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Certain subclasses of harmonic functions involving $q-$Mittag-Leffler Function

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Abstract

In this article, the $q-$ differential operator for harmonic function related with Mittag-Leffler function is described to familiarise a new class of complex-valued harmonic functions which are orientation preserving, univalent in the open unit disc . We conquer certain significant aspects, such as distortion limits, preservation of convolution and convexity constraints, which are also addressed. Furthermore, with the use of sufficiency criteria, we calculate sharp bounds of the real parts of the ratios of harmonic functions to its sequences of partial sums. Besides, some of the interesting consequences of our investigation are also included.

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1. Introduction and Definitions

Let $\mathfrak{h} = u + iv$ is continuous and complex- valued harmonic function in Ω a complex domain whenever u and v are real and harmonic in Ω. In $\mathbb{D} \subset \Omega$ any simply-connected domain, we uniquely represent $\mathfrak{h} = \mathfrak{f} + \overline{\mathfrak{g}}$, where $\mathfrak{f}$ and $\mathfrak{g}$ are analytic in $\mathbb{D}$. We call $\mathfrak{f}$ the analytic part and $\mathfrak{g}$ the co-analytic part of $\mathfrak{h}$. Also $\mathfrak{h}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if $|\mathfrak{f}'(z)| > |\mathfrak{g}'(z)|$ in $\mathbb{D}$ (see [11]). Symbolize by $\mathcal{H}$ the family of functions

$$\mathfrak{h} = \mathfrak{f} + \overline{\mathfrak{g}} \tag{1.1}$$

which are harmonic, univalent and sense preserving in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ so that $\mathfrak{h}$ is normalized by $\mathfrak{h}_{\overline{z}}(0) = \mathfrak{h}'_z(0) - 1 = 0$. Thus, for $\mathfrak{h} = \mathfrak{f} + \overline{\mathfrak{g}} \in \mathcal{H}$, the functions $\mathfrak{f}$ and $\mathfrak{g}$ analytic in $\mathbb{U}$ can be articulated in the ensuing forms:

$$\mathfrak{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; \mathfrak{g}(z) = \sum_{n=1}^{\infty} b_n z^n \;\; (0 \leqslant b_1 < 1),$$

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and $\mathfrak{h}(z)$ is then given by

$$\mathfrak{h}(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad (0 \leqslant b_1 < 1). \tag{1.2}$$

We annotate the family $\mathcal{H} \equiv \mathcal{S}$ if $\mathfrak{g} \equiv 0$. Denote by $\overline{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions of the form

$$\mathfrak{h}(z) = z - \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \;\; (0 \leqslant b_1 < 1). \tag{1.3}$$

teremed the class of harmonic functions with negative coefficients (see [33]).

For $\mathfrak{h} \in \mathcal{H}$ assumed as in (1.1) and $\mathfrak{H} \in \mathcal{H}$ assumed by

$$\mathfrak{H}(z) = \mathfrak{F}(z) + \overline{\mathfrak{G}(z)} = z + \sum_{n=2}^{\infty} u_n z^n + \overline{\sum_{n=1}^{\infty} v_n z^n}, \tag{1.4}$$

we evoke the Hadamard product (or convolution) of $\mathfrak{h}$ and $\mathfrak{H}$ by

$$(\mathfrak{h} * \mathfrak{H})(z) = z + \sum_{n=2}^{\infty} a_n u_n z^n + \overline{\sum_{n=1}^{\infty} b_n v_n z^n} \;\; (z \in \mathbb{U}). \tag{1.5}$$

We concisely evoke here the concept of *q-operators* i.e. *q-difference operator* fascinated and inspired many scholars due its use in various areas of the quantitative sciences. The application of *q-calculus* was initiated by Jackson [16] (also see [8, 23, 38]). Kanas and Răducanu [21] have used the fractional *q-calculus operators* in investigations of certain classes of functions which are analytic in $\mathbb{U}$.

For $0 < q < 1$ the Jackson's *q-derivative* of a function $\mathfrak{f} \in \mathcal{S}$ is, by definition, given as follows [16]

$$\mathcal{D}_q \mathfrak{f}(z) = \begin{cases} \dfrac{\mathfrak{f}(z) - \mathfrak{f}(qz)}{(1-q)z} & \text{for} \quad z \neq 0, \\ \mathfrak{f}'(0) & \text{for} \quad z = 0, \end{cases} \tag{1.6}$$

and $\mathcal{D}_q^2 \mathfrak{f}(z) = \mathcal{D}_q(\mathcal{D}_q \mathfrak{f}(z))$. From (1.6), we have $\mathcal{D}_q \mathfrak{f}(z) = 1 + \sum_{n=2}^{\infty} [n] a_n z^{n-1}$ where $[n] = \frac{1-q^n}{1-q}$, is sometimes called *the basic number* n. If $q \to 1^-, [n] \to n$. For a function $\mathfrak{f}(z) = z^m$, we obtain $\mathcal{D}_q \mathfrak{f}(z) = \mathcal{D}_q z^m = \frac{1-q^m}{1-q} z^{m-1} = [m] z^{m-1}$, and

$$\lim_{q \to 1^-} \mathcal{D}_q \mathfrak{f}(z) = \lim_{q \to 1^-} \left([m] z^{m-1} \right) = m z^{m-1} = \mathfrak{f}'(z),$$

where $\mathfrak{f}'$ is the ordinary derivative.

Research work in linking with function theory and $q-$ theory together was first presented by Ismail et al. [15]. Till now only non-significant interest in this area was shown although it deserves more attention. The $q-$difference operator related to the $q-$calculus was introduced by Andrews et al. (see[9], Ch. 10). Now we recall the well Mittag-Leffler function $\mathfrak{M}_{\vartheta}(z)$ studied by Mittag-Leffler [25] and given by

$$\mathfrak{M}_{\vartheta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\vartheta n + 1)}, \quad (z \in \mathbb{C}, \Re() > 0\ ).$$

A more universal function $\mathfrak{M}_{\vartheta,\rho}$ generalizing $\mathfrak{M}_{\vartheta}(z)$ was familiarised by Wiman [39, 40] and defined by

$$\mathfrak{M}_{\vartheta,\rho}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\vartheta n + \rho)}, \quad (z, \vartheta, \rho \in \mathbb{C},\ \Re() > 0,\ \Re() > 0). \tag{1.7}$$

Recent attention has been drawn to Mittag-Leffer function research, as this kind of function can be widely applied across engineering, chemical and biological sciences, physics and in applied science. Various factors in applying such functions are evident within chaotic, stochastic and dynamic systems, fractional differential equations, and distribution of statistics. The geometric characteristics such as convexity, close-to-convexity and starlikeness, of the functions investigated here have been broadly examined by many authors. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [6, 10, 13, 23].Observe that Mittag-Leffler function $\mathcal{E}_{\vartheta,\rho}$ does not belong to the family $\mathcal{A}$. Therefore, we consider the following normalization of the Mittag-Leffler function (see,[10, 30])

$$\begin{aligned} \mathcal{E}_{\vartheta,\rho}(z) &= \Gamma(\rho) z \mathfrak{M}_{\vartheta,\rho}(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\rho)}{\Gamma(\vartheta(n-1)+\rho)} z^n, \end{aligned} \tag{1.8}$$

