d_1(\theta\zeta^1_n, \theta\zeta^1_{n+2m+1}) + d_1(\theta\zeta^2_n, \theta\zeta^2_{n+2m+1}) + \cdots + d_1(\theta\zeta^n_n, \theta\zeta^n_{n+2m+1}) \\
&\leqslant s(\delta_n + \delta_{n+1}) + s^2(\delta_{n+2} + \delta_{n+3}) + \cdots \\
&+ s^m(\delta_{n+2m-2} + \delta_{n+2m-1})) + s^m\lambda^{n+2m} \leqslant s(\lambda^n + \lambda^{n+1})\delta_0 + \\
&+ s^2(\lambda^{n+2} + \lambda^{n+3})\delta_0 + \cdots + s^m(\lambda^{n+2m-2} + \lambda^{n+2m-1})\delta_0 + s^m\lambda^{n+2m}\delta_0 \\
&= s\lambda^n(1+\lambda)\left[1 + (s\lambda^2) + (s\lambda^2)^2 + \cdots + (s\lambda^2)^{m-1}\right]\delta_0 + s^m\lambda^{n+2m}\delta_0, \qquad (2.13)
\end{aligned}$$

from (2.13) we get

$$= \begin{cases} s\lambda^n(1+\lambda)(m)\delta_0 + s^m\lambda^{n+2m}\delta_0, & \textit{if } s\lambda^2 = 1, \\ s\lambda^n(1+\lambda).\dfrac{1-(s\lambda^2)^m}{1-s\lambda^2} + s^m\lambda^{n+2m}\delta_0, & \textit{if } s\lambda^2 \neq 1. \end{cases}$$

$$\leqslant \begin{cases} s\lambda^n(1+\lambda)(m)\delta_0 + s^m\lambda^{n+2m}\max\{\delta_0, \delta_0^*\}, & \textit{if } s\lambda^2 = 1, \\ \dfrac{s\lambda^n(1+\lambda)}{1-s\lambda^2}\delta_0 + s^m\lambda^{n+2m}\max\{\delta_0, \delta_0^*\}, & \textit{if } s\lambda^2 \neq 1. \end{cases} \qquad (2.14)$$

Case 2. p is an even number. Assume that $p = 2m$.

$$\begin{aligned}
& d_1(\theta\zeta^1_n, \theta\zeta^1_{n+p}) = d_1(\theta\zeta^1_n, \theta\zeta^1_{n+2m}) \\
&\leqslant s\left[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2}) + d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+2m})\right] \\
&\leqslant s\left[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1}) + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2})\right] + s^2\left[d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+3})\right. \\
&\left. + d_1(\theta\zeta^1_{n+3}, \theta\zeta^1_{n+4}) + d_1(\theta\zeta^1_{n+4}, \theta\zeta^1_{n+2m})\right] \leqslant \cdots \leqslant s\left[d_1(\theta\zeta^1_n, \theta\zeta^1_{n+1})\right. \\
&\left. + d_1(\theta\zeta^1_{n+1}, \theta\zeta^1_{n+2})\right] + s^2\left[d_1(\theta\zeta^1_{n+2}, \theta\zeta^1_{n+3})\right. \\
&\left. + d_1(\theta\zeta^1_{n+3}, \theta\zeta^1_{n+4})\right] + \cdots + s^{m-1}\left[d_1(\theta\zeta^1_{n+2m-4}, \theta\zeta^1_{n+2m-3}) + d_1(\theta\zeta^1_{n+2m-3}, \theta\zeta^1_{n+2m-2})\right] \\
&+ s^{m-1}d_1(\theta\zeta^1_{n+2m-2}, \theta\zeta^1_{n+2m}). \qquad (2.15)
\end{aligned}$$

By similar arguments as above,

$$\begin{aligned}
& d_1(\theta\zeta^2_n, \theta\zeta^2_{n+p}) = d_1(\theta\zeta^2_n, \theta\zeta^2_{n+2m}) \\
&\leqslant s\left[d_1(\theta\zeta^2_n, \theta\zeta^2_{n+1}) + d_1(\theta\zeta^2_{n+1}, \theta\zeta^2_{n+2})\right] + s^2\left[d_1(\theta\zeta^2_{n+2}, \theta\zeta^2_{n+3}) + d_1(\theta\zeta^2_{n+3}, \theta\zeta^2_{n+4})\right] + \cdots \\
&+ s^{m-1}\left[d_1(\theta\zeta^2_{n+2m-4}, \theta\zeta^2_{n+2m-3}) + d_1(\theta\zeta^2_{n+2m-3}, \theta\zeta^2_{n+2m-2})\right] \\
&+ s^{m-1}d_1(\theta\zeta^2_{n+2m-2}, \theta\zeta^2_{n+2m}), \qquad (2.16)
\end{aligned}$$

$$\cdots$$

$$\begin{aligned}
& d_1(\theta\zeta^n_n, \theta\zeta^n_{n+p}) = d_1(\theta\zeta^n_n, \theta\zeta^n_{n+2m}) \leqslant s\left[d_1(\theta\zeta^n_n, \theta\zeta^n_{n+1}) + d_1(\theta\zeta^n_{n+1}, \theta\zeta^n_{n+2})\right] + s^2\left[d_1(\theta\zeta^n_{n+2}, \theta\zeta^n_{n+3})\right. \\
&\left. + d_1(\theta\zeta^n_{n+3}, \theta\zeta^n_{n+4})\right] + \cdots \\
&+ s^{m-1}\left[d_1(\theta\zeta^n_{n+2m-4}, \theta\zeta^n_{n+2m-3})\right. \\
&\left. + d_1(\theta\zeta^n_{n+2m-3}, \theta\zeta^n_{n+2m-2})\right] + s^{m-1}d_1(\theta\zeta^n_{n+2m-2}, \theta\zeta_{n+2m}). \qquad (2.17)
\end{aligned}$$

