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Some Integral Inequalities Involving Exponential Type Convex Functions and Applications

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Abstract

In this present case, we focus and explore the idea of a new family of convex function namely exponential type m–convex functions. To support this newly introduced idea, we elaborate some of its nice algebraic properties. Employing this, we investigate the novel version of Hermite–Hadamard type integral inequality. Furthermore, to enhance the paper, we present several new refinements of Hermite–Hadamard $(H-H)$ inequality. Further, in the manner of this newly introduced idea, we investigate some applications of special means. These new results yield us some generalizations of the prior results in the literature. We believe, the methodology investigated in this paper will further inspire intrigued researchers.

Keywords: Convex function, Hölder's inequality, Power-mean integral inequality, m–type Convexity, Exponential convex function.
2010 MSC: 26A51, 26A33, 26D07, 26D10, 26D15.

1. Introduction

The theory of convexity is very important in the theoretical aspects of mathematicians and economists and also for physicists. Mathematicians use this theory, to provide the solution of problems that arise in different branches of sciences. This theory touches almost all branches of mathematics. Convex functions play an important role in many

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areas of mathematics, as well as in other areas of science, economy, engineering, medicine, industry, and business. It is especially important in the study of optimization problems, where it is distinguished by a number of convenient properties (for example, any minimum of a convex function is a global minimum, or the maximum is attained at a boundary point). This explains why there is a very rich theory of convex functions and convex sets. Optimization of convex functions has many practical applications (circuit design, controller design, modeling, etc.). Due to a lot of importance, the term “convexity" has become a rich source of inspiration and absorbing field for researchers. Interested readers are referred to [1, 2, 3, 4, 5, 6, 7, 8, 9].

During the last few decades, the concept of convex analysis has played crucial and consequential role in the generalizations and extensions of theory of inequalities. Both the theory of convexity and the theory of inequality are closely related to each other. The integral inequalities have elegant and effective importance in information technology, integral operator theory, numerical integration, optimization theory, statistics, probability, and stochastic process. During the last few decades, many mathematicians and research scholars concentrated their great contributions and attentions on the study of this inequality. Thus a rich and meaningful literature on inequalities can be found for the convexity, see the references [10, 11, 12, 13, 14, 15, 16, 17, 18].

2. Preliminaries

In this section we recall some known concepts.

Definition 2.1. [1] Let $Q : \mathbb{X} \to \mathbb{R}$ be a real valued function. A function Q is said to be convex, if

$$Q\left(d_1\theta + (1-\theta)\,d_2\right) \leqslant \theta Q\left(d_1\right) + (1-\theta)\,Q\left(d_2\right), \tag{2.1}$$

holds for all $d_1, d_2 \in \mathbb{X}$ and $\theta \in [0, 1]$.

Any paper on Hermite inequalities seems to be incomplete without mentioning the well-known Hermite–Hadamard inequality. This inequality states that, if $Q : \mathbb{X} \subset \mathbb{R} \to \mathbb{R}$ is convex in $\mathbb{X}$ for $d_1, d_2 \in \mathbb{X}$ and $d_1 < d_2$, then

$$Q\left(\frac{d_1 + d_2}{2}\right) \leqslant \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} Q(\chi)\,d\chi \leqslant \frac{Q(d_1) + Q(d_2)}{2}. \tag{2.2}$$

Interested readers can refer to [19, 20, 21, 22].
The family of m–convex functions was first time explored and introduced by G. Toader in [23].

Definition 2.2. [23] A function $Q : [0, d_2] \to \mathbb{R}$, $d_2 > 0$, is said to be m–convex, where $m \in (0, 1]$, if

$$Q\left(\theta d_1 + m\,(1-\theta)\,d_2\right) \leqslant \theta Q\left(d_1\right) + m\,(1-\theta)\,Q\left(d_2\right) \tag{2.3}$$

holds $\forall\, d_1, d_2 \in [0, d_2]$ and $\theta \in [0, 1]$. Otherwise Q is m–concave if $(-Q)$ is m–convex.

