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On Equivalence of the ap-Sequential Henstock and ap-Sequential Topological Henstock Integrals

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Abstract

Let $\mathbb{X}$ be a topological space and $\mathbb{K} \subset \mathbb{X}$. Suppose $f : \mathbb{K} \to \mathbb{X}$ is a function defined in $\mathbb{K}$ and τ is a vector in $\mathbb{R}$ taking values in $\mathbb{X}$. Suppose f is ap-Sequential Henstock integrable with respect to τ, is f ap-Sequential Topological Henstock integrable with respect to τ? It is the purpose of this paper to proffer affirmative answer to this question and give an applicable example.

Keywords: ap-Sequential Henstock Integrals, ap-Sequential Topological Henstock Integral, Topological Space, guages, equi-integrability, Absolutely Continuous (AC).
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1. Introduction

Henstock integral was introduced in the early twentieth century by Henstock R. and Kursweil J. in 1955 and 1957 respectively (see [1] and [2]). The techniques in this integration concept have some difficulties as that of the Lebesgue integral. Some have sought to redefine the rigorous nature of these techniques with a view of avoiding its difficulties by introducing new and reliable integrals. One of the most popular integrals is the Henstock integral. Its definition is obtained by a slight modification of the Riemann's definition. It is popularly known as the Generalised Riemann integral, which is considerably simpler than the Lebesgue's definition see e.g.,[1]-[15]; and has been shown by Paxton [10] to be equivalent to the Sequential Henstock integral. In the recent past, Sergio [12] examined the theory, relationship and equivalence of ϕ-integral and Henstock integrals for Topological vector space-valued functions. The authors [7, 8] studied the equivalence of Henstock and Certain Sequential Henstock Integrals, established the equivalence of Sequential Topological Henstock to Sequential Henstock integral and proved the equivalence of p-Henstock-Type integrals. In of this paper, we present the equivalence of ap-Sequential Henstock integrals to ap-Sequential Topological Henstock integral and give an applicable example.

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2. Preliminaries

Throughout this paper, we use $\mathbb{X}$ as a topological space, which is a subset of the real line $\mathbb{R}$, $\mathbb{K}$ as subspace of $\mathbb{X}$, $\mathbb{N}$ as set of natural numbers, $\{\delta_n(x)\}_{n=1}^{\infty}$, as set of gauge functions, P_n, as set of partitions of subintervals of a compact interval $[a, b]$, and $\ll$ as much more smaller, where a gauge on $[a, b]$ is a positive real-valued function $\delta : [a, b] \to \mathbb{R}^+$ and this gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$. A tagged partition P of [a,b] is a finite collection of ordered pairs $P = \{(u_{i-1} u_i), t_i : i = 1, ..., m\}$, $a = u_0 < u_1 <, ..., < u_m = b, t_i \in [u_{i-1}, u_i]$. δ_n as set of gauge functions in $\mathbb{X}$, P_n as set of partitions of the subspace $\mathbb{K}$(Hausdorff) in $\mathbb{X}$.

We firstly recall the following definitions(see [8, 9]).

Let E be a measurable set and let $c \in \mathbb{R}$. The density of E at c is defined by

$$d_c E = \lim_{h \to 0+} \frac{\mu(E \cap (c - h, c + h))}{2h},$$

provided the limit exists. The point c is called a point of density of E if $d_c E = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