Combining (2.6), (2.9), (2.15), (2.16) and (2.17), we have

$$\begin{aligned}
d_1(\theta\zeta_n^1, \theta\zeta_{n+p}^1) + d_1(\theta\zeta_n^2, \theta\zeta_{n+p}^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+p}^n) &= d_1(\theta\zeta_n^1, \theta\zeta_{n+2m}^1) \\
&+ d_1(\theta\zeta_n^2, \theta\zeta_{n+2m}^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+2m}^n) \\
&\leqslant s(\delta_n + \delta_{n+1}) + s^2(\delta_{n+2} + \delta_{n+3}) + \cdots \\
&+ s^{m-1}(\delta_{n+2m-4} + \delta_{n+2m-3})) + s^{m-1}\delta^{\star}_{n+2m-2} \\
&\leqslant s(\lambda^n + \lambda^{n+1})\delta_0 + s^2(\lambda^{n+2} + \lambda^{n+3})\delta_0 + \cdots \\
&+ s^{m-1}(\lambda^{n+2m-4} + \lambda^{n+2m-3}))\delta_0 + s^{m-1}\lambda^{n+2m-2}\max\{\delta_0, \delta_0^{\star}\}
\end{aligned}$$

$$= s\lambda^n(1+\lambda)\left[1 + (s\lambda^2) + (s\lambda^2)^2 + \cdots + (s\lambda^2)^{m-2}\right]\delta_0 + s^{m-1}\lambda^{n+2m-2}\max\{\delta_0, \delta_0*\},$$

$$= \begin{cases} s\lambda^n(1+\lambda)\left[(m-1)\right]\delta_0 + s^{m-1}\lambda^{n+2m-2}\max\{\delta_0, \delta_0*\}, \ \textit{if}\ s\lambda^2 = 1, \\ s\lambda^n(1+\lambda).\dfrac{1-(s\lambda^2)^{m-1}}{1-s\lambda^2}\delta_0 + s^{m-1}\lambda^{n+2m-2}\max\{\delta_0, \delta_0*\}, \ \textit{if}\ s\lambda^2 \neq 1. \end{cases}$$

$$\leqslant \begin{cases} s\lambda^n(1+\lambda)(m-1)\delta_0 + s^{m-1}\lambda^{n+2m-2}\max\{\delta_0, \delta_0^*\}, \ \textit{if}\ s\lambda^2 = 1, \\ \dfrac{s\lambda^n(1+\lambda)}{1-s\lambda^2}\delta_0 + s^{m-1}\lambda^{n+2m-2}\max\{\delta_0, \delta_0^*\}, \ \textit{if}\ s\lambda^2 \neq 1. \end{cases} \tag{2.18}$$

Since $\lambda \in [0,1)$, so $\lambda^n \to 0$ as $n \to \infty$. Taking limit as $n \to \infty$ in (2.14) and (2.18), we get

$$\lim_{n\to\infty}\left[d_1(\theta\zeta_n^1, \theta\zeta_{n+p}^1) + d_1(\theta\zeta_n^2, \theta\zeta_{n+p}^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+p}^n)\right] = 0.$$

Which implies that $\{\theta\zeta_n^1\}$, $\{\theta\zeta_n^2\} \cdots \{\theta\zeta_n^n\}$ are Cauchy sequences in $\theta(\Delta)$ with respect to rectangular metric. Since $\theta(\Delta)$ is complete, therefore there exist $\zeta^1, \zeta^2, \cdots \zeta^n \in \Delta$ such that

$$\lim_{n\to\infty}\theta\zeta_n^1 = \theta\zeta^1, \lim_{n\to\infty}\theta\zeta_n^2 = \theta\zeta^2, \cdots, \lim_{n\to\infty}\theta\zeta_n^n = \theta\zeta^n.$$