where $z, \vartheta, \rho \in \mathbb{C}; \rho \neq 0, -1, -2, \cdots$ and $\Re() > 0$, $\Re() > 0$. Whilst formula (1.8) holds for complex-valued ϑ, ρ and $z \in \mathbb{C}$, however in this paper, we shall restrict our attention to the case of real-valued ϑ, ρ and $z \in \mathbb{U}$. Observe that the function $\mathfrak{M}_{\vartheta,\rho}$ contains many well-known functions as its special case, for example,
$\mathfrak{M}_{2,1}(z) = z \cosh \sqrt{z}$, $\mathfrak{M}_{2,2}(z) = \sqrt{z} \sinh \sqrt{z}$, $\mathfrak{M}_{2,3}(z) = 2[\cosh \sqrt{z} - 1]$ and
$\mathfrak{M}_{2,4}(z) = 6[\sinh \sqrt{z} - \sqrt{z}]/\sqrt{z}$. Now we define the following new linear operator based on convolution (or Hadamard) product. In 2014 Sharma and Jain [37] introduced the q-analogue of generalized Mittag-Leffler function

$$\mathcal{E}^{\sigma}_{\vartheta,\rho}(z, q) = \sum_{n=0}^{\infty} \frac{(q^{\sigma}; q)_n}{(q; q)_n} \frac{\Gamma(\rho)}{\Gamma(\vartheta n + \rho)} z^n, \quad (|q| < 1]) \tag{1.9}$$

where $\Gamma_q(n)$ is the $q-$gamma function and $q \to 1^-$ and $\Gamma_q(n) = \Gamma(n)$. The $q-$analogue of the Pochhammer symbol ($q-$shifted factorial) is defined by (see [14])

$$(\kappa, q)_n = \begin{cases} (1-\kappa)(1-\kappa q)...(1-\kappa q^{n-1}) & \text{for} \quad n = 1, 2, 3, ..., \\ 1 & \text{for} \quad n = 0, \end{cases} \tag{1.10}$$

Further, the $q-$ gamma function $\Gamma_q(z)$ satisfies the functional equation (see [5, 14]))

$$\Gamma_q(n+1) = \frac{1-q^n}{1-q} \Gamma_q(n) = [n]_q \Gamma_q(n)$$

Also,

$$(q^{\lambda}, q)_n = \frac{(1-q)^n \Gamma_q(\lambda+n)}{\Gamma_q(\lambda)} \quad (n > 0).$$

We define

$$\begin{aligned} \mathcal{M}^{\sigma}_{\vartheta,\rho}(z,q) &= \Gamma(\rho) z \mathfrak{M}_{\vartheta,\rho}(z;q) \\ &= z + \sum_{n=2}^{\infty} \frac{(q^{\sigma};q)_{n-1}}{(q;q)_{n-1}} \frac{\Gamma(\rho)}{\Gamma(\vartheta(n-1)+\rho)} z^n. \end{aligned} \tag{1.11}$$

For real parameters ϑ, ρ, with $\vartheta, \rho, \notin \{0, -1, -2, \dots\}$ and $\mathcal{E}_{\vartheta,\rho}$ be given by (1.8), we define the linear operator $\Lambda_{\vartheta,\rho} : \mathcal{A} \to \mathcal{A}$ with the aid of the convolution product

$$\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{f}(z) := \mathfrak{f}(z) * \mathcal{M}^{\sigma}_{\vartheta,\rho}(z,q) = z + \sum_{n=2}^{\infty} \frac{(q^{\sigma};q)_{n-1}}{(q;q)_{n-1}} \frac{\Gamma(\rho)}{\Gamma(\vartheta(n-1)+\rho)} a_n z^n, \; z \in \mathbb{D}, \tag{1.12}$$

and

$$\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{g}(z) := \mathfrak{g}(z) * \mathcal{M}^{\sigma}_{\vartheta,\rho}(z,q) = \sum_{n=1}^{\infty} \frac{(q^{\sigma};q)_{n-1}}{(q;q)_{n-1}} \frac{\Gamma(\rho)}{\Gamma(\vartheta(n-1)+\rho)} b_n z^n, \; z \in \mathbb{D}, \tag{1.13}$$

where $*$ denote the convolution or Hadamard product of two series. Lately,Abdeljawad and Baleanu[1] have studied the $q-$ analogue Mittag–Leffler function, a nabla guantum analogue of a Mittag-Leffler function with two parameters,and Caputo $q-$fractional derivatives are introduced and studied in recent past (refer to [2, 3]and references cited therin) for different perspective. For the determination of this article, we familarize a new operator

$$z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{h}(z)) = z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{f}(z)) - \overline{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{g}(z))} \tag{1.14}$$

and describe a subclass of $\mathcal{H}$ symbolized by $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\lambda, \gamma)$ which comprises the convolution (1.5) and consist of all functions of the form (1.1) sustaining the inequality:

$$\Re\left(\frac{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{h}(z))}{\lambda z' + (1-\mu)(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{h}(z))}\right) = \Re\left(\frac{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{f}(z)) - \overline{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{g}(z))}}{\mu z' + (1-\mu)(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{h}(z))}\right) \geqslant \xi \tag{1.15}$$

where $z \in \mathbb{U}$, $0 \leqslant \mu \leqslant 1$ and $z' = \frac{\partial}{\partial \theta} z = re^{i\theta}$ where $0 = \theta < 2\pi$. Also denote $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi) = \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi) \cap \overline{\mathcal{H}}$.

We deem it appropriate to remark underneath some of the function classes which transpire from the function class $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$ defined above. Indeed, we observe that if we fix the parameters μ suitably and $q \to 1^-$. Denote the reliable reducible new classes of $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$, as illustrated below:

(i) If $\mu = 0$ we let $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(0, \xi) = \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\xi)$ satisfying the criteria

$$\Re\left(\frac{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{h}(z))}{\Lambda^{\sigma}_{\vartheta,\rho} \mathfrak{h}(z)}\right) \geqslant \xi.$$

(ii) If $\mu = 1$ we let $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(1,\xi) = \mathcal{NH}^{q,\sigma}_{\vartheta,\rho}(\xi)$ satisfying the criteria

$$\Re\left(\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z))\right) \geqslant \xi.$$

(iii) Letting $q \to 1$ we let $\mathcal{MH}^{\sigma}_{\vartheta,\rho}$ satisfying the criteria

$$\Re\left(\frac{z(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z))'}{\mu z' + (1-\mu)\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z)}\right) \geqslant \xi.$$

(iv) Letting $q \to 1$, we let $\mathcal{MH}^{\sigma}_{\vartheta,\rho}(0,\xi) = \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\xi)$ [18] satisfying the criteria

$$\Re\left(\frac{z(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z))'}{\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z)}\right) \geqslant \xi.$$

(v) Letting $q \to 1$ and taking $\mu = 1$, we let $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(0,\xi) = \mathcal{RH}^{\sigma}_{\vartheta,\rho}(\mu,\xi)$ satisfying the criteria

$$\Re\left(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z)\right)' \geqslant \xi.$$

Stirred by the prior works (see [7, 11, 17, 18, 19, 20, 22, 27, 28, 29, 33]) on the subject of harmonic functions, in this paper we obtain a sufficiency criteria for functions $\mathfrak{h}$ given by (1.2) to be in the class $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$. It is shown that this criteria is also necessary for $\mathfrak{h}\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$. Further, distortion limits, convexity conditions,extreme points and partial sums problems $f \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ are also obtained.

2. Coefficient bounds

For the sake of brevity we denote

$$\Psi^{\sigma}_{\vartheta,\rho}(n,q) = \frac{(q^{\sigma};q)_{n-1}}{(q;q)_{n-1}}\frac{\Gamma(\rho)}{\Gamma(\vartheta(n-1)+\rho)} \tag{2.1}$$

throughout our study unless otherwise stated.