A function $F : [a, b] \to \mathbb{R}$ is said to be approximately differentiable at $c \in [a, b]$ if there exists a measurable set $E \subseteq [a, b]$ such that $c \in E^d$ and

$$\lim_{t \to c, t \in E} \frac{F(t) - F(c)}{t - c},$$

exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

The concept of sequence of approximate neighborhoods(or ap-nbds) of $t_{i_n} \in [a, b]$ is a measurable set $S_{t_{i_n}} \subseteq [a, b]$ containing t_{i_n} as a sequence of points of density. For every $t_{i_n} \in E \subseteq [a, b]$, choose an ap-nbd $S_{t_{i_n}} \subseteq [a, b]$ of t_{i_n}. Then we say that $S = \{S_{t_{i_n}} : t_{i_n} \in E\}$ is a choice on E. A tagged interval $(t_{i_n}, [c_{i_n}, d_{i_n}])$ is said to be subordinate to the choice $S = S_{t_{i_n}}$ if $c_{i_n}, d_{i_n} \in S_{t_{i_n}}$. Let $P_n = \{(t_{i_n}, [c_{i_n}, d_{i_n}]) : 1 \leqslant i \leqslant m, m \in \mathbb{N}\}$ be a finite collection of non-overlapping tagged intervals. If $(t_{i_n}, [c_{i_n}, d_{i_n}])$ is subordinate to a choice S for each i_n for $i = 1, ..., m$, then we say that P_n is subordinate to S. If P_n is subordinate to S and $[a, b] = \bigcup_{i=1}^{n} [c_{i_n}, d_{i_n}]$, then we say that P_n is a tagged partition of $[a, b]$ that is subordinate to S.

We introduce the concept of approximate Lusin function. This function is used to define the ap-Henstock and ap-Topological Henstock integrals.

Definition 2.1 [8] A function $F : [a, b] \to \mathbb{R}$ is an approximate Lusin function(or F is an AL function) on $[a, b]$ if for every measurable set $E \subseteq [a, b]$ of measure zero and for any $\varepsilon > 0$ there exists a choice S on E such that

$$|P \sum_{i=1}^{n} F(I)| < \varepsilon,$$

for every finite collection P of non-overlapping tagged intervals that is subordinate to S.

Definition 2.2 [8] A function $F : [a, b] \to \mathbb{R}$ is AC_s on a measurable set $E \subseteq [a, b]$ if for any $\varepsilon > 0$ there exists a positive number $\eta \in \mathbb{R}$ and a choice S on E such that

$$|P \sum_{i=1}^{n} F(I)| < \varepsilon,$$

for every finite collection P of non-overlapping tagged intervals that is subordinate to S and satisfies $(P) \sum_{i=1}^{n} |I| < \eta$, where $|I|$ is the lebesgues measure of the interval I. The function F is ACG_S on E if E can be expressed as a countable union of measurable sets on each of which F is AC_S

Definition 2.3 [9] A function $f : [a, b] \to \mathbb{R}$ is Henstock integrable($H_f[a, b]$) to $\alpha \in \mathbb{R}$ on $[a, b]$ if for any $\varepsilon > 0$ there exists a gauge $\delta(t) > 0$ such that for any $\delta - fine$ tagged partitions $P = \{(u_{i-1}, u_i), t_i\}$ on [a,b] where $[u_{i-1}, u_i] \in [a, b]$ and $u_{i-1} \leqslant t_i \leqslant u_i$ we have

$$|S(f, P) - \alpha| < \varepsilon.$$

We write $(H) \int_a^b f(t)d(t) = \alpha$ and $f \in H_f[a, b]$, where $S(f, P) = \sum_{i=1}^{n} f(t_i)[(u_i) - (u_{i-1})]$.

Definition 2.4 [12] (ap-Henstock integral) A function $f : [a, b] \to \mathbb{R}$ is ap-Henstock integrable(ap-$H_f[a, b]$)to a vector $\alpha \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a choice S on $[a, b]$ such that for any S-fine tagged partitions $P = \{(u_{(i-1)}, u_i), t_i\}$ we have $|\sum_{i=1}^{n} f(t_i)[u_i - u_{(i-1)}] - \alpha| < \varepsilon$. i.e ap-$H_f[a, b] = \alpha = \int_{[a,b]} f$.