Definition 2.3. [24] Let Q be a nonnegative function. Then $Q : \mathbb{X} \to \mathbb{R}$, is said exponential type convex, if

$$Q(\theta d_1 + (1-\theta) d_2) \leqslant \left(e^{\theta} - 1\right) Q(d_1) + \left(e^{(1-\theta)} - 1\right) Q(d_2) \tag{2.4}$$

holds $\forall\, d_1, d_2 \in \mathbb{X}$ and $\theta \in [0,1]$.

Inspired by the above results and literatures of inequality theory, we organize the paper as follow : In section 3, we elaborate the concept and algebraic properties of exponential type m–convex function. In section 4, we deduce new generalization of $(H-H)$ type inequality for the exponential type m–convex function. Next, in section 5, we establish some refinements of the $H-H$ inequality, whose first derivative in absolute value at certain power are exponential type m–convex. Further, in section 6, in the manner of this newly introduced idea, we investigate some applications of special means. Finally, in section 7, we give a briefly conclusion.

3. Algebraic properties of exponential type m–convex functions

The principal focus of this section, we will present our main definition of exponential type m–convex function and its associated properties.

Definition 3.1. Let Q be a nonnegative function, then $Q : \mathbb{X} \to \mathbb{R}$, is said exponential type m–convex, if

$$Q(\theta d_1 + m(1-\theta) d_2) \leqslant \left(e^{\theta} - 1\right) Q(d_1) + m\left(e^{(1-\theta)} - 1\right) Q(d_2) \tag{3.1}$$

holds $\forall\, d_1, d_2 \in \mathbb{X}$, $m \in [0,1]$, and $\theta \in [0,1]$.

Remark 3.2. For $m = 1$, we attain exponential type convexity, which is explored by İşcan in [24].

Remark 3.3. The range of the exponential type m–convex functions for $m \in [0,1]$ is $[0,+\infty)$.

Proof. The proof is obvious. □

We explore some relations between the class exponential type m–convex functions and other class of generalized convex functions.

Lemma 3.4. *The following inequalities* $(e^{\theta} - 1) \geqslant \theta$ *and* $(e^{(1-\theta)} - 1) \geqslant (1-\theta)$ *hold,* $\forall \theta \in [0,1]$.

Proof. The proof is clearly seen and hence omitted. □

Proposition 3.5. *If* $m \in [0,1]$*, then every nonnegative* m*–convex function is exponential type* m*–convex function.*

Proof. Since $m \in [0,1]$, and by using Lemma 3.4, we have

$$\begin{aligned} Q(\theta d_1 + m(1-\theta) d_2) &\leqslant \theta Q(d_1) + m(1-\theta) Q(d_2) \\ &\leqslant \left(e^{\theta} - 1\right) Q(d_1) + m\left(e^{(1-\theta)} - 1\right) Q(d_2). \end{aligned}$$

□

Theorem 3.6. *Let* $Q, \phi : [d_1, d_2] \to \mathbb{R}$. *If* Q *and* P *are exponential type* m*–convex functions for* $m \in [0, 1]$, *then*

1. $Q + P$ *is exponential type* m*–convex function;*
2. *For nonnegative real number* c, cQ *is exponential type* m*–convex function.*

Proof. The proof is obvious and hence omitted. □

Theorem 3.7. *Let* $Q : [0, d_2] \to J$ *be* m*–convex function for* $d_2 > 0$ *and* $m \in [0, 1]$ *and* $Q : \mathbb{X} \to \mathbb{R}$ *is non-decreasing and exponential type* m*–convex function. Then for the same fixed numbers* $m \in (0, 1]$, *the function* $P \circ Q : [0, d_2] \to \mathbb{R}$ *is exponential type* m*–convex.*

Proof. $\forall d_1, d_2 \in [0, d_2]$, $m \in [0, 1]$, and $\theta \in [0, 1]$, we have

$$(\phi \circ Q)(\theta d_1 + m(1-\theta) d_2) = P(Q(\theta d_1 + m(1-\theta) d_2)) \leqslant \phi(\theta Q(d_1) + m(1-\theta) Q(d_2))$$
$$\leqslant \left(e^{s\theta} - 1\right)(P \circ Q)(d_1) + m\left(e^{(1-\theta)} - 1\right)(P \circ Q)(d_2).$$

□

Theorem 3.8. *Let* $Q_i : [d_1, d_2] \to \mathbb{R}$ *be a class of exponential type* m*–convex functions for* $m \in [0, 1]$ *and let* $Q(d) = \sup_i Q_i(d)$. *If* $E = \{d \in [d_1, d_2] : Q(d) < +\infty\} \neq \emptyset$, *then* E *is an interval and* Q *is exponential type* m*–convex function on* E.