In the following theorem, we obtain a sufficient criteria for $\mathfrak{h} \in \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$. Let $\mathfrak{h} = \mathfrak{f} + \overline{\mathfrak{g}}$ be given by (1.2). If

$$\sum_{n=1}^{\infty}\left[\frac{[n]_q-(1-\mu)\xi}{1-\xi}|a_n| + \frac{[n]_q+(1-\mu)\xi}{1-\xi}|b_n|\right]\Psi^{\sigma}_{\vartheta,\rho}(n,q) \leqslant 2 \tag{2.2}$$

where $a_1 = 1$ and $0 \leqslant \xi < 1$, then $\mathfrak{h} \in \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$.

Proof. In order to achieve the result,it is sufficient to determine $\mathfrak{h} \in \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ validates the relationship (2.2) . From (1.15) we can write

$$\Re\left(\frac{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{f}(z)) - \overline{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{g}(z))}}{(1-\mu)z' + \mu(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z))}\right) \geqslant \xi$$

$$= \Re\left(\frac{A(z)}{B(z)}\right) \geqslant \xi$$

where

$$\begin{aligned} A(z) &= z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{f}(z)) - \overline{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{g}(z))} \\ &= z + \sum_{n=2}^{\infty}[n]_q\Psi^{\sigma}_{\vartheta,\rho}(n,q)a_n z^n - \sum_{n=1}^{\infty}[n]_q\Psi^{\sigma}_{\vartheta,\rho}(n,q)\overline{b}_n\overline{z}^n \\ \text{and } B(z) &= \mu z' + (1-\mu)(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z)) \\ &= z + \sum_{n=2}^{\infty}(1-\mu)\Psi^{\sigma}_{\vartheta,\rho}(n,q)a_n z^n + \sum_{n=1}^{\infty}(1-\mu)\Psi^{\sigma}_{\vartheta,\rho}(n,q)\overline{b}_n\overline{z}^n. \end{aligned}$$

Consuming the fact that Re $\{w\} \geqslant \xi$ if and only if $|1-\xi+w| \geqslant |1+\xi-w|$, it suffices to show that

$$|A(z) + (1-\xi)B(z)| - |A(z) - (1+\xi)B(z)| \geqslant 0. \tag{2.3}$$

Bartering for $A(z)$ and $B(z)$ in (2.3), we get

$$\begin{aligned} & |A(z) + (1-\xi)B(z)| - |A(z) - (1+\xi)B(z)| \\ = & \ \Big|\ (2-\xi)z + \sum_{n=2}^{\infty}[[n]_q + (1-\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)a_n z^n - \sum_{n=1}^{\infty}[[n]_q - (1-\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)\overline{b}_n\ \overline{z}^n\ \Big| \\ & - \Big| -\xi z + \sum_{n=2}^{\infty}[[n]_q - (1+\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)a_n z^n - \sum_{n=1}^{\infty}[[n]_q + (1+\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)\overline{b}_n\overline{z}^n\ \Big| \\ \geqslant & \ (2-\xi)|z| - \sum_{n=2}^{\infty}[[n]_q + (1-\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)|a_n||z|^n - \sum_{n=1}^{\infty}[[n]_q - (1-\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)|b_n|\ |z|^n \\ & -\xi|z| - \sum_{n=2}^{\infty}[[n]_q - (1+\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)|a_n|\ |z|^n - \sum_{n=1}^{\infty}[[n]_q + (1+\xi)(1-\mu)]\Psi^{\sigma}_{\vartheta,\rho}(n,q)|b_n|\ |z|^n \\ \geqslant & \ 2(1-\xi)|z|\left\{2 - \sum_{n=1}^{\infty}\left[\frac{[n]_q - (1-\mu)\xi}{1-\xi}|a_n| + \frac{[n]_q + (1-\mu)\xi}{1-\xi}|b_n|\right]\Psi^{\sigma}_{\vartheta,\rho}(n,q)|z|^{n-1}\right\} \\ \geqslant & \ 2(1-\xi)\left\{2 - \sum_{n=1}^{\infty}\left[\frac{[n]_q - (1-\mu)\xi}{1-\xi}|a_n| + \frac{[n]_q + (1-\mu)\xi}{1-\xi}|b_n|\right]\Psi^{\sigma}_{\vartheta,\rho}(n,q)\right\}. \end{aligned}$$

The above condition is non negative by (2.2), and so $\mathfrak{f} \in \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$.

The harmonic function

$$\mathfrak{h}(z) = z + \sum_{n=2}^{\infty}\frac{1-\xi}{([n]_q - \xi(1-\mu))\Psi^{\sigma}_{\vartheta,\rho}(n,q)}x_n z^n + \sum_{n=1}^{\infty}\frac{1-\xi}{([n]_q + \xi(1-\mu))\Psi^{\sigma}_{\vartheta,\rho}(n,q)}\overline{y}_n(\overline{z})^n \tag{2.4}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$ shows that the coefficient bound given by (2.2) is sharp.
The functions of the form (2.4) are in $\mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ because

$$\begin{aligned} & \sum_{n=1}^{\infty}\left(\frac{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi}|a_n| + \frac{([n]_q + (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi}|b_n|\right) \\ = & \ 1 + \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 2. \end{aligned}$$

□

Next theorem launches that such coefficient constraints cannot be enhanced further. For $a_1 = 1$ and $0 \leqslant \xi < 1$, $\mathfrak{h} = \mathfrak{f} + \overline{\mathfrak{g}} \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$ if and only if

$$\sum_{n=1}^{\infty} \left[\frac{[n]_q - (1-\mu)\xi}{1-\xi} |a_n| + \frac{[n]_q + (1-\mu)\xi}{1-\xi} |b_n| \right] \Psi^{\sigma}_{\vartheta,\rho}(n, q) \leqslant 2. \tag{2.5}$$

Proof. Since $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi) \subset \mathcal{HS}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$, we only essential to prove the "only if" part of the theorem. To this end, for $\mathfrak{h} \in \mathfrak{f}$ of the form (1.3), we sign that the condition

$$\Re \left(\frac{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z)) - \overline{z\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{g}(z))}}{(1-\mu)z' + \mu(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(z))} \right) \geqslant \xi$$

Equivalently,

$$\Re \left(\frac{(1-\xi)z - \sum\limits_{n=2}^{\infty} ([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n, q) a_n z^n - \sum\limits_{n=1}^{\infty} [[n]_q + (1-\mu)\xi]\Psi^{\sigma}_{\vartheta,\rho}(n, q)\overline{b}_n \overline{z}^n}{z - \sum\limits_{n=2}^{\infty} (1-\mu)\Psi^{\sigma}_{\vartheta,\rho}(n, q) a_n z^n + \sum\limits_{n=1}^{\infty} (1-\mu)\Psi^{\sigma}_{\vartheta,\rho}(n, q)\overline{b}_n \overline{z}^n} \right) \geqslant 0.$$

The above mandatory condition must hold for all values of z in $\mathbb{U}$. Upon taking the values of z on the positive real axis where $0 \leqslant z = r < 1$, we must have

$$\frac{(1-\xi) - \sum\limits_{n=2}^{\infty} ([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n, q) a_n r^{n-1} - \sum\limits_{n=1}^{\infty} ([n]_q + (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n, q) b_n r^{n-1}}{1 - \sum\limits_{n=2}^{\infty} (1-\mu)\Psi^{\sigma}_{\vartheta,\rho}(n, q) a_n r^{n-1} + \sum\limits_{n=1}^{\infty} (1-\mu)\Psi^{\sigma}_{\vartheta,\rho}(n, q) b_n r^{n-1}} \geqslant 0. \tag{2.6}$$

If the condition (2.5) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the quotient of (2.6) is negative. This contradicts the required condition for $\mathfrak{h} \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$. This completes the proof of the theorem. □

3. Distortion bounds and Extreme Points

The subsequent theorem provides the distortion limits for functions in $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$ which yields a covering result for the class $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$.