Implies from (2.1) and (2.6), we have

$$\begin{aligned}
&d_1(\theta\zeta_{n+1}^1, \Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n)) + d_1(\theta\zeta_{n+1}^2, \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n)) + \cdots \\
&\qquad + d_1(\theta\zeta_{n+1}^n, \Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1})) \\
&= d_1(\Theta(\zeta_n^1, \zeta_n^2, \zeta_n^3, \cdots, \zeta_n^n), \Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n)) + d_1(\Theta(\zeta_n^2, \zeta_n^3, \zeta_n^4, \cdots, \zeta_n^1), \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n)) \\
&\qquad + \cdots + d_1(\Theta(\zeta_n^n, \zeta_n^1, \zeta_n^2, \cdots, \zeta_n^{n-1}), \Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1})) \\
&\leqslant \lambda_1\left[d_2(\theta\zeta_n^1, \theta\zeta^1) + d_2(\theta\zeta_n^2, \theta\zeta^2) + \cdots + d_2(\theta\zeta_n^n, \theta\zeta^n)\right] \\
&\qquad + \lambda_2\left[d_2(\theta\zeta_n^1, \Theta(\zeta_n^1, \zeta_n^2, \zeta_n^3, \cdots, \zeta_n^n)) + d_2(\theta\zeta_n^2, \Theta(\zeta_n^2, \zeta_n^3, \zeta_n^4, \cdots, \zeta_n^1)) + \cdots \right. \\
&\qquad \left. + d_2(\theta\zeta_n^n, \Theta(\zeta_n^n, \zeta_n^1, \zeta_n^2, \cdots, \zeta_n^{n-1}))\right] \\
&\qquad + \lambda_3\left[d_2(\theta\zeta^1, \Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n)) + d_2(\theta\zeta^2, \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n)) + \cdots \right. \\
&\qquad \left. + d_2(\theta\zeta^n, \Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1}))\right] \\
&\leqslant \lambda_1\left[d_1(\theta\zeta_n^1, \theta\zeta^1) + d_1(\theta\zeta_n^2, \theta\zeta^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta^n)\right] \\
&\qquad + \lambda_2\left[d_1(\theta\zeta_n^1, \theta\zeta_{n+1}^1) + d_1(\theta\zeta_n^2, \theta\zeta_{n+1}^2) + \cdots + d_1(\theta\zeta_n^n, \theta\zeta_{n+1}^n)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_3\big[d_1(\theta\zeta^1,\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta\zeta^2,\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+d_1(\theta\zeta^n,\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]\\
&\leqslant \lambda_1\big[d_1(\theta\zeta^1_n,\theta\zeta^1)+d_1(\theta\zeta^2_n,\theta\zeta^2)+\cdots+d_1(\theta\zeta^n_n,\theta\zeta^n)\big]\\
&+\lambda_2\big[d_1(\theta\zeta^1_n,\theta\zeta^1_{n+1})+d_1(\theta\zeta^2_n,\theta\zeta^2_{n+1})+\cdots+d_1(\theta\zeta^n_n,\theta\zeta^n_{n+1})\big]+\\
&\lambda_3\big[d_1(\theta\zeta^1,\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta\zeta^2,\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots+d_1(\theta\zeta^n,\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]\\
&=\lambda_1\big[d_1(g\zeta^1_n,\theta\zeta^1)+d_1(\theta\zeta^2_n,\theta\zeta^2)+\cdots+d(\theta\zeta^n_n,\theta\zeta^n)\big]\\
&+\lambda_2\delta_n+\lambda_3\big[d_1(\theta\zeta^1,\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta\zeta^2,\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))\\
&+\cdots+d_1(\theta\zeta^n,\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]\\
&\leqslant \lambda_1\big[d_1(\theta\zeta^1_n,\theta\zeta^1)+d_1(\theta\zeta^2_n,\theta\zeta^2)+\cdots+d_1(\theta\zeta^n_n,\theta\zeta^n)\big]+\lambda_2\lambda^n\delta_0\\
&+\lambda_3\big[d_1(\theta\zeta^1,\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta\zeta^2,\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots\\
&+d_1(\theta\zeta^n,\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]. \qquad (2.19)
\end{aligned}
$$

Using (2.19) and (2.6) we have

$$
\begin{aligned}
&d_1(\theta\zeta^1,\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta\zeta^2,\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots+d_1(\theta\zeta^n,\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\\
&\leqslant s\big[d_1(\theta\zeta^1,\theta\zeta^1_n)+d_1(\theta\zeta^1_n,\theta\zeta^1_{n+1})+d_1(\theta\zeta^1_{n+1},\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))\big]\\
&+s\big[d_1(\theta\zeta^2,\theta\zeta^2_n)+d_1(\theta\zeta^2_n,\theta\zeta^2_{n+1})+d_1(\theta\zeta^2_{n+1},\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))\big]\\
&+\cdots.+s\big[d_1(\theta\zeta^n,\theta\zeta^n_n)+d_1(\theta\zeta^n_n,\theta\zeta^n_{n+1})+d_1(\theta\zeta^n_{n+1},\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]\\
&=s\big[d_1(\theta\zeta^1,\theta\zeta^1_n)+d_1(\theta\zeta^2,\theta\zeta^2_n)+\cdots+d_1(\theta\zeta^n,\theta\zeta^n_n)\big]+s\delta_n+s\big[d_1(\theta\zeta^1_{n+1},\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))\\
&+d_1(\theta\zeta^2_{n+1},\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots+d_1(\theta\zeta^n_{n+1},\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]\\
&\leqslant s(1+\lambda_1)\big[d_1(\theta\zeta^1_n,\theta\zeta^1)+d_1(\theta\zeta^2_n,\theta\zeta^2)+\cdots+d_1(\theta\zeta^n_n,\theta\zeta^n)\big]\\
&+s(1+\lambda_2)\lambda^n\delta_0+s\lambda_3\big[d_1(\theta\zeta^1,\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))\\
&+d_1(\theta\zeta^2,\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots+d(\theta\zeta^n,\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]. \qquad (2.20)
\end{aligned}
$$