Proof. For all $d_1, d_2 \in E$, $m \in [0, 1]$, and $\theta \in [0, 1]$, we have

$$Q(\theta d_1 + m(1-Q) d_2) = \sup_i Q_i(\theta d_1 + m(1-\theta) d_2)$$
$$\leqslant \sup_i \left[\left(e^{\theta} - 1\right) Q_i(d_1) + m\left(e^{(1-\theta)s} - 1\right) Q_i(d_2)\right]$$
$$\leqslant \left(e^{\theta} - 1\right) \sup_i Q_i(d_1) + m\left(e^{(1-\theta)} - 1\right) \sup_i Q_i(d_2)$$
$$= \left(e^{\theta} - 1\right) Q(d_1) + m\left(e^{(1-\theta)} - 1\right) Q(d_2) < +\infty.$$

□

Theorem 3.9. *If the function* $Q : [d_1, d_2] \to \mathbb{R}$ *is exponential type* m*–convex for* $m \in [0, 1]$, *then* Q *is bounded on* $[d_1, md_2]$.

Proof. Suppose $x \in [d_1, d_2]$ be a point and $m \in [0, 1]$ and $L = \max\left\{Q(d_1), mQ(d_2)\right\}$ and Then $\exists\ \theta \in [0, 1]$ such that $x = \theta d_1 + m(1-\theta) d_2$. Thus, since $e^{\theta} \leqslant e$ and $e^{(1-\theta)} \leqslant e$, we have

$$Q(x) = Q(\theta d_1 + m(1-\theta) d_2) \leqslant \left(e^{s\theta} - 1\right) Q(d_1) + m\left(e^{(1-\theta)} - 1\right) Q(d_2)$$
$$\leqslant (e-1) L + m(e-1) L = L(m+1)(e-1) = M.$$

□

4. New generalization of $(H-H)$ type inequality using exponential type m–convex function

The subject of this section is to deduce new generalizations of $(H-H)$ type integral inequality involving exponential type m–convex function.

Theorem 4.1. *Let* $Q:[d_1, md_2]\to\mathbb{R}$ *be exponential type* m*–convex function for* $m\in(0,1]$ *and* $d_1<md_2$. *If* $Q\in L_1([d_1,md_2])$, *then*

$$\frac{1}{(\sqrt{e}-1)}Q\left(\frac{d_1+md_2}{2}\right)\leqslant\frac{1}{(md_2-d_1)}\left\{\int_{d_1}^{md_2}Q(x)dx+m\int_{\frac{d_1}{m}}^{d_2}Q(x)dx\right\}$$
$$\leqslant(e-2)\left[Q(d_1)+Q(d_2)+m\left(Q\left(\frac{d_1}{m^2}\right)+Q(d_2)\right)\right],\quad(4.1)$$

Proof. Let denote

$$a_1=\theta d_1+m(1-\theta)d_2,\quad a_2=(1-\theta)\frac{d_1}{m}+\theta d_2,\quad\forall\,\theta\in[0,1],$$

respectively.
Using the definition of exponential type m–convexity of Q, we have

$$Q\left(\frac{d_1+md_2}{2}\right)=Q\left(\frac{a_1+ma_2}{2}\right)$$
$$=Q\left(\frac{[\theta d_1+m(1-\theta)d_2]+[(1-\theta)d_1+m\theta d_2]}{2}\right)$$
$$\leqslant(\sqrt{e}-1)\Big[Q(\theta d_1+m(1-\theta)d_2)+Q((1-\theta)d_1+m\theta d_2)\Big].$$