Let $\mathfrak{h} \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$. Then for $|z| = r < 1$, we have

$$\begin{aligned} &(1-b_1)r - \frac{\Gamma_q(\vartheta, \rho)}{\Gamma_q(\rho)[\sigma]_q} \left(\frac{1-\xi}{[2]_q - (1-\mu)\xi} - \frac{1+\xi}{[2]_q - (1-\mu)\xi} b_1 \right) r^2 \leqslant |\mathfrak{f}(z)| \\ &\leqslant (1+b_1)r + \frac{\Gamma_q(\vartheta, \rho)}{\Gamma_q(\rho)[\sigma]_q} \left(\frac{1-\xi}{[2]_q - (1-\mu)\xi} - \frac{1+\xi}{[2]_q - (1-\mu)\xi} b_1 \right) r^2. \end{aligned}$$

when for $n = 2$ in (2.1) we get

$$\Psi^{\sigma}_{\vartheta,\rho}(n, q) = \frac{(q^{\sigma}; q)_{n-1}}{(q; q)_{n-1}} \frac{\Gamma(\rho)}{\Gamma(\vartheta(n-1) + \rho)} = \Psi^{\sigma}_{\vartheta,\rho}(2, q) = \frac{\Gamma_q(\vartheta, \rho)}{\Gamma_q(\rho)[\sigma]_q}. \tag{3.1}$$

Proof. We only show the right hand inequality. Taking the absolute value of $\mathfrak{h}(z)$, we obtain

$$\begin{aligned}|\mathfrak{h}(z)| &= \left|z+\sum_{n=2}^{\infty}a_n z^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n\right| \\ &\leqslant (1+b_1)|z|+\sum_{n=2}^{\infty}(a_n+b_n)|z|^n \\ &= (1+b_1)r+\sum_{n=2}^{\infty}(a_n+b_n)r^2 \\ &= (1+b_1)r+\frac{(1-\xi)\Gamma_q(\vartheta,\rho)}{([2]_q-(1-\mu)\xi)\Gamma_q(\rho)[\sigma]_q} \\ &\times \sum_{n=2}^{\infty}\left(\frac{([2]_q-(1-\mu)\xi)\Gamma_q(\rho)[\sigma]_q}{(1-\xi)\Gamma_q(\vartheta,\rho)}a_n+\frac{([2]_q-(1-\mu)\xi)\Gamma_q(\rho)[\sigma]_q}{(1-\xi)\Gamma_q(\vartheta,\rho)}b_n\right)r^2 \\ &= (1+b_1)r+\frac{(1-\xi)\Gamma_q(\vartheta,\rho)}{([2]_q-(1-\mu)\xi)\Gamma_q(\rho)[\sigma]_q}\left(1-\frac{1+\xi}{1-\xi}b_1\right)r^2 \\ &\leqq (1+b_1)r+\frac{\Gamma_q(\vartheta,\rho)}{\Gamma_q(\rho)[\sigma]_q}\left(\frac{1-\xi}{[2]_q-(1-\mu)\xi}-\frac{1+\xi}{[2]_q-(1-\mu)\xi}b_1\right)r^2.\end{aligned}$$

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality. □

The covering result follows from the left hand inequality given in Theorem 3. If $\mathfrak{h}\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$, then

$$\left\{w:|w|<\frac{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)-(1-\xi)}{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)}-\frac{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)-(1+\xi)}{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)}|b_1|\right\}\subset\mathfrak{h}(\mathbb{U})$$

when for $n=2$ we let

$$\Psi^{\sigma}_{\vartheta,\rho}(2,q)=\frac{\Gamma_q(\vartheta,\rho)}{\Gamma_q(\rho)[\sigma]_q}.$$

Proof. Using the left hand inequality of Theorem 3 and letting $r\to 1$, we prove that

$$\begin{aligned}&(1-b_1)-\frac{1}{\Psi^{\sigma}_{\vartheta,\rho}(2,q)}\left(\frac{1-\xi}{[2]_q-(1-\mu)\xi}-\frac{1+\xi}{[2]_q-(1-\mu)\xi}b_1\right) \\ &= (1-b_1)-\frac{1}{\Psi^{\sigma}_{\vartheta,\rho}(2,q)([2]_q-(1-\mu)\xi)}[1-\xi-(1+\xi)b_1] \\ &= \frac{(1-b_1)\Psi^{\sigma}_{\vartheta,\rho}(2,q)([2]_q-(1-\mu)\xi)-(1-\xi)+(1+\xi)b_1}{\Psi^{\sigma}_{\vartheta,\rho}(2,q)([2]_q-(1-\mu)\xi)} \\ &= \left\{\frac{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)-(1-\xi)}{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)}-\frac{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)-(1+\xi)}{([2]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(2,q)}|b_1|\right\}. \\ &\subset\mathfrak{h}(\mathbb{U}).\end{aligned}$$

□

Next we regulate the extreme points of closed convex hulls of $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ symbolized by $clco\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$.

A function $\mathfrak{h} \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ if and only if

$$\mathfrak{h}(z) = \sum_{n=1}^{\infty} (X_n \mathfrak{h}_n(z) + Y_n \mathfrak{g}_n(z))$$

where

$$\mathfrak{f}_1(z) = z, \mathfrak{f}_n(z) = z - \frac{1-\xi}{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)} z^n; \qquad (n \geqslant 2), \tag{3.2}$$

$$\mathfrak{g}_n(z) = z + \frac{1-\xi}{([n]_q + (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)} \bar{z}^n; \qquad (n \geqslant 2), \tag{3.3}$$

$\sum\limits_{n=1}^{\infty} (X_n + Y_n) = 1, \quad X_n \geqslant 0 \text{ and } Y_n \geqslant 0$. In particular, the extreme points of $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ are $\{\mathfrak{f}_n\}$ and $\{\mathfrak{g}_n\}$.

Proof. We annotate that for $\mathfrak{h}$ as in above theorem , we may state

$$\begin{aligned}
\mathfrak{h}(z) &= \sum_{n=1}^{\infty} (X_n \mathfrak{f}_n(z) + Y_n \mathfrak{g}_n(z)) \\
&= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\xi}{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)} X_n z^n \\
&\quad + \sum_{n=1}^{\infty} \frac{1-\xi}{([n]_q + (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)} Y_n \bar{z}^n.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi} |a_n| + \sum_{n=1}^{\infty} \frac{([n]_q + (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi} |b_n| \\
&= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n \\
&= 1 - X_1 \leqslant 1,
\end{aligned}$$

and so $\mathfrak{h} \in clco\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$.

Conversely, suppose that $\mathfrak{h} \in clco\overline{\mathcal{HS}}^{\sigma}_{\vartheta,\rho}(\mu,\xi)$ Setting

$$X_n = \frac{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi} |a_n|, \quad (0 \leqslant X_n \leqslant 1, n \geqslant 2)$$

$$Y_n = \frac{([n]_q + (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi} |b_n|, \;\; (0 \leqslant Y_n \leqslant 1, n \geqslant 1)$$

and $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$.
Therefore ,$\mathfrak{h}$ can be rewritten as

$$\begin{aligned}
\mathfrak{h}(z) &= z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \\
&= z - \sum_{n=2}^{\infty} \frac{1-\xi}{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)} X_n z^n + \sum_{n=1}^{\infty} \frac{1-\xi}{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)} Y_n \overline{z}^n \\
&= z + \sum_{n=2}^{\infty} (\mathfrak{f}_n(z) - z) X_n + \sum_{n=1}^{\infty} (\mathfrak{g}_n(z) - z) Y_n \\
&= z\{1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n\} + \sum_{n=2}^{\infty} \mathfrak{f}_n(z) X_n + \sum_{n=1}^{\infty} \mathfrak{g}_n(z) Y_n \\
&= \sum_{n=1}^{\infty} (X_n \mathfrak{f}_n(z) + Y_n \mathfrak{g}_n(z)) \text{ as required.}
\end{aligned}$$

□

4. Inclusion Results

Now we inspect convinced closure properties for $f \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ below convex combinations and integral transform.