By taking $n\to\infty$ in the above inequality (2.20), we have

$$
\begin{aligned}
&d_1(\theta(\zeta^1),\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta(\zeta^2),\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots\\
&+d_1(\theta(\zeta^n),\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1)})\\
&\leqslant s\lambda_3\big[d_1(\theta(\zeta^1),\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta(\zeta^2),\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots\\
&+d_1(\theta(\zeta^n),\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))\big]. \qquad (2.21)
\end{aligned}
$$

By the condition $0\leqslant s\lambda_3<1$ and (2.21), we can easily obtain that

$$
\begin{aligned}
&d_1(\theta(\zeta^1),\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n))+d_1(\theta(\zeta^2),\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n))+\cdots\\
&+d_1(\theta(\zeta^n),\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))=0.
\end{aligned}
$$

Which implies that $\theta(\zeta^1)\in\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n),\theta(\zeta^2)\in\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^n),\cdots,\theta(\zeta^n)\in\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1}))$. Therefore, we conclude that $(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n)$ is the N-tupled coincidence point of θ and Θ. Next, we show the uniqueness of the N-tupled point of coincidence of θ and Θ. Assume that $(\zeta^{1*}, \zeta^{2*}, \cdots, \zeta^{n*})$ is another N-tupled coincidence point of mappings θ and Θ. By (2.1), we have

$$\begin{aligned}
&d_1(\theta\zeta^1, \theta\zeta^{1*}) + d_1(\theta\zeta^2, \theta\zeta^{2*}) + \cdots + d_1(\theta\zeta^n, \theta\zeta^{n*}) \\
&= d_1(\Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n), \Theta(\zeta^{1*}, \zeta^{2*}, \zeta^{3*}, \cdots, \zeta^{n*})) \\
&+ d_1(\Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^1), \Theta(\zeta^{2*}\zeta^{3*}, \zeta^{4*}, \cdots, \zeta^{n*})) + \cdots \\
&+ d_1(\Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1}), \Theta(\zeta^{n*}, \zeta^{1*}, \zeta^{2*}\zeta^{(n-1)*}))) \\
&\leqslant \lambda_1\left[d_1(\theta\zeta^1, \theta\zeta^{1*}) + d_1(\theta\zeta^2, \theta\zeta^{2*}) + \cdots + d_1(\theta\zeta^n, \theta\zeta^{n*})\right] \\
&+ \lambda_2\left[d_1(\theta\zeta^1, \Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n)) + d_1(\theta\zeta^2, \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n))\right. \\
&\left.+ \cdots + d_1(\theta\zeta^n, \Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1}))\right] \\
&+ \lambda_3\left[d_1(\theta\zeta^{1*}, \Theta(\zeta^{1*}, \zeta^{2*}, \zeta^{3*}, \cdots, \zeta^{n*})) + d_1(\theta\zeta^{2*}, \Theta(\zeta^{2*}\zeta^{3*}, \zeta^{4*}, \cdots, \zeta^{n*}))\right. \\
&\left.+ \cdots + d_1(\theta\zeta^{n*}, \Theta(\zeta^{n*}, \zeta^{1*}, \zeta^{2*}\zeta^{(n-1)*}))\right] \\
&= \lambda_1\left[d_1(\theta\zeta^1, \theta\zeta^{1*}) + d_1(\theta\zeta^2, \theta\zeta^{2*}) + \cdots + d_1(\theta\zeta^n, \theta\zeta^{n*})\right]. \quad (2.22)
\end{aligned}$$

By virtue of $0 \leqslant \lambda_1 \leqslant \lambda_1 + \lambda_2 + \lambda_3 < 1$ and (2.22), we deduce

$$d(\theta\zeta^1, \theta\zeta^{1*}) + d(\theta\zeta^2, \theta\zeta^{2*}) + \cdots + d(\theta\zeta^n, \theta\zeta^{n*}) = 0.$$

This implies that $\theta\zeta^1 = \theta\zeta^{1*}, \theta\zeta^2 = \theta\zeta^{2*}, \cdots, \theta\zeta^n = \theta\zeta^{n*}$. So that the N-tupled point of coincidence of θ and Θ is unique. Next, we show that $\theta\zeta^1 = \theta\zeta^2 = \cdots = \theta\zeta^n$ it follows from (2.1) that

$$\begin{aligned}
&d_1(\theta\zeta^1, \theta\zeta^2) + d_1(\theta\zeta^2, \theta\zeta^3) + \cdots . + d_1(\theta\zeta^{n-1}, \theta\zeta^n) \\
&= d_1(\Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n), \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n)) + d_1(\Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n), \Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1})) \\
&+ \cdots . + d_1(\Theta(\zeta^{n-1}, \zeta^1, \zeta^2, \cdots, \zeta^n), \Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1}))
\end{aligned}$$

$$\begin{aligned}
&\leqslant \lambda_1\left[d_1(\theta\zeta^1, \theta\zeta^2) + d_1(\theta\zeta^2, \theta\zeta^3) + \cdots . + d_1(\theta\zeta^{n-1}, \theta\zeta^n)\right] \\
&+ \lambda_2\left[d_1(\theta\zeta^1, \Theta(\zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^n)) + d_1(\theta\zeta^2, \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n)) + \cdots\right. \\
&\left.+ d_1(\theta\zeta^{n-1}, \Theta(\zeta^{n-1}, \zeta^1, \zeta^2, \cdots, \zeta^n))\right] \\
&+ \lambda_3\left[d_1(\theta\zeta^2, \Theta(\zeta^2, \zeta^3, \zeta^4, \cdots, \zeta^n)) + d_1(\theta\zeta^3, \Theta(\zeta^3, \zeta^1, \zeta^2, \cdots, \zeta^{n-1})) + \cdots\right. \\
&\left.+ d_1(\theta\zeta^n, \Theta(\zeta^n, \zeta^1, \zeta^2, \cdots, \zeta^{n-1}))\right] \\
&= \lambda_1\left[d_1(\theta\zeta^1, \theta\zeta^2) + d_1(\theta\zeta^2, \theta\zeta^3) + \cdots . + d_1(\theta\zeta^{n-1}, \theta\zeta^n)\right]. \quad (2.23)
\end{aligned}$$