The following definitions are well known to the case of functions defined in a Topological space(see [2])
Let $\mathbb{X}$ be a locally compact Hausdorff space with subspace $\mathbb{K} \subset \mathbb{X}$. We denote the closure of $\mathbb{K}$ as $\bar{\mathbb{K}}$ and the interior as $Int\mathbb{K}$. Let $\{O_i : i \in I\}$ be a family of subsets of $\mathbb{X}$ such that
i. If $\mathbb{K} \in O$, then $\bar{\mathbb{K}}$ is compact.
ii. For each $t \in \mathbb{X}$, the collection $A(t) = \{\mathbb{K} \in A | t \in \text{Int } \mathbb{K}\}$ is a neighbourhood base at t.
iii. If $A, B \in O$, then $A \cap B \in O$. and there exist disjoint sets $C_1, ..., C_n \in O$ such that $A - B = \bigcup_{i=1}^{n} C_i$.

A gauge (topological) on $\mathbb{K}$ is a map U assigning to each $t \in \bar{\mathbb{K}}$ a neighbourhood $U(t)$ of t contained in $\mathbb{X}$.

A division (topological) of $\mathbb{K}$ is a disjoint collection $\{A_1, ..., A_n\} \subset \mathbb{K}$ such that $\bigcup_{i=1}^{n} A_i = A$.

A partition (topological) of $\mathbb{K}$ is a set $P = \{(A_1, t_1), ..., (A_k, t_n)\}$ such that $\{A_1, ..., A_n\}$ is a division of $\mathbb{K}$ and $\{t_1, ..., t_n\} \subset \bar{\mathbb{K}}$. If U is a gauge on $\mathbb{K}$, we say the partition P is U-fine if $A_i \subset U(t_i)$, for $i = 1, 2, ..., n$.

A volume is a non-negative function such that $V(A) = \sum_{i=1}^{n} \nu(A_i)$.

Note: Volume here can intuitively be defined to represent the "length" of the "interval" $\in I$.

Definition 2.5 [9](Topological Henstock integral) Let $\mathbb{X}$ be a locally compact Hausdorff space and let $\mathbb{K} \subset \mathbb{X}$ and $\bar{\mathbb{K}}$(a closure of $\mathbb{K}$) with $f : \bar{\mathbb{K}} \to \mathbb{R}$, then f is Topological Henstock integrable ($TH_f[a,b]$) on $\bar{\mathbb{K}}$ if for any $\varepsilon > 0$ there exists a neighbourhood $U(t)$ such that

$$|\sum_{i=1}^{n} f(t_i)v(\tau_i) - \int_{\mathbb{K}} f| = |\sigma(f,P) - \int_{\mathbb{K}} f| < \varepsilon,$$

For every $U(t) - fine$ partition P of $\mathbb{K}$, where $\tau \subset \mathbb{K}$
The concept of compactment of a Hausdorff space, gauge function, partition, neighbourhood and volume of a topological space are defined in [8].

The following definitions are introduced [9] to show that the points in the subinterval $[u_{i-1}, u_i] \in [a,b]$ could be redefined in such a way that the Henstock integral is defined sequentially.

Definition 2.6 [9] A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable($SH_f[a,b]$) to $\alpha \in \mathbb{R}$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a sequence of gauge functions $\delta_\mu \in \{\delta_n(t)\}_{n=1}^{\infty}$ such that for every $\delta_n(t) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ we have

$$S(f,P_n) = \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) \to \alpha, n \to \infty,$$

where $[u_{(i-1)_n}, u_{i_n}] \in [a,b]$ and $u_{(i-1)_n} \leqslant t_{i_n} \leqslant u_{i_n}$ i.e $\alpha = \int_{[a,b]} f$.

This integral was established by Laramie[9]. It is well known that he proved that Sequential Henstock integral is equivalent to the Henstock integral. Moreso, it has the potential of expanding its overall theory into more abstract mathematical settings which may lead to further applications.