The family $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ is closed under convex combinations.

Proof. For $i = 1, 2, \ldots,$ suppose that $\mathfrak{h}_i \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ where

$$\mathfrak{h}_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + \sum_{n=2}^{\infty} \overline{b}_{i,n} \overline{z}^n.$$

Then, by Theorem 2

$$\sum_{n=2}^{\infty} \frac{([n]_q - (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi} a_{i,n} + \sum_{n=1}^{\infty} \frac{([n]_q + (1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi} b_{i,n} \leqslant 1. \quad (4.1)$$

For $\sum_{i=1}^{\infty} \tau_i = 1$, $0 \leqslant \tau_i \leqslant 1$, the convex combination of $\mathfrak{h}_i$ may be written as

$$\sum_{i=1}^{\infty} \tau_i \mathfrak{h}_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} \tau_i a_{i,n} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \tau_i \overline{b}_{i,n} \right) \overline{z}^n.$$

Using the inequality (2.5), we obtain

$$\sum_{n=2}^{\infty}\frac{([n]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi}\left(\sum_{i=1}^{\infty}\tau_i a_{i,n}\right)+\sum_{n=1}^{\infty}\frac{([n]_q+(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi}\left(\sum_{i=1}^{\infty}\tau_i b_{i,n}\right)$$
$$=\sum_{i=1}^{\infty}\tau_i\left(\sum_{n=2}^{\infty}\frac{([n]_q-(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi}a_{i,n}+\sum_{n=1}^{\infty}\frac{([n]_q+(1-\mu)\xi)\Psi^{\sigma}_{\vartheta,\rho}(n,q)}{1-\xi}b_{i,n}\right)$$
$$\leqslant\sum_{i=1}^{\infty}\tau_i=1,$$

and therefore $\sum_{i=1}^{\infty}\tau_i\mathfrak{h}_i\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$. □

Now,let the generalized q− Bernardi-Libera -Livingston integral operator$\mathcal{L}_{q,\kappa}(h)$ be defined by

$$\mathcal{L}_{q,\kappa}(\mathfrak{h})=\frac{[\kappa+1]_q}{z^{\kappa}}\left[\int_0^z t^{\kappa-1}f(t)d_qt+\int_0^z t^{\kappa-1}\overline{\mathfrak{g}(t)}d_qt,\right]\qquad(\kappa>-1).$$

Let $\mathfrak{h}\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$. Then $\mathcal{L}_{\kappa}(\mathfrak{h}(z))\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$.

Proof. From the representation of $\mathcal{L}_{\kappa}(\mathfrak{h}(z))\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$, it follows that

$$\mathcal{L}_{q,\kappa}(\mathfrak{h})=\frac{[\kappa+1]_q}{z^{\kappa}}\int_0^z t^{\kappa-1}\left[\mathfrak{f}(t)+\overline{\mathfrak{g}(t)}\right]dt.$$

$$=\frac{[\kappa+1]_q}{z^{\kappa}}\left(\int_0^z t^{\kappa-1}\left(t-\sum_{n=2}^{\infty}a_nt^n\right)d_qt+\overline{\int_0^z t^{\kappa-1}\left(\sum_{n=1}^{\infty}b_nt^n\right)d_qt}\right)$$
$$=z-\sum_{n=2}^{\infty}\frac{[\kappa+1]_q}{[\kappa+n]_q}a_nz^n+\sum_{n=1}^{\infty}\frac{[\kappa+1]_q}{[\kappa+n]_q}b_nz^n.$$

Using the inequality (2.5), we get

$$\sum_{n=1}^{\infty}\left(\frac{[n]_q-(1-\mu)\xi}{1-\xi}\left(\frac{[\kappa+1]_q}{[\kappa+n]_q}|a_n|\right)+\frac{[n]_q+(1-\mu)\xi}{1-\xi}\left(\frac{[\kappa+1]_q}{[\kappa+n]_q}|b_n|\right)\right)\Psi^{\sigma}_{\vartheta,\rho}(n,q)$$
$$\leqslant\sum_{n=1}^{\infty}\left(\frac{[n]_q-(1-\mu)\xi}{1-\xi}|a_n|+\frac{[n]_q+(1-\mu)\xi}{1-\xi}|b_n|\right)\Psi^{\sigma}_{\vartheta,\rho}(n,q)$$
$$\leqslant 2(1-\xi),\ \text{since }\mathfrak{h}(z)\in\overline{\mathcal{HS}}^{\sigma}_{\vartheta,\rho}(\mu,\xi).$$

Hence by Theorem 2, $\mathcal{L}_{q,\kappa}(\mathfrak{h}(z))\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$. □

For $0\leqslant\delta\leqslant\xi<1$, let $\mathfrak{h}\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ and $\mathfrak{H}\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$. Then $\mathfrak{h}(z)*\mathfrak{H}(z)\in\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)\subset\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\delta)$.

Proof. Let $\mathfrak{h} \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ and $\mathfrak{H}(z) \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\delta)$. Then $\mathfrak{h}(z) * \mathfrak{H}(z)$ is given by (1.5).

For $\mathfrak{h}(z) * \mathfrak{H}(z) \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\delta)$ we note that $|u_m| \leqslant 1$ and $|v_m| \leqslant 1$. Now by Theorem 2 we have

$$\sum_{n=2}^{\infty} \frac{[n]_q - (1-\mu)\delta}{1-\delta} \Psi^{\sigma}_{\vartheta,\rho}(n,q)|a_n|\ |u_n| + \sum_{n=1}^{\infty} \frac{[n]_q - (1-\mu)\delta}{1-\delta} \Psi^{\sigma}_{\vartheta,\rho}(n,q)|b_n|\ |v_n|$$
$$\leqslant \sum_{n=2}^{\infty} \frac{[n]_q - (1-\mu)\delta}{1-\delta} \Psi^{\sigma}_{\vartheta,\rho}(n,q)|a_n| + \sum_{n=1}^{\infty} \frac{[n]_q - (1-\mu)\delta}{1-\delta} \Psi^{\sigma}_{\vartheta,\rho}(n,q)|b_n|$$

and since $0 \leqslant \delta \leqslant \xi < 1$

$$\leqslant \sum_{n=2}^{\infty} \frac{[n]_q - (1-\mu)\xi}{1-\xi} \Psi^{\sigma}_{\vartheta,\rho}(n,q)|a_n| + \sum_{n=1}^{\infty} \frac{[n]_q - (1-\mu)\xi}{1-\xi} \Psi^{\sigma}_{\vartheta,\rho}(n,q)|b_n| \leqslant 1,$$

by Theorem 2 , we get desired result. □

5. Partial Sums results

Many researchers studied and indiscriminate the results on partial sums for various classes of analytic functions based on the results given by Silvia [37], Silverman [35] but analogues results on harmonic functions have not explored in the literature. Lately, in [31] Porwal fill this gap by inspecting exciting results on the partial sums of starlike harmonic univalent functions (see[32]) . In this section we investigate the partial sums results for $\mathfrak{h} \in \overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$.