Now, integrating on both sides in the last inequality with respect to θ over $[0,1]$, we get

$$Q\left(\frac{d_1+md_2}{2}\right)\leqslant(\sqrt{e}-1)$$
$$\times\left[\int_0^1Q(\theta d_1+m(1-\theta)d_2)d\theta+\int_0^1Q\left((1-\theta)\frac{d_1}{m}+\theta d_2\right)d\theta\right]$$
$$=\frac{(\sqrt{e}-1)}{(md_2-d_1)}\left\{\int_{d_1}^{md_2}Q(x)dx+m\int_{\frac{d_1}{m}}^{d_2}Q(x)dx\right\},$$

This completes the left side inequality. For the right side inequality, using exponential type m–convexity of Q, we obtain

$$\frac{1}{(md_2-d_1)}\left\{\int_{d_1}^{md_2}Q(x)dx+m\int_{\frac{d_1}{m}}^{d_2}Q(x)dx\right\}$$
$$=\int_0^1Q(\theta d_1+m(1-\theta)d_2)d\theta+\int_0^1Q\left((1-\theta)\frac{d_1}{m}+\theta d_2\right)d\theta$$

$$\leqslant \int_0^1 \left[\left(e^{\theta}-1\right) Q(d_1) + m\left(e^{(1-\theta)}-1\right) Q(d_2) \right] d\theta$$

$$+ \int_0^1 \left[\left(e^{\theta}-1\right) Q(d_2) + m\left(e^{(1-\theta)}-1\right) Q\left(\frac{d_1}{m^2}\right) \right] d\theta$$

$$= (e-2)\left[Q(d_1) + Q(d_2) + m\left(Q\left(\frac{d_1}{m^2}\right) + Q(d_2) \right) \right].$$

The proof is completed. □

Corollary 4.2. *If $m = 1$ in Theorem 4.1, we get (Theorem 3.1, [24]).*

5. Refinements of $(\mathbf{H}-\mathbf{H})$ type inequality via exponential type m–convex function

Let us establish some refinements of the $(H-H)$ inequality for functions whose first derivative in absolute value at certain power is exponential type m–convex. First we need some new useful lemmas.

Lemma 5.1. *Let $0 < k \leqslant 1$ and a mapping $Q : \left[d_1, \frac{d_2}{k}\right] \to \mathbb{R}$ is differentiable on $\left(d_1, \frac{d_2}{k}\right)$ with $0 < d_1 < d_2$. If $Q' \in L_1\left[d_1, \frac{d_2}{k}\right]$ and $m \in [0,1]$, then*

$$\frac{Q(d_1) + Q\left(\frac{md_2}{k}\right)}{2} - \frac{k}{md_2 - kd_1} \int_{d_1}^{\frac{md_2}{k}} Q(\theta)\, d\theta = \left(\frac{md_2 - kd_1}{2k}\right) \tag{5.1}$$

$$\times \int_0^1 (1-2\theta) Q'\left(\theta d_1 + m(1-\theta)\frac{d_2}{k}\right) d\theta.$$

Proof. Using the integrating by parts, we have

$$\left(\frac{md_2 - kd_1}{2k}\right) \int_0^1 (1-2\theta) Q'\left(\theta d_1 + m(1-\theta)\frac{d_2}{k}\right) d\theta$$

$$= \left(\frac{md_2 - kd_1}{2k}\right) \left\{ \frac{(1-2\theta) Q\left(\theta d_1 + m(1-\theta)\frac{d_2}{k}\right)}{d_1 - \frac{md_2}{k}} \Bigg|_0^1 - \int_0^1 \frac{Q\left(\theta d_1 + m(1-\theta)\frac{d_2}{k}\right)}{d_1 - \frac{md_2}{k}} (-2)\, d\theta \right\}$$

$$= \left(\frac{md_2 - kd_1}{2k}\right) \left\{ \frac{-Q(d_1) - Q\left(\frac{md_2}{k}\right)}{\frac{kd_1 - md_2}{k}} + \frac{2k}{kd_1 - md_2} \int_0^1 Q\left(\theta d_1 + m(1-\theta)\frac{d_2}{k}\right) d\theta \right\}$$