Definition 2.7 [9](Sequential Topological Henstock integral) Let $\mathbb{X}$ be a locally compact Hausdorff space and $\mathbb{K} \subset \mathbb{X}$. Let $\tau \in \mathbb{K}$ be a topology of the family of $\mathbb{K}$, a subset of $\mathbb{X}$ with $f : \mathbb{K} \to \mathbb{X}$. Then, f is Sequential Topological Henstock integrable ($STH_f[a,b]$) on $\bar{\mathbb{K}}$ to α if for each $\varepsilon > 0$, there exists a sequence $U_n(t)$ such that

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})v(\tau_{i_n}) - \int_{\mathbb{K}} f| = |S(f_\tau, P_n) - \int_{\mathbb{K}} f| < \varepsilon,$$

For every $U_n(t) - fine$ partition P_n of $\mathbb{K}$.
We say f is Sequential Topological Henstock integrable to α and denote it by $\alpha = H_{f_\tau}([\mathbb{K}, \mathbb{R}]) = (SH)\int_{\mathbb{K}} f$ and $S(f_\tau, P_n) = \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})v(\tau_{i_n})$ and $\mathbb{X} \subset \mathbb{R}$

Next, we prove the equivalence theorems of $H_{f_\tau}([\mathbb{K}, \mathbb{R}])$, the family of Henstock integrals with particular interest in the sequential approach by first stating the following useful lemmas which were proved by Sergio[3] and Park, Jung, Kim and Lee[8] respectively.

Lemma 2.8 If U and V are $\theta-$ neighbourhoods with $V \subseteq U$, then $\tau_u(t) \leqslant \tau_v(t)$ for all $t \in \mathbb{X}$

Lemma 2.9 If $F : [a, b] \to \mathbb{R}$ is ACG_S on $[a, b]$, then F is an AL function on $[a, b]$

The equivalence results in Laramie[9], Ying [10], Chartfield [3] and Iluebe and Mogbademu[7] and [8] were the motivations for this work.

Remark 2.10 : Clearly, from those results, $H_f[a, b] = TH_f[s, b] = DH_f[a, b] = A_P H_f[a, b] = SH_f[a, b]$.
The following new concepts are analogous of definitions 2.5, 2.6, 2.7 and 2.9.

Definition 2.11(ap-Sequential Henstock integral) A function $f : [a, b] \to \mathbb{R}$ is ap-Sequential Henstock ($ap\text{-}SH_f[a, b]$) integrable to a vector $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a sequence of choice functions $\{S_n(t)\}_{n=1}^{\infty}$ such that for each $S_n(t) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ we have

$$|\sum_{i=1}^{n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon.$$

We write $ap\text{-}SH_f[a, b]$ and $\int_{[a,b]} f dt = \alpha$.
Hence, We introduce the following definition which is analogous to definitions 2.7 and 2.11.

Definition 2.12(ap-Sequential Topological Henstock integral) Let $\mathbb{X}$ be a locally compact Hausdorff space and $\mathbb{K} \subset \mathbb{X}$. Let $\tau \in \mathbb{K}$ be a topology of the family of $\mathbb{K}$, a subset of $\mathbb{X}$ with $f : \bar{\mathbb{K}} \to \mathbb{R}$, then f is ap-Sequential Topological Henstock integrable ($ap\text{-}STH_f[a, b]$) to $\alpha \in \mathbb{R}$ if for each $\varepsilon > 0$, there exists an AL function F of sequence of neighbourhoods $U_\mu(t) \in \{U_n(t)\}_{n=1}^{\infty}$ on $\mathbb{K}$ such that F is approximately differentiable a.e(almost everywhere) on Ω and $F'_{ap} = f$ a.e on $\mathbb{K}$ and

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n}) v(\tau_{i_n}) - \int_{\mathbb{K}} f| = |\sigma(f, P_n) - \int_{\mathbb{K}} f| < \varepsilon.$$

For every $U_n(t) - fine$ partition P_n of $\mathbb{X}$, where $\alpha = H_{f_\tau}([\mathbb{K}, \mathbb{R}]) = (SH)\int_{\mathbb{K}} f$ and $S(f_\tau, P_n) = \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n}) v(\tau_{i_n})$ and $\mathbb{X} \subset \mathbb{R}, \tau \in \mathbb{K}$. We say f is ap-Sequential Topological Henstock integrable with respect to τ on a measurable set $E \subseteq \mathbb{K}$ if $f\chi_E$ is ap-Sequential Topological Henstock integrable with respect to τ_1 on a $\mathbb{K}$.