By making use of $0 \leqslant \lambda_1 \leqslant \lambda_1 + \lambda_2 + \lambda_3 < 1$ and (2.23), we deduce

$$d_1(\theta\zeta^1, \theta\zeta^2) + d_1(\theta\zeta^2, \theta\zeta^3) + \cdots . + d_1(\theta\zeta^{n-1}, \theta\zeta^n) = 0.$$

This means that

$$\theta\zeta^1 = \theta\zeta^2 = \theta\zeta^3 = \cdots = \theta\zeta^{n-1} = \theta\zeta^n.$$

Finally, if θ and Θ are w-compatible, then we have $\theta(\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n)) \subseteq \Theta(\theta\zeta^1,\theta\zeta^2,\theta\zeta^3,\cdots,\theta\zeta^n)$. Therefore, by taking $u = \theta\zeta^1$, we have

$$u = \theta\zeta^1 = \Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n),\ \theta\zeta^2 = \Theta(\zeta^2,\zeta^3,\zeta^4,\ \cdots,\zeta^n),\cdots,\ \theta\zeta^n = \Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1})),$$

hence we have $\theta u = \theta\theta\zeta^1 = \theta(\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n)) \subseteq \Theta(\theta\zeta^1,\theta\zeta^2,\theta\zeta^3,\cdots,\theta\zeta^n) = \Theta(u,u,\cdots,u)$. Thus, $(\theta u,\theta u,\cdots,\theta u)$ is N-tupled point of coincidence of θ and Θ, and by its uniqueness, we get $\theta u = \theta\zeta^1$. Thus, we obtain $u = \theta u = \Theta(u,u,\cdots,u)$. Therefore, $(u,u,\cdots,u)$ is the unique common N-tupled fixed point of θ and Θ. □

Corollary 2.3. *Assume that d_1 and d_2 be two rectangular b-metrices on Δ with coefficient $s \geqslant 1$ $(d_2(\zeta,\xi) \leqslant d_1(\zeta,\xi))$. and $\Theta : \Delta^n \to \Delta$ and $\theta : \Delta \to \Delta$ be two mappings. Suppose that there exist λ_1,λ_2 and $\lambda_3 \in [0,1)$ with $0 \leqslant \lambda_1+\lambda_2+\lambda_3 < 1$ and $0 \leqslant s\lambda_3 < 1$ such that the following condition holds.*

$$\begin{aligned} d_1(\Theta(\zeta^1,\zeta^2,\zeta^3,..,\zeta^n),\Theta(\eta^1,\eta^2,\eta^3,\cdots,\eta^n)) + d_1(\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^1),\Theta(\eta^2,\eta^3,\eta^4,\cdots,\eta^n)) + \cdots \\ + d_1(\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1},\Theta(\eta^n,\eta^1,\eta^2,\cdots,\eta^{n-1})) \\ \leqslant \lambda_1\big[d_2(\theta\zeta^1,\theta\eta^1) + d_2(\theta\zeta^2,\theta\eta^2) + \cdots + d_2(\theta\zeta^n,\theta\eta^n)\big] \\ + \lambda_2\big[d_2(\theta(\zeta^1),\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n)) + d_2(\theta(\zeta^2),\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^1)) + \cdots \\ + d_2(\theta(\zeta^n),\Theta(\zeta^n,\zeta^2,\zeta^3,\cdots,\zeta^{n-1}))\big] + \\ \lambda_3\big[d_2(\theta(\eta^1),\Theta(\eta^1,\eta^2,\eta^3,\cdots,\eta^n)) + d_2(\theta(\eta^2),\Theta(\eta^2,\eta^1,\eta^3\cdots\eta^n)) + \cdots \\ + d_2(\theta(\eta^n),\Theta(\eta^n,\eta^2,\eta^3,\cdots,\eta^1))\big], \end{aligned}$$

for all $(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n),(\eta^1,\eta^2,\eta^3,\cdots,\eta^n) \in \Delta^n$. If $\Theta(\Delta^n) \subseteq \theta(\Delta)$ and $\theta(\Delta)$ is complete, then θ and Θ have a N-tupled coincidence point $(\zeta^1,\zeta^2,\cdots,\zeta^n) \in \Delta^n$, satisfying that $\theta(\zeta^1) = \Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n)$, $\theta(\zeta^2) = \Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^1)$,$\cdots$, $\theta(\zeta^n) = \Theta(\zeta^n,\zeta^2,\zeta^3\cdots\zeta^{n-1})$. Moreover, if θ and Θ are w-compatible, then θ and Θ have a unique common N-tupled fixed point of the form $(\eta,\eta,\cdots\eta)$, which satisfies that $\eta = \theta\eta \in \Theta(\eta,\eta,\cdots\eta)$.