$$= \left(\frac{md_2 - kd_1}{2k}\right) \left\{ \frac{k\left(Q(d_1) + Q\left(\frac{md_2}{k}\right)\right)}{md_2 - kd_1} - \frac{2k}{md_2 - kd_1} \int_0^1 Q\left(\theta d_1 + m(1-\theta)\frac{d_2}{k}\right) d\theta \right\}$$

$$= \frac{Q(d_1) + Q\left(\frac{md_2}{k}\right)}{2} - \frac{k}{md_2 - kd_1} \int_{d_1}^{\frac{md_2}{k}} Q(d)\, d\theta.$$

This completes the proof. □

Remark 5.2. If $m = 1$ in Lemma 5.1, we have

$$\frac{Q(d_1) + Q\left(\frac{d_2}{k}\right)}{2} - \frac{k}{d_2 - kd_1} \int_{d_1}^{\frac{d_2}{k}} Q(\theta)\, d\theta = \left(\frac{d_2 - kd_1}{2k}\right) \times \int_0^1 (1 - 2\theta)\, Q'\left(\theta d_1 + (1-\theta)\frac{d_2}{k}\right) d\theta. \tag{5.2}$$

Remark 5.3. If $k = 1$ in Lemma 5.1, we have

$$\frac{Q(d_1) + Q(md_2)}{2} - \frac{1}{md_2 - d_1} \int_{d_1}^{md_2} Q(\theta)\, d\theta = \left(\frac{md_2 - d_1}{2}\right) \times \int_0^1 (1 - 2\theta)\, Q'(\theta d_1 + m(1-\theta) d_2)\, d\theta. \tag{5.3}$$

Remark 5.4. If $m = k = 1$ in Lemma 5.1, then we have a Lemma 2.1 in [25].

Lemma 5.5. *Let* $0 < k \leqslant 1$ *and* $Q : [kd_1, d_2] \to \mathbb{R}$ *is differentiable on* (kd_1, d_2) *with* $0 < d_1 < d_2$. *If* $Q' \in L_1[kd_1, a_2]$ *and* $m \in [0,1]$, *then*

$$\frac{Q(mkd_1) + Q(d_2)}{2} - \frac{1}{d_2 - kd_1} \int_{mkd_1}^{d_2} Q(\theta)\, d\theta = \left(\frac{d_2 - mkd_1}{2}\right) \times \int_0^1 (2\theta - 1)\, Q'(\theta d_2 + mk(1-\theta) d_1)\, d\theta. \tag{5.4}$$

Proof. Using the integrating by parts, we have

$$\begin{aligned}
&\left(\frac{d_2 - mkd_1}{2}\right) \int_0^1 (2\theta - 1)\, Q'(\theta d_2 + mk(1-\theta) d_1) \\
&= \left(\frac{d_2 - mkd_1}{2}\right) \\
&\times \left\{ \frac{(2\theta - 1)\, Q(\theta d_2 + mk(1-\theta) d_1)}{d_2 - mkd_1}\Bigg|_0^1 - \int_0^1 \frac{Q(\theta d_2 + mk(1-\theta) d_1)}{d_2 - mkd_1} (2)\ d\theta \right\} \\
&= \left(\frac{d_2 - mkd_1}{2}\right) \left\{ \frac{Q(d_2) + Q(mkd_1)}{d_2 - mkd_1} - \frac{2}{d_2 - mkd_1} \int_0^1 Q(\theta d_2 + mk(1-\theta) d_1)\, d\theta \right\} \\
&= \left(\frac{d_2 - mkd_1}{2}\right) \left\{ \frac{Q(d_2) + Q(mkd_1)}{d_2 - mkd_1} - \frac{2}{d_2 - kd_1} \int_0^1 Q(\theta d_2 + mk(1-\theta) d_1)\, d\theta \right\} \\
&= \frac{Q(mkd_1) + Q(d_2)}{2} - \frac{1}{d_2 - mkd_1} \int_{mkd_1}^{d_2} Q(\theta)\, d\theta,
\end{aligned}$$