Next, we prove theorems to establish the equivalence of the class of ap-Sequential Henstock integral to a new class of ap-Sequential Topological Henstock integrals; which is of significant importance to the family of Sequential Henstock integral introduced in Laramie[9] in our main results.

3. Main Results

Theorem 3.1 If $f : [a, b] \to \mathbb{R}$ is ap-Sequential Henstock integrable in $[a, b]$, then it is Sequential Henstock integrable there. Infact,

$$(ap\text{-}SH) \int_a^b f = (SH) \int_a^b f.$$

Proof. Let $f \in ap\text{-}SH_f[a, b]$, we want to show that $f \in SH_f[a, b]$.
Suppose that $\{\delta_n(t)\}_{n=1}^{\infty}$ is a decreasing sequence of gauge functions such that $\delta_{n+1} < \delta_n$ for all $t \in [a, b]$. Let $\varepsilon > 0$, there exists a $\delta_\mu \in \{\delta_n(t)\}_{n=1}^{\infty}$ for $\mu \leqslant n \in \mathbb{N}$ and $\delta_n(t) - fine$ partitions $P_n = \{[u_{(i-1)_n}, u_{i_n}, t_{i_n}]\}$, then
$|S(f, P_n) - \alpha| < \varepsilon$, where $\alpha = \int_a^b f$ (2)
For $n = 1, 2, ...,$. Let ε_n be a rational ε such that $0 < \varepsilon < 1$. By definition 2.6, there exists $\delta_n(t)$ for each ε_n satisfying (2). Since Q is a rational number which is countable, then $\{\delta_n(t)\}_{n=1}^{\infty}$ is a sequence. Given $\varepsilon > 0$, there exists a $\delta_{\mu_\nu} \in \{\delta_{n_\nu}(t)\}_{n=1}^{\infty}$ such that for $\mu \leqslant n \in \mathbb{N}$ and $\delta_n - fine$ partitions $P_n = [(u_{(i-1)_n}, u_{i_n}), t_{i_n}]$, then

$$|S(f, P_n) - \alpha| < \varepsilon.$$

Thus,

$$ap\text{-}SH_f[a, b] = SH_f[a, b].$$

Theorem 3.2 If $f : [a, b] \to \mathbb{R}$ is Sequential Henstock integrable in $[a, b]$, then it is ap-Sequential Henstock integrable there. Infact,

$$(SH) \int_a^b f = (ap\text{-}SH) \int_a^b f.$$

Proof. Suppose $f \in SH_f[a, b]$, we want to show that $f \in ap\text{-}SH_f[a, b]$.
Suppose there exists a $\delta_\mu \in \{\delta_n(t)\}_{n=1}^{\infty}$ for $\mu \leqslant n \in \mathbb{N}$ such that $P_n = \{[(u_{(i-1)_n}, u_{i_n}), t_{i_n}]\}$ where $u_{(i-1)_n} \leqslant t_{i_n} \leqslant u_{i_n}$ is any for δ_n-fine partition of $[a, b]$, then

$$|S(f, P_n) - \alpha| < \frac{1}{N^*}.$$

Let $\lambda \in \mathbb{R} > 0$, we choose a $\{\delta_n(t)\}_{n=1}^{\infty}$ such that for a given $\delta_n(t) > 0$, we have