Let $\Delta = R$. d_1 and d_2 be two rectangular b-metrics such that

$$d_1(\zeta,\xi) = (\zeta-\xi)^2$$

and

$$d_2(\zeta,\xi) = \frac{(\zeta-\xi)^2}{N}.$$

Define $\Theta : \Delta^n \to \Delta$ and $\theta : \Delta \to \Delta$ by

$$\Theta(\zeta^1,\zeta^2,\zeta^3,..,\zeta^n) = \frac{\zeta^1+\zeta^2+\zeta^3+..+\zeta^n}{N}, \theta(\zeta) = N\zeta$$

where $N = 2,3,4...$

$$\begin{aligned} &d_1(\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n),\Theta(\eta^1,\eta^2,\eta^3,\cdots,\eta^n)) \\ &= \Big(\frac{\zeta^1+\zeta^2+\zeta^3+..+\zeta^n}{N} - \frac{\eta^1+\eta^2+\eta^3+..+\eta^n}{N}\Big)^2 \end{aligned}$$

$$= \left(\frac{\zeta^1-\eta^1}{N}+\frac{\zeta^2-\eta^2}{N}+\frac{\zeta^3-\eta^3}{N}+..+\frac{\zeta^n-\eta^n}{N}\right)^2$$
$$\leqslant N\Big[\left(\frac{\zeta^1-\eta^1}{N}\right)^2+\left(\frac{\zeta^2-\eta^2}{N}\right)^2+\left(\frac{\zeta^3-\eta^3}{n}\right)^2+..+\left(\frac{\zeta^n-\eta^n}{n}\right)^2\Big]$$
$$= N\Big[\frac{\left(\zeta^1-\eta^1\right)^2}{N^2}+\frac{\left(\zeta^2-\eta^2\right)^2}{N^2}+\frac{\left(\zeta^3-\eta^3\right)^2}{N^2}+..+\frac{\left(\zeta^n-\eta^n\right)^2}{N^2}\Big]$$
$$= \frac{\left(\zeta^1-\eta^1\right)^2}{N}+\frac{\left(\zeta^2-\eta^2\right)^2}{N}+\frac{\left(\zeta^3-\eta^3\right)^2}{N}+..+\frac{\left(\zeta^n-\eta^n\right)^2}{N}$$
$$= \frac{1}{N^3}\Big[\left(N\zeta^1-N\eta^1\right)^2+\left(N\zeta^2-N\eta^2\right)^2+\left(N\zeta^3-N\eta^3\right)^2+..+\left(N\zeta^n-N\eta^n\right)^2\Big]$$
$$= \frac{1}{N^3}\Big[d_2(\theta\zeta^1,\theta\eta^1)+d_2(\theta\zeta^2,\theta\eta^2)+\cdots+d_2(\theta\zeta^n,\theta\eta^n)\Big]$$

Which implies

$$d_1(\Theta(\zeta^1,\zeta^2,\zeta^3,\cdots,\zeta^n),\Theta(\eta^1,\eta^2,\eta^3,\cdots,\eta^n)) \leqslant \frac{1}{N^3}\Big[d_2(\theta\zeta^1,\theta\eta^1)+d_2(\theta\zeta^2,\theta\eta^2)+\cdots+d_2(\theta\zeta^n,\theta\eta^n)\Big]$$

By similar process we obtain

$$d_1(\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^1),\Theta(\eta^2,\eta^3,\eta^4,\cdots,\eta^n)) \leqslant \frac{1}{N^3}\Big[d_2(\theta\zeta^1,\theta\eta^1)+d_2(\theta\zeta^2,\theta\eta^2)+\cdots+d_2(\theta\zeta^n,\theta\eta^n)\Big]$$
$$\vdots$$
$$d_1(\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1},\Theta(\eta_n,\eta_1,\eta_2,\cdots,\eta^{n-1})) \leqslant \frac{1}{N^3}\Big[d_2(\theta\zeta^1,\theta\eta^1)+d_2(\theta\zeta^2,\theta\eta^2)+\cdots+d_2(\theta\zeta^n,\theta\eta^n)\Big]$$

Combining the above inequalities, we get

$$\begin{aligned} d_1(\Theta(\zeta^1,\zeta^2,\zeta^3,..,\zeta^n),\Theta(\eta^1,\eta^2,\eta^3,\cdots,\eta^n))+d_1(\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^1),\Theta(\eta^2,\eta^3,\eta^4,\cdots,\eta^n))+\cdots \\ +d_1(\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1},\Theta(\eta_n,\eta_1,\eta_2,\cdots,\eta^{n-1})) \\ \leqslant \lambda_1\big[d_2(\theta\zeta^1,\theta\eta^1)+d_2(\theta\zeta^2,\theta\eta^2)+\cdots+d_2(\theta\zeta^n,\theta\eta^n)\big], \end{aligned}$$

by taking $\lambda_2=\lambda_3=0$ and $\lambda_1=\frac{1}{N^2}$ then from theorem (2.2) θ and Θ have common N-tupled fixed point.