which completes the proof. □

Remark 5.6. If $m = 1$ in Lemma 5.5, we have

$$\frac{Q(kd_1) + Q(d_2)}{2} - \frac{1}{d_2 - kd_1}\int_{kd_1}^{d_2} Q(\theta)\, d\theta = \left(\frac{d_2 - kd_1}{2}\right) \tag{5.5}$$

$$\times \int_0^1 (2\theta - 1)\, Q'(\theta d_2 + k(1-\theta) d_1)\, d\theta.$$

Remark 5.7. If $k = 1$ in Lemma 5.5, we have

$$\frac{Q(md_1) + Q(d_2)}{2} - \frac{1}{d_2 - d_1}\int_{md_1}^{d_2} Q(\theta)\, d\theta = \left(\frac{d_2 - md_1}{2}\right) \tag{5.6}$$

$$\times \int_0^1 (2\theta - 1)\, Q'(\theta d_2 + m(1-\theta) d_1)\, d\theta.$$

Remark 5.8. If $m = k = 1$ in Lemma 5.5, then we have a Lemma 2.1 in [25].

Theorem 5.9. *Let $Q : \mathbb{X} \to \mathbb{R}$ is differentiable on $\mathbb{X}$ with $0 < d_1 < d_2$. and $0 < k \leqslant 1$. If $|Q'|^q$ is exponential type m–convex function on $\mathbb{X}$ for $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$, then for $m \in [0, 1]$, the following inequality holds:*

$$\left|\frac{Q(d_1) + Q\left(\frac{md_2}{k}\right)}{2} - \frac{k}{md_2 - kd_1}\int_{d_1}^{\frac{md_2}{k}} Q(\theta)\, d\theta\right| \leqslant \left(\frac{md_2 - kd_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{5.7}$$

$$\times \left\{(e-2)\left(|Q'(d_1)|^q + m\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 5.1, Hölder's inequality and exponential type m–convexity of $|Q'|^q$, we have

$$\left|\frac{Q(d_1) + Q\left(\frac{md_2}{k}\right)}{2} - \frac{k}{md_2 - kd_1}\int_{d_1}^{\frac{md_2}{k}} Q(\theta)\, d\theta\right|$$

$$\leqslant \left(\frac{md_2 - kd_1}{2k}\right)\left(\int_0^1 |1 - 2\theta|^p \ d\theta\right)^{\frac{1}{p}} \left\{\int_0^1 \left|Q'\left(\theta d_1 + m(1-\theta)\frac{d_2}{k}\right)\right|^q \ d\theta\right\}^{\frac{1}{q}}$$

$$\leqslant \left(\frac{md_2 - kd_1}{2k}\right)\left(\int_0^1 |1 - 2\theta|^p \ d\theta\right)^{\frac{1}{p}}$$

$$\times \left\{\int_0^1 \left[(e^{\theta} - 1)\,|Q'(d_1)|^q + m\left(e^{(1-\theta)} - 1\right)\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right] d\theta\right\}^{\frac{1}{q}}$$

$$= \left(\frac{md_2 - kd_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{(e-2)\left(|Q'(d_1)|^q + m\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

This completes the proof. □

Remark 5.10. If $m = 1$ in Theorem 5.9, we have

$$\left|\frac{Q(d_1)+Q\left(\frac{d_2}{k}\right)}{2}-\frac{k}{d_2-kd_1}\int_{d_1}^{\frac{d_2}{k}}Q(\theta)\,d\theta\right| \leqslant \left(\frac{d_2-kd_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{5.8}$$
$$\times\left\{(e-2)\left(\left|Q'(d_1)\right|^q+\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Remark 5.11. If $k = 1$ in Theorem 5.9, we have

$$\left|\frac{Q(d_1)+Q(md_2)}{2}-\frac{1}{md_2-d_1}\int_{d_1}^{md_2}Q(\theta)\,d\theta\right| \leqslant \left(\frac{md_2-d_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{5.9}$$
$$\times\left\{(e-2)\left(\left|Q'(d_1)\right|^q+m\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Remark 5.12. If $m = k = 1$ in Theorem 5.9, we have