$$|\delta_{\mu_\nu}(t) - \delta_\mu(t)| < \lambda,$$

for all $t \in [a, b]$. where $\delta_{\mu_\nu}(t)$ is a gauge function in $[a, b]$.
Now, let $\lambda \to 0$, then, our choice of $\delta_\mu(t)$ guarantees that $P_\mu \ll \delta_\mu(t)$. Hence, for a gauge then $P_\mu \ll \delta_{\mu_\nu}(t)$, We can make the Riemann sums for P_{μ_ν} and P_μ arbitrarily close (using the similar tags on each partition) such that

$$|S(f, P_\mu) - S(f, P_n)| < \frac{\varepsilon}{2}.$$

Let $\varepsilon > 0$, there exists $\{\delta_n(t)\}_{n=1}^{\infty}$, such that $P_n \ll \delta_n(t)$. Moving down the sequence $\{\delta_n(t)\}_{n=1}^{\infty}$ and denoting its' new position as N^*, so that $\frac{1}{N^*} < \frac{\varepsilon}{2}$, then

$$\begin{aligned} |S(f, P_n) - \alpha| &= |S(f, P_n) - S(f, P_\mu) + S(f, P_\mu) - \alpha| \\ &\leqslant |S(f, P_n) - S(f, P_\mu)| + |S(f, P_\mu) - \alpha| \\ &< \frac{\varepsilon}{2}. + \frac{1}{N^*} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus,

$$SH_f[a,b] = ap\text{-}SH_f[a,b].$$

Corollary 3.3 If $f : [a,b] \to \mathbb{R}$ is ap-Sequential Henstock integrable in $[a,b]$, if and only if, it is Sequential Henstock integrable there and

$$ap\text{-}SH_f[a,b] = SH_f[a,b].$$

Proof. It follows easily from Theorems 3.1 and 3.2. This completes the proof.

Theorem 3.4 Let $\mathbb{X}$ be a locally compact Hausdorff space and $\mathbb{K} \subset \mathbb{X}$. Let $f : \bar{\mathbb{K}} \to \mathbb{R}$ be ap-Sequential Topological Henstock integrable on $\mathbb{K}$ and let $F(x) = \int_{\mathbb{K}} f$ for each $x \in \mathbb{K} \subset \mathbb{X}$ then,
(a) the function F approximately differentiable a.e on $\mathbb{K}$ and $F'_{ap} = f$ a.e on $\mathbb{K}$ and
(b) the function F and f are measurable.

Proof. (a) follows from the definition of the ap-Sequential Topological integral. Since F is approximately continuous a.e on $\mathbb{K}$, F is measurable by ([4], Theorem 14.7). It follows from ([4], Theorem 14.7). that f is measurable. This completes the proof.

Theorem 3.5 Let $\mathbb{X}$ be a locally compact Hausdorff space. The ap-Sequential Topological Henstock integral is equivalent to Sequential Henstock integral on $I = [a,b] \subset \mathbb{R}$. In fact, $(ap\text{-}STH)\int_a^b f = (ap\text{-}SH)\int_{\mathbb{K}} f$.

Since $\mathbb{X}$ is a locally compact Hausdorff space, then by Heine - Borel's theorem, each $[u_{i-1}, u_i] \subset [u_{(i-1)_n}, u_{i_n}] \subset [a,b]$ is compact. Hence, any choice $x \in \mathbb{R}$ contained in the open interval (a,b) in which $F'_{ap} = f$ a.e on $\mathbb{K}$ is defined on the compact subspace $[u_{(i-1)_n}, u_{i_n}] \subset \mathbb{X}$, so that $\mathbb{R}$ is made into a locally compact Hausdorff space-valued measurable functions F and f which are approximately differentiable a.e on $[a,b]$. Therefore our choice $\{[u_{(i-1)_n}, u_{i_n}] \subset [a,b] : a,b \in \mathbb{R}, a < b\}$ is proved. This shows that under this condition, the ap-Sequential Topological Henstock integral reduces to Sequential Henstock integral. Thus,