3. Application

In this section, we give an existence theorem for the solution of the system of N-integral equations. We assume that $\Delta = C[\kappa_1,\kappa_2]$ is the set of all continuous functions. Define a rectangular b-metric respectively by

$$d(\zeta,\xi)=\max|\zeta(\nu_1)-\xi(\nu_1)|^k, \forall\zeta,\xi\in\Delta, (k\geqslant 1), \nu_1\in[\kappa_1,\kappa_2].$$

Then the coefficient of rectangular b-metric is $s = 3^{k-1}$. Consider the following non-linear system of N-integral equations

$$\left.\begin{aligned}
\zeta^1(u_1) &= K(u_1) + \int_{\kappa_1}^{\kappa_2} \Phi(u_1, v_1)\big[\varphi_1(u_1, \zeta^1(v_1)) + \varphi_2(u_1, \zeta^2(v_1)) + \cdots + \varphi_n(u_1, \zeta^n(v_1))\big]\, dv_1, \\
\zeta^2(u_1) &= K(u_1) + \int_{\kappa_1}^{\kappa_2} \Phi(u_1, v_1)\big[\varphi_1(u_1, \zeta^2(v_1)) + \varphi_2(u_1, \zeta^3(v_1)) + \cdots + \varphi_n(u_1, \zeta^1(v_1))\big]\, dv_1, \\
&\cdots , \\
\zeta^n(u_1) &= K(u_1) + \int_{\kappa_1}^{\kappa_2} \Phi(u_1, v_1)\big[\varphi_1(u_1, \zeta^n(v_1)) + \varphi_2(u_1, \zeta^1(v_1)) + \cdots + \varphi_n(u_1, \zeta^{n-1}(v_1))\big]\, dv_1.
\end{aligned}\right\} \tag{3.1}$$

Theorem 3.1. *Suppose the following hypotheses hold:*

1. $\varphi_1, \varphi_2, \cdots \varphi_n : [\kappa_1, \kappa_2] \times \Delta \to \mathbb{R}$ *are* N- *continuous functions.;*
2. $K : [\kappa_1, \kappa_2] \to \mathbb{R}$ *is continuous function,* $\Phi : [\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2] \to \mathbb{R}^+$ *is a continuous function.;*
3. *There exist* $M_i > 0$ $(i = 1, 2, 3, \cdots n)$ *such that for all* $\zeta, \xi \in \Delta$

$$\begin{aligned}
&|\varphi_1(u_1, \zeta(v_1)) - \varphi_1(u_1, \xi(v_1))| \leqslant M_1|\zeta - \xi|, \\
&\qquad |\varphi_2(u_1, \zeta(v_1)) - \varphi_2(u_1, \xi(v_1))| \leqslant M_2|\zeta - \xi|, \\
&\qquad\qquad \cdots , |\varphi_n(u_1, \zeta(v_1)) - \varphi_n(u_1, \xi(v_1))| \leqslant M_n|\zeta - \xi|.
\end{aligned}$$

4. $$\left(\int_{\kappa_1}^{\kappa_2} \Phi(u_1, v_1) dv_1\right)^k \leqslant \frac{1}{n^{2k+1} M^k}, \text{ where } n \geqslant 3. \tag{3.2}$$

Then under conditions 1-4, the integral equation (3.1) has a unique common solution on $[\kappa_1, \kappa_2]$.

Proof. Define $\Theta : \Delta^n \to \Delta$ and $g : \Delta \to \Delta$ respectively by,

$$\left.\begin{aligned}
\Theta(\zeta^1, \zeta^2, \cdots \zeta^n)(u_1) = K(u_1) + \int_{\kappa_1}^{\kappa_2} \Phi(u_1, v_1)\big[\varphi_1(u_1, \zeta^1(v_1)) + \varphi_2(u_1, \zeta^2(v_1)) \\
+ \cdots + \varphi_n(u_1, \zeta^n(v_1))\big]\, dv_1,
\end{aligned}\right\}$$

$g(\zeta) = \zeta$ for $\zeta \in \Delta$. Now, we have

$$\begin{aligned}
&\Big| d(\Theta(\zeta^1, \zeta^2, \cdots \zeta^n)(u_1)) - \Theta(\eta^1, \eta^2, \cdots \eta^n)(u_1) \Big|^k \\
&= \Big| K(u_1) + \int_{\kappa_1}^{\kappa_2} \Phi(u_1, v_1)\big[\varphi_1(u_1, \zeta^1(v_1)) + \varphi_2(u_1, \zeta^2(v_1)) + \cdots + \varphi_n(u_1, \zeta^n(v_1))\big]\, dv_1 \\
&- K(u_1) + \int_{\kappa_1}^{\kappa_2} \Phi(u_1, v_1)\big[\varphi_1(u_1, \eta^1(v_1)) + \varphi_2(u_1, \eta^2(v_1)) + \cdots + \varphi_n(u_1, \eta^n(v_1))\big]\, dv_1 \Big|^k
\end{aligned}$$

$$= \Big| \int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1)\big[\varphi_1(u_1,\zeta^1(v_1)) - \varphi_1(u_1,\eta^1(v_1))\big] + \int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1)\big[\varphi_2(u_1,\zeta^2(v_1)) - \varphi_2(u_1,\eta^2(v_1))\big] + \cdots + \int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1)\big[\varphi_n(u_1,\zeta^n(v_1)) - \varphi_n(u_1,\eta^n(v_1))\big]\big] dv_1 \Big|^k$$