$$\left|\frac{Q(d_1)+Q(d_2)}{2}-\frac{1}{d_2-d_1}\int_{d_1}^{d_2}Q(\theta)\,d\theta\right| \leqslant \left(\frac{d_2-d_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{5.10}$$
$$\times\left\{(e-2)\left(\left|Q'(d_1)\right|^q+\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Theorem 5.13. *Let* $Q : \mathbb{X} \to \mathbb{R}$ *is differentiable on* $\mathbb{X}$ *with* $0 < d_1 < d_2$. *and* $0 < k \leqslant 1$. *If* $|Q'|^q$ *is exponential type* m*–convex function on* $\mathbb{X}$ *for* $q \geqslant 1$, *and* $m \in [0, 1]$,, *we have*

$$\left|\frac{Q(d_1)+Q\left(\frac{md_2}{k}\right)}{2}-\frac{k}{md_2-kd_1}\int_{d_1}^{\frac{md_2}{k}}Q(\theta)\,d\theta\right| \leqslant \left(\frac{md_2-kd_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \tag{5.11}$$
$$\times\left\{\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left(\left|Q'(d_1)\right|^q+m\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 5.1, power mean inequality and exponential type m–convexity of

$|Q'|^q$, we have

$$\begin{aligned}
&\left|\frac{Q(d_1)+Q\left(\frac{md_2}{k}\right)}{2}-\frac{k}{md_2-kd_1}\int_{d_1}^{\frac{md_2}{k}}Q(\theta)\,d\theta\right| \\
&\leqslant\left(\frac{md_2-kd_1}{2k}\right)\left\{\int_0^1|1-2\theta|\left|Q'\left(\theta d_1+m(1-\theta)\frac{d_2}{k}\right)\right|d\theta\right\} \\
&\leqslant\left(\frac{md_2-kd_1}{2k}\right)\left(\int_0^1|1-2\theta|\,d\theta\right)^{1-\frac{1}{q}}\left\{\int_0^1|1-2\theta|\left|Q'\left(\theta d_1+m(1-\theta)\frac{d_2}{k}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \\
&\leqslant\left(\frac{md_2-kd_1}{2k}\right)\left(\int_0^1|1-2\theta|\,d\theta\right)^{1-\frac{1}{q}} \\
&\times\left\{\int_0^1|1-2\theta|\left[\left(e^{\theta}-1\right)\left|Q'(d_1)\right|^q+m\left(e^{(1-\theta)}-1\right)\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right]d\theta\right\}^{\frac{1}{q}} \\
&=\left(\frac{md_2-kd_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left(\left|Q'(d_1)\right|^q+m\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Remark 5.14. If $m=1$ in Theorem 5.13, we get

$$\begin{aligned}
&\left|\frac{Q(d_1)+Q\left(\frac{d_2}{k}\right)}{2}-\frac{k}{d_2-kd_1}\int_{d_1}^{\frac{d_2}{k}}Q(\theta)\,d\theta\right|\leqslant\left(\frac{d_2-kd_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left(\left|Q'(d_1)\right|^q+\left|Q'\left(\frac{d_2}{k}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned} \tag{5.12}$$

Remark 5.15. If $k=1$ in Theorem 5.13, we get

$$\begin{aligned}
&\left|\frac{Q(d_1)+Q(md_2)}{2}-\frac{1}{md_2-d_1}\int_{d_1}^{md_2}Q(\theta)\,d\theta\right|\leqslant\left(\frac{md_2-d_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left(\left|Q'(d_1)\right|^q+m\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned} \tag{5.13}$$

Remark 5.16. If $m=k=1$ in Theorem 5.13, we get

$$\begin{aligned}
&\left|\frac{Q(d_1)+Q(d_2)}{2}-\frac{1}{d_2-d_1}\int_{d_1}^{d_2}Q(\theta)\,d\theta\right|\leqslant\left(\frac{d_2-d_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left(\left|Q'(d_1)\right|^q+\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned} \tag{5.14}$$