Let $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ denote the subclass of $\mathcal{H}$ consisting of functions $\mathfrak{h} = \mathfrak{f} + \overline{\mathfrak{g}}$ of the form(1.2) which satisfy the inequality

$$\sum_{n=2}^{\infty} \mathbf{M}_n |a_n| + \sum_{n=1}^{\infty} \mathbf{Q}_n |b_n| \leqslant 1 \tag{5.1}$$

where

$$\mathbf{M}_n = \frac{[n]_q - (1-\mu)\xi}{1-\xi} \text{ and } \mathbf{Q}_n = \frac{[n]_q + (1-\mu)\xi}{1-\xi}$$

unless otherwise stated.

Now, we discuss the ratio of a function of the form (1.5) with $b_1 = 0$ are

$$\mathfrak{h}_\ell(z) = z + \sum_{n=2}^{\ell} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$$

,

$$\mathfrak{h}_k(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{k} b_n z^n}$$

,

$$\mathfrak{h}_{\ell,k}(z) = z + \sum_{n=2}^{m} a_n z^n + \overline{\sum_{n=2}^{k} b_n z^n}.$$

We first obtain the sharp bounds for $\Re\left\{\frac{\mathfrak{h}(z)}{\mathfrak{h}_\ell(z)}\right\}$. *If* $\mathfrak{h}$ *of the form (1.5) with* $b_1 = 0$ *satisfies the condition (5.1), then*

$$\Re\left\{\frac{\mathfrak{h}(z)}{\mathfrak{h}_\ell(z)}\right\} \geqslant \frac{\mathbf{M}_{\ell+1}-(1-\xi)}{\mathbf{M}_{\ell+1}}, \qquad (z \in \mathbb{U}) \tag{5.2}$$

where

$$\mathbf{M}_n \geqslant \begin{cases} 1-\xi, & \text{if} \quad n = 2,3,\dots,\ell \\ \mathbf{M}_{\ell+1}, & \text{if} \quad n = \ell+1,\ell+2,\dots. \end{cases}$$

$$\mathbf{Q}_n \geqslant 1-\xi, \qquad \text{if} \quad n = 2,3,\dots$$

The result (5.2) is sharp with the function given by

$$\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{M}_{\ell+1}} z^{\ell+1}. \tag{5.3}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\mathbf{M}_{\ell+1}}{1-\xi}\left[\frac{\mathfrak{h}(re^{i\theta})}{\mathfrak{h}_m(re^{i\theta})} - \frac{\mathbf{M}_{\ell+1}-(1-\xi)}{\mathbf{M}_{\ell+1}}\right]$$

$$= \frac{1+\sum\limits_{n=2}^{\ell} a_n r^{n-1} e^{i(n-1)\theta} + \sum\limits_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta} - \frac{\mathbf{M}_{\ell+1}}{1-\xi}\left(\sum\limits_{n=\ell+1}^{\infty} a_n r^{n-1} e^{i(n-1)\theta}\right)}{1+\sum\limits_{n=2}^{\ell} a_n r^{n-1} e^{i(n-1)\theta} + \sum\limits_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta}}. \tag{5.4}$$

It suffices to show that $|w(z)| \leqslant 1$. Now, from (5.4) we can write

$$w(z) = \frac{\frac{\mathbf{M}_{\ell+1}}{1-\xi}\left(\sum\limits_{n=\ell+1}^{\infty} a_n r^{n-1} e^{i(n-1)\theta}\right)}{2+2\left(\sum\limits_{n=2}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} + \sum\limits_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta}\right) + \frac{\mathbf{M}_{\ell+1}}{1-\xi}\left(\sum\limits_{n=\ell+1}^{\infty} a_n r^{n-1} e^{i(n-1)\theta}\right)}.$$

Hence we obtain

$$|w(z)| \leqslant \frac{\frac{\mathbf{M}_{\ell+1}}{1-\xi}\left(\sum\limits_{n=\ell+1}^{\infty} |a_n|\right)}{2-2\left[\sum\limits_{n=2}^{\ell} |a_n| + \sum\limits_{n=2}^{\infty} |b_n|\right] - \frac{\mathbf{M}_{\ell+1}}{1-\xi} \sum\limits_{n=\ell+1}^{\infty} |a_n|}.$$

Now $|w(z)| \leqslant 1$ if

$$\sum_{n=2}^{\infty} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\mathbf{M}_{\ell+1}}{1-\xi} \sum_{n=\ell+1}^{\infty} |a_n| \leqslant 1.$$

From the condition (5.1), it is sufficient to show that

$$\sum_{n=2}^{\infty} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\mathbf{M}_{\ell+1}}{1-\xi} \sum_{n=\ell+1}^{\infty} |a_n| \leqslant \sum_{n=2}^{\infty} \frac{\mathbf{M}_n}{1-\xi} |a_n| + \sum_{n=2}^{\infty} \frac{\mathbf{Q}_n}{1-\xi} |a_n|$$

which is equivalently to

$$\sum_{n=2}^{\ell}\left(\frac{\mathbf{M}_n-(1-\xi)}{1-\xi}\right)|a_n|+\sum_{n=2}^{\infty}\left(\frac{\mathbf{Q}_n-(1-\xi)}{1-\xi}\right)|b_n|+\sum_{n=\ell+1}^{\infty}\left(\frac{\mathbf{M}_n-\mathbf{M}_{n+1}}{1-\xi}\right)|a_n|\geqslant 0.$$

To see that the function given by (5.3) gives the sharp result, we observe that for $z = re^{i\pi/n}$

$$\begin{aligned}\frac{\mathfrak{h}(z)}{\mathfrak{h}_m(z)} &= 1+\frac{1-\xi}{\mathbf{M}_{\ell+1}}z^{\ell}\to 1-\frac{1-\xi}{\mathbf{M}_{\ell+1}}\\ &= \frac{\mathbf{M}_{\ell+1}-(1-\xi)}{\mathbf{M}_{\ell+1}} \quad \text{when } r\to 1^{-}.\end{aligned}$$

□

We next determine bounds for $\Re\left\{\mathfrak{h}_m(z)/\mathfrak{h}(z)\right\}$. *If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then*

$$\Re\left\{\frac{\mathfrak{h}_{\ell}(z)}{h(z)}\right\}\geqslant\frac{\mathbf{M}_{\ell+1}}{\mathbf{M}_{\ell+1}+1-\xi}, \qquad (z\in\mathbb{U}) \tag{5.5}$$

where

$$\mathbf{M}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n=2,3,\dots,\ell \\ \mathbf{M}_{\ell+1}, & \text{if } \ n=\ell+1,\ell+2,\dots. \end{cases}$$

$$B_n \geqslant 1-\xi, \qquad \text{if } \ n=2,3,\dots \tag{5.6}$$

The result (5.5) is sharp with the function given by

$$\mathfrak{h}(z) = z+\frac{1-\xi}{\mathbf{M}_{\ell+1}}z^{\ell+1}. \tag{5.7}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)}=\frac{\mathbf{M}_{\ell+1}+1-\xi}{1-\xi}\left[\frac{\mathfrak{h}_{\ell}(re^{i\theta})}{\mathfrak{h}(re^{i\theta})}-\frac{\mathbf{M}_{\ell+1}}{\mathbf{M}_{\ell+1}+1-\xi}\right]$$

$$=\frac{1+\sum\limits_{n=2}^{\ell}a_n r^{n-1}e^{i(n-1)\theta}+\sum\limits_{n=2}^{\infty}\overline{b_n}r^{n-1}e^{-i(n+1)\theta}-\frac{\mathbf{M}_{\ell+1}}{1-\xi}\left(\sum\limits_{n=\ell+1}^{\infty}a_n r^{n-1}e^{i(n-1)\theta}\right)}{1+\sum\limits_{n=2}^{\ell}a_n r^{n-1}e^{i(n-1)\theta}+\sum\limits_{n=2}^{\infty}\overline{b_n}r^{n-1}e^{-i(n+1)\theta}}.$$