$$(ap\text{-}STH)\int_{\mathbb{K}} f = (SH)\int_a^b f.$$

Proof. For $\mathbb{K} \in \mathbb{X}$, let $\phi(\mathbb{K}_{i_n}) = u_{i_n} - u_{(i-1)_n}$ and define a choice sequence of positive gauges $U_{S_n}(t_{i_n})$ on $\mathbb{K}$ such that $U_{S_n}(t_{i_n}) = \min(t_{i_n} - S_n(t_{i_n}), t_{i_n} + S_n(t_{i_n})), (i = 1,2,...k)$, is a U_n- fine partitions P_n on $\mathbb{K}$ with $S_\mu \in \{S_n(x)\}_{n=1}^{\infty}$ for $\mu \leqslant n$ and with measurable functions F and f which are approximately differentiable a.e on $\mathbb{K}$ for all $x \in [a,b]$. Thus, in this case, we can say $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ where $u_{(i-1)_n} \leqslant t_{i_n} \leqslant u_{i_n}$ is $S_n(x) - fine$. Since, each subinterval $[u_{(i-1)_n}, u_{i_n}] \in (t_{i_n} - S_n(t_{i_n}), t_{i_n} + S_n(t_{i_n})), i = 1,2,...k$. Therefore, for $[a,b] \subset \mathbb{R}$ and by Definition 2.2, for any $\varepsilon > 0$, there exists a choice $S_\mu \in \{S_n(x)\}_{n=1}^{\infty}$ for $\mu \leqslant n$ for each $x \in [a,b]$ such that for all $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$, we have

$$\left|\sum_{i=1}^{k} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \int_a^b f\right| = \left|S(f,P_n) - \int_a^b f\right| < \varepsilon.$$

Hence, the Sequential Henstock integral reduces to the ap-Sequential Topological Henstock integral on $\mathbb{X} \subset \mathbb{R}$. Thus,

$$(SH) \int_{\mathbb{K}} f = (ap\text{-}STH) \int_a^b f,$$

when the measurable sets in $\mathbb{X}$ are $\mathbb{K} = [a, b] \subset \mathbb{R}$. This completes the proof. **Corollary 3.6** If $f : [a, b] \to \mathbb{R}$ is ap-Sequential Topological Henstock integrable in $[a, b]$, if and only if, it is Sequential Henstock integrable there and

$$ap\text{-}STH_f[a, b] = SH_f[a, b].$$

Proof. It follows easily from Theorems 3.5. This completes the proof.

Remark 3.7 Clearly, from Corollaries 3.3 and 3.6, we have

$$ap\text{-}SH_f[a, b] = SH_f[a, b] = ap\text{-}STH_f[a, b].$$

Application: The results in this paper is applicable in example 2.7 in Iluebe and Mogbademu[7]
Hence, f is Sequential Henstock integral on $[0, 1]$ and that $\int_0^1 f(x)dx = 0$. So by Theorems 4.4 and 4.8, it follows that $f(x)$ is also ap-Sequential Henstock integrable and ap-Sequential Topological Henstock integrable on $[0, 1]$.

4. Conclusion and Suggestion for Further Study

In this paper, we studied the equivalence of ap-Sequential Henstock integral to ap-Sequential Topological Henstock integral by incorporating the concept of ap-Sequential Henstock to generate a new concept for ap-Sequential Topological Henstock integral and proving theorems on their equivalence to the Sequential Henstock integral. The results obtained show that there is equivalence between these family of Henstock integrals. To this end, an example to show the applicability of the result is also given.
However, the results of this research can now be extended to studies in more abstract spaces. In conclusion, can equivalence of ap-Sequential Henstock and ap-Sequential Topological Henstock Integrals hold for classes of functions, such as complex valued functions, step functions, measurable functions, absolutely integrable functions? It is of the view of the authors that these problems could be considered for further studies.

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