$$\leqslant n^{k-1} \Big| \int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1)\big[\varphi_1(u_1,\zeta^1(v_1)) - \varphi_1(u_1,\eta^1(v_1))\big] \Big|^k + \Big| \int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1)\big[\varphi_2(u_1,\zeta^2(v_1)) - \varphi_2(u_1,\eta^2(v_1))\big] \Big|^k + \cdots + \Big| \int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1)\big[\varphi_n(u_1,\zeta^n(v_1)) - \varphi_n(u_1,\eta^n(v_1))\big]\big] dv_1 \Big|^k$$

$$\leqslant n^{k-1} M_1^k \Big| \max|\zeta^1(v_1) - \eta^1(v_1)|)\Big|^k + M_2^k \Big| \max|\zeta^2(v_1)) - \eta^2(v_1)|\Big|^k + \cdots + M_n^k \Big| \max|\zeta^n(v_1)) - \eta^n(v_1)|\Big|^k \Big(\int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1) dv_1 \Big)^k$$

$$\leqslant n^{k-1} M^k \Bigg[\Big| \max|\zeta^1(v_1) - \eta^1(v_1)|\Big|^k + \Big| \max|\zeta^2(v_1)) - \eta^2(v_1)|\Big|^k + \cdots + \Big| \max|\zeta^n(v_1)) - \eta^n(v_1)|\Big|^k \Bigg] \Big(\int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1) dv_1 \Big)^k$$

$$\leqslant n^{k-1} M^k \Bigg[d(\zeta^1(v_1),\eta^1(v_1)) + d(\zeta^2(v_1)),\eta^2(v_1)) + \cdots + d(\zeta^n(v_1)),\eta^n(v_1)) \Bigg] \Big(\int_{\kappa_1}^{\kappa_2} \Phi(u_1,v_1) dv_1 \Big)^k$$

$$\leqslant n^{k-1} M^k \Bigg[d(\zeta^1(v_1),\eta^1(v_1)) + d(\zeta^2(v_1)),\eta^2(v_1)) + \cdots + d(\zeta^n(v_1)),\eta^n(v_1)) \Bigg] \frac{1}{n^{2k+1} M^k}$$

$$\leqslant \quad \frac{1}{n^{k+2}} \Bigg[d(\zeta^1(v_1),\eta^1(v_1)) \; + \; d(\zeta^2(v_1)),\eta^2(v_1)) \; + \; \cdots \; + \; d(\zeta^n(v_1)),\eta^n(v_1)) \Bigg]$$

It follows from the above inequality that

$$d((\Theta(\zeta^1,\zeta^2,\cdots\zeta^n)(u_1)),\Theta(\eta^1,\eta^2,\cdots\eta^n)(u_1)) \leqslant \frac{1}{n^{k+2}} \Bigg[d(\zeta^1(v_1),\eta^1(v_1)) + d(\zeta^2(v_1)),\eta^2(v_1)) + \cdots + d(\zeta^n(v_1)),\eta^n(v_1)) \Bigg] \quad (3.3)$$

By similar arguments as above,

$$d((\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^1)(u_1)),\Theta(\eta^2,\eta^3,\eta^4,\cdots,\eta^1)(u_1))$$
$$\leqslant \frac{1}{n^{k+2}}\Big[d(\zeta^2(v_1),\eta^2(v_1))+d(\zeta^3(v_1)),\eta^3(v_1))+\cdots+d(\zeta^1(v_1)),\eta^1(v_1))\Big] \quad (3.4)$$

$$\cdots$$

$$d((\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1})(u_1)),\Theta(\eta^n,\eta^1,\eta^2,\cdots,\eta^{n-1})(u_1))$$
$$\leqslant \frac{1}{n^{k+2}}\Big[d(\zeta^n(v_1),\eta^n(v_1))+d(\zeta^1(v_1)),\eta^1(v_1))+\cdots+d(\zeta^{n-1}(v_1)),\eta^{n-1}(v_1))\Big] \quad (3.5)$$

It follows from (3.3), (3.4) and (3.5)

$$d(\Theta(\zeta^1,\zeta^2,\cdots\zeta^n),\Theta(\eta^1,\eta^2,\cdots\eta^n))+d((\Theta(\zeta^2,\zeta^3,\zeta^4,\cdots,\zeta^1)(u_1)),\Theta(\eta^2,\eta^3,\eta^4,\cdots,\eta^1)(u_1))$$
$$+\cdots+d((\Theta(\zeta^n,\zeta^1,\zeta^2,\cdots,\zeta^{n-1})(u_1)),\Theta(\eta^n,\eta^1,\eta^2,\cdots,\eta^{n-1})(u_1))$$
$$\leqslant \frac{1}{n^{k+2}}[d(\zeta^1(v_1),\eta^1(v_1))+d(\zeta^2(v_1)),\eta^2(v_1))+\cdots+d(\zeta^n(v_1)),\eta^n(v_1))]$$

By taking $\lambda_2=\lambda_3=0$ in equation (2.2) and $d_1=d_2=d$ we have that there exists $\zeta\in\Delta$ such that $\Theta(\zeta,\zeta,\cdots.\zeta)=\theta\zeta=\zeta$. Which implies that ζ is the unique solution of equation set (3.1).

□

4. conclusion

We have obtained N-tupled fixed point result in rectangular b-metric. Also, existence and uniqueness results for non-linear N-integral equations have been given. Our results can further be applied to the system of fractional differential equations and system of matrix equations containing N number of equations.

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