Hence we obtain

$$|w(z)|\leqslant\frac{\frac{\mathbf{M}_{\ell+1}+1-\xi}{1-\xi}\left(\sum\limits_{n=\ell+1}^{\infty}|a_n|\right)}{2-2\left[\sum\limits_{n=2}^{\ell}|a_n|+\sum\limits_{n=2}^{\infty}|b_n|\right]-\frac{\mathbf{M}_{\ell+1}-(1-\xi)}{1-\xi}\sum\limits_{n=\ell+1}^{\infty}|a_n|}\leqslant 1.$$

The last inequality is equivalent to

$$\sum_{n=2}^{\ell} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\mathbf{M}_{\ell+1}}{1-\xi} \sum_{n=\ell+1}^{\infty} |a_n| \leqslant 1.$$

Making use of (5.1) and the condition (5.6) we obtain (5). Finally equality holds in (5.5) for the extremal function $\mathfrak{h}(z)$ given by (5.7). □

We next turns to ratios for the for $\Re\left\{\mathfrak{h}'(z)/\mathfrak{h}_\ell'(z)\right\}$ and $\Re\left\{\mathfrak{h}_\ell'(z)/\mathfrak{h}'(z)\right\}$. *If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then*

$$\Re\left\{\frac{\mathfrak{h}'(z)}{\mathfrak{h}_\ell'(z)}\right\} \geqslant \frac{\mathbf{M}_{\ell+1} - (\ell+1)(1-\xi)}{\mathbf{M}_{\ell+1}}, \qquad (z \in \mathbb{U}) \tag{5.8}$$

where

$$\mathbf{M}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n = 2, 3, \ldots, \ell \\ \mathbf{M}_{\ell+1}, & \text{if } \ n = \ell+1, \ell+2, \ldots. \end{cases}$$

$$\mathbf{Q}_n \geqslant 1-\xi, \qquad \text{if } \ n = 2, 3, \ldots$$

The result (5.8) is sharp with the function given by $\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{M}_{\ell+1}} z^{\ell+1}$.

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\mathbf{M}_{\ell+1}}{(\ell+1)1-\xi}\left[\frac{f'(z)}{f_\ell'(z)} - \frac{\mathbf{M}_{\ell+1} - (\ell+1)(1-\xi)}{\mathbf{M}_{\ell+1}}\right]$$

$$= \frac{1 + \sum_{n=2}^{\ell} n a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^{\infty} n\overline{b_n} r^{n-1} e^{-i(n+1)\theta} - \frac{\mathbf{M}_{\ell+1}}{(\ell+1)1-\xi}\left(\sum_{n=\ell+1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta}\right)}{1 + \sum_{n=2}^{\ell} n a_n r^{n-1} e^{i(n-1)\theta} - \sum_{n=2}^{\infty} n\overline{b_n} r^{n-1} e^{-i(n+1)\theta}}.$$

The result(5.8) follows by using the techniques as used in Theorem 5. □

Proceeding exactly as in the proof of Theorem 5, we can prove the following theorem . *If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then*

$$\Re\left\{\frac{h_\ell'(z)}{h'(z)}\right\} \geqslant \frac{\mathbf{M}_{\ell+1}}{\mathbf{M}_{\ell+1} + (\ell+1)(1-\xi)}, \qquad (z \in \mathbb{U}). \tag{5.9}$$

The result is sharp for $\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{M}_{\ell+1}} z^{\ell+1}$.

We next determine bounds for $\Re\{\mathfrak{h}(z)/\mathfrak{h}_k(z)\}$ and $\Re\{\mathfrak{h}_k(z)/\mathfrak{h}(z)\}$. *If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then*

$$\Re\left\{\frac{\mathfrak{h}(z)}{\mathfrak{h}_k(z)}\right\} \geqslant \frac{\mathbf{Q}_{k+1} - (1-\xi)}{\mathbf{Q}_{k+1}}, \qquad (z \in \mathbb{U}) \tag{5.10}$$

where

$$\mathbf{Q}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n=2,3,\dots,k \\ \mathbf{Q}_{k+1}, & \text{if } \ n=k+1,k+2,\dots. \end{cases}$$

$$\mathbf{M}_n \geqslant 1-\xi, \qquad \text{if } \ n=2,3,\dots$$

The result (5.10) is sharp with the function given by $\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{Q}_{k+1}}\overline{z}^{k+1}$. If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then

$$\Re\left\{\frac{\mathfrak{h}_k(z)}{\mathfrak{h}(z)}\right\} \geqslant \frac{\mathbf{Q}_{k+1}}{\mathbf{Q}_{k+1}+1-\xi}, \qquad (z \in \mathbb{U}) \tag{5.11}$$

where

$$\mathbf{Q}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n=2,3,\dots,k \\ \mathbf{Q}_{k+1}, & \text{if } \ n=k+1,k+2,\dots. \end{cases}$$

$$\mathbf{M}_n \geqslant 1-\xi, \qquad \text{if } \ n=2,3,\dots$$

The result (5.11) is sharp with the function given by $\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{Q}_{k+1}}\overline{z}^{k+1}$.

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\mathbf{Q}_{k+1}+1-\xi}{1-\xi}\left[\frac{f_k(re^{i\theta})}{f(re^{i\theta})} - \frac{\mathbf{Q}_{k+1}}{\mathbf{Q}_{k+1}+1-\xi}\right]$$

$$= \frac{1+\sum\limits_{n=2}^{\infty} a_n r^{n-1}e^{i(n-1)\theta} + \sum\limits_{n=2}^{k} \overline{b_n} r^{n-1}e^{-i(n+1)\theta} - \frac{\mathbf{Q}_{k+1}}{\mathbf{Q}_{k+1}+1-\xi}\sum\limits_{n=2}^{k+1} \overline{b_n} r^{n-1}e^{-i(n-1)\theta}}{1+\sum\limits_{n=2}^{\infty} a_n r^{n-1}e^{i(n-1)\theta} + \sum\limits_{n=2}^{k} \overline{b_n} r^{n-1}e^{-i(n+1)\theta}}.$$

We omit the details of proof ,because it runs parallel to that from Theorem 5. □

If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then

$$\Re\left\{\frac{\mathfrak{h}(z)}{\mathfrak{h}_{\ell,k}(z)}\right\} \geqslant \frac{\mathbf{M}_{\ell+1}-(1-\xi)}{\mathbf{M}_{\ell+1}}, \qquad (z \in \mathbb{U}) \tag{5.12}$$

where

$$\mathbf{M}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n=2,3,\dots,\ell,\ell+1 \\ \mathbf{M}_{\ell+1}, & \text{if } \ n=\ell+1,\ell+2,\dots. \end{cases}$$

$$\mathbf{Q}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n=2,3,\dots,\ell \\ \mathbf{M}_{\ell+1}, & \text{if } \ n=\ell+1,\ell+2,\dots. \end{cases}$$

The result (5.12) is sharp with the function given by $\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{M}_{\ell+1}}z^{\ell+1}$. If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then

$$\Re\left\{\frac{\mathfrak{h}(z)}{\mathfrak{h}_{\ell,k}(z)}\right\} \geqslant \frac{\mathbf{Q}_{k+1}-(1-\xi)}{\mathbf{Q}_{k+1}}, \qquad (z \in \mathbb{U}) \tag{5.13}$$

where

$$\mathbf{Q}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n=2,3,\dots,k \\ \mathbf{Q}_{k+1}, & \text{if } \ n=k+1,k+2,\dots. \end{cases}$$

$$\mathbf{M}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n = 2,3,\dots,k \\ \mathbf{Q}_{k+1}, & \text{if } \ n = k+1, k+2, \dots. \end{cases}$$

If $\mathfrak{h}$ *of the form (1.5) with* $b_1 = 0$ *satisfies the condition (5.1), then*

$$\Re\left\{\frac{\mathfrak{h}_{\ell,k}(z)}{\mathfrak{h}(z)}\right\} \geqslant \frac{\mathbf{M}_{\ell+1}}{\mathbf{M}_{\ell+1}+1-\xi}, \qquad (z \in \mathbb{U}). \tag{5.14}$$

If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then

$$\Re\left\{\frac{\mathfrak{h}_{\ell,k}(z)}{\mathfrak{h}(z)}\right\} \geqslant \frac{\mathbf{Q}_{k+1}}{\mathbf{Q}_{k+1}+1-\xi}, \qquad (z \in \mathbb{U}). \tag{5.15}$$

The result(5.15)is sharp with the function given by $\mathfrak{h}(z) = z + \frac{1-\xi}{B_{k+1}}\bar{z}^{k+1}$. *If* $\mathfrak{h}$ *of the form (1.5) with* $b_1 = 0$ *satisfies the condition (5.1), then*

$$\Re\left\{\frac{\mathfrak{h}'(z)}{\mathfrak{h}_\ell'(z)}\right\} \geqslant \frac{\mathbf{M}_{\ell+1}-(\ell+1)(1-\xi)}{\mathbf{M}_{\ell+1}}, \qquad (z \in \mathbb{U}) \tag{5.16}$$

where

$$\mathbf{M}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n = 2,3,\dots,\ell \\ \mathbf{M}_{\ell+1}, & \text{if } \ n = \ell+1, \ell+2, \dots. \end{cases}$$

$$\mathbf{Q}_n \geqslant \begin{cases} 1-\xi, & \text{if } \ n = 2,3,\dots,\ell \\ \mathbf{M}_{\ell+1}, & \text{if } \ n = \ell+1, \ell+2, \dots. \end{cases}$$

The result (5.16) is sharp with the function given by $\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{M}_{\ell+1}} z^{\ell+1}$. If $\mathfrak{h}$ of the form (1.5) with $b_1 = 0$ satisfies the condition (5.1), then

$$\Re\left\{\frac{\mathfrak{h}_{\ell,k}'(z)}{\mathfrak{h}'(z)}\right\} \geqslant \frac{\mathbf{M}_{\ell+1}}{\mathbf{M}_{\ell+1}+(\ell+1)(1-\xi)}, \qquad (z \in \mathbb{U}). \tag{5.17}$$

The result (5.17) is sharp with the function given by $\mathfrak{h}(z) = z + \frac{1-\xi}{\mathbf{M}_{\ell+1}} z^{\ell+1}$.

6. Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu,\xi)$ due to Dziok[12]and Silverman [34]. [24] If the functions ϕ and φ are analytic in $\mathbb{D}$ with $\varphi \prec \phi$, then for $\eta > 0$, and $0 < r < 1$,

$$\int_0^{2\pi} \left|\varphi(re^{i\theta})\right|^\eta d\theta \leqslant \int_0^{2\pi} \left|\phi(re^{i\theta})\right|^\eta d\theta. \tag{6.1}$$

Due to recent work of Dziok [12], we suppose $\mathfrak{h} \in \overline{\mathcal{HS}}^{\sigma}_{\vartheta,\rho}(\mu,\xi)$ $\eta > 0$, $0 \leqslant \mu < 1$, $0 \leqslant \xi \leqslant 1$, and $\mathfrak{f}_2(z)$ is defined by

$$\begin{aligned} \mathfrak{f}_2(z) &= z - \frac{1-\xi}{([2]_q-(1-\mu)\xi)\Psi^\sigma_{\vartheta,\rho}(2,q)} z^2 \quad (n \geqslant 2), \\ \mathfrak{g}_2(z) &= z + \frac{1-\xi}{([2]_q+(1-\mu)\xi)\Psi^\sigma_{\vartheta,\rho}(2,q)} \bar{z}^2; \quad (n \geqslant 2). \end{aligned}$$

where $\Psi^{\sigma}_{\vartheta,\rho}(2, q)$ is given by (3.1). Since

$$\frac{\mathfrak{f}^*_n(z)}{z} \prec \frac{\mathfrak{f}^*_2(z)}{z} \quad \text{and} \quad \frac{\bar{\mathfrak{g}}(z)}{z} \prec \frac{\mathfrak{f}^*_2(z)}{z}$$

by Lemma 6 we have

$$\int_0^{2\pi} \left|\frac{\mathfrak{f}^*_n(z)}{z}\right|^{\eta} d\theta \leqslant \int_0^{2\pi} \left|\frac{\mathfrak{f}^*_2(z)}{z}\right|^{\eta} d\theta, \quad (z = re^{i\theta})$$

$$\int_0^{2\pi} \left|\frac{\mathfrak{g}^*_n(z)}{z}\right|^{\eta} d\theta = \int_0^{2\pi} \left|\frac{\bar{\mathfrak{g}}_n(z)}{z}\right|^{\eta} d\theta \leqslant, \int_0^{2\pi} \left|\frac{\mathfrak{f}^*_2(z)}{z}\right|^{\eta} d\theta, \quad (z = re^{i\theta}).$$

Thus we have the following result: Let $0 < r < 1, \eta > 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathfrak{f}^*_n(re^{i\theta})\right|^{\eta} d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\mathfrak{f}^*_2(re^{i\theta})\right|^{\eta} d\theta, \quad (n = 1, 2, 3, \cdot)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathfrak{g}^*_n(re^{i\theta})\right|^{\eta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\mathfrak{f}^*_2(re^{i\theta})\right|^{\eta} d\theta, \quad (n = 2, 3, 4, \cdot).$$

where $\mathfrak{f}^*_n$ and $\mathfrak{g}^*_n$ are defined by (3.2) and (3.3). By Lemma 6 and Theorem 3 we have the following : Let $0 < r < 1, \eta > 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathfrak{h}(re^{i\theta})\right|^{\eta} d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\mathfrak{f}^*_2(re^{i\theta})\right|^{\eta} d\theta, \quad (n = 1, 2, 3, \cdot)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{h}(re^{i\theta}))\right|^{\eta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{D}_q(\Lambda^{\sigma}_{\vartheta,\rho}\mathfrak{f}^*_2(re^{i\theta}))\right|^{\eta} d\theta, \quad (n = 2, 3, 4, \cdot).$$

where $\mathfrak{f}^*_n$ and $\mathfrak{g}^*_n$ are defined by (3.2) and (3.3).

Concluding Remarks: For appropriate selections of μ, as we piercing out the $\overline{\mathcal{HS}}^{q,\sigma}_{\vartheta,\rho}(\mu, \xi)$. When $\mu = 0$ and $q \to 1^-$, as the many results existing in this paper would afford motivating extensions and simplifications of those deliberated earlier simpler harmonic function classes(see [19, 20, 27]) linked with Mittag-Leffler functions. Correspondingly by fixing $\mu = 1$ one can provide interesting results for Noshiro-type harmonic functions studied in [7]. The facts intricate in the origins of such specialism of the significance obtainable in this paper are fairly straight- forward, hence omitted. By making use of the generalized struve functions(see[4] and references cited therein) one can study certain inclusion results for $\mathfrak{h} \in \overline{\mathcal{HS}}$ also results on subordination involving polynomials induced by lower triangular matrices(see[26]).

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