Theorem 5.17. *Let* $Q:\mathbb{X}\to\mathbb{R}$ *is differentiable on* $\mathbb{X}$ *with* $0<d_1<d_2$. *and* $0<k\leqslant 1$. *If* $|Q'|^q$ *is exponential type* m*–convex function on* $\mathbb{X}$ *for* $q>1$ *and* $\frac{1}{q}+\frac{1}{p}=1$, *then for* $m\in[0,1]$, *we have:*

$$\left|\frac{Q(mkd_1)+Q(d_2)}{2}-\frac{1}{d_2-mkd_1}\int_{mkd_1}^{d_2}Q(\theta)\,d\theta\right|\leqslant\left(\frac{d_2-mkd_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{5.15}$$
$$\times\left\{(e-2)\left(m\left|Q'(kd_1)\right|^q+\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 5.5, Hölder's inequality and exponential type m–convexity of $|Q'|^q$, we have

$$\begin{aligned}&\left|\frac{Q(kd_1)+Q(d_2)}{2}-\frac{1}{d_2-kd_1}\int_{kd_1}^{d_2}Q(\theta)\,d\theta\right|\\
&\leqslant\left(\frac{d_2-kd_1}{2}\right)\left(\int_0^1|2\theta-1|^p\ d\theta\right)^{\frac{1}{p}}\left\{\int_0^1\left|Q'(\theta d_2+mk(1-\theta)d_1)\right|^q d\theta\right\}^{\frac{1}{q}}\\
&\leqslant\left(\frac{d_2-mkd_1}{2}\right)\left(\int_0^1|2\theta-1|^p\ d\theta\right)^{\frac{1}{p}}\\
&\times\left\{\int_0^1\left[\left(e^\theta-1\right)\left|Q'(d_2)\right|^q+m\left(e^{(1-\theta)}-1\right)\left|Q'(kd_1)\right|^q\right]d\theta\right\}^{\frac{1}{q}}\\
&=\left(\frac{d_2-mkd_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\
&\times\left\{(e-2)\left(m\left|Q'(kd_1)\right|^q+\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.\end{aligned}$$

This completes the proof. □

Remark 5.18. If $m=1$ in Theorem 5.17, we obtain

$$\left|\frac{Q(kd_1)+Q(d_2)}{2}-\frac{1}{d_2-kd_1}\int_{kd_1}^{d_2}Q(\theta)\,d\theta\right|\leqslant\left(\frac{d_2-kd_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{5.16}$$
$$\times\left\{(e-2)\left(\left|Q'(kd_1)\right|^q+\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Remark 5.19. If $k=1$ in Theorem 5.17, we obtain

$$\left|\frac{Q(md_1)+Q(d_2)}{2}-\frac{1}{d_2-md_1}\int_{md_1}^{d_2}Q(\theta)\,d\theta\right|\leqslant\left(\frac{d_2-md_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{5.17}$$
$$\times\left\{(e-2)\left(m\left|Q'(d_1)\right|^q+\left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Remark 5.20. If $m = k = 1$ in Theorem 5.17, we obtain

$$\left|\frac{Q(d_1)+Q(d_2)}{2} - \frac{1}{d_2-d_1}\int_{d_1}^{d_2} Q(\theta)\,d\theta\right| \leqslant \left(\frac{d_2-d_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left\{(e-2)\left(\left|Q'(d_1)\right|^q + \left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}. \tag{5.18}$$

Theorem 5.21. *Let* $Q : \mathbb{X} \to \mathbb{R}$ *is differentiable on* $\mathbb{X}$ *with* $0 < d_1 < d_2$. *and* $0 < k \leqslant 1$. *If* $|Q'|^q$ *is exponential type* m*–convex function on* $\mathbb{X}$ *for* $q \geqslant 1$, *and* $m \in [0, 1]$, *then we have:*

$$\left|\frac{Q(mkd_1)+Q(d_2)}{2} - \frac{1}{d_2-mkd_1}\int_{mkd_1}^{d_2} Q(\theta)\,d\theta\right| \leqslant \left(\frac{d_2-kd_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times \left\{\frac{8\sqrt{e}-2e-7}{2}\left(m\left|Q'(kd_1)\right|^q + \left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}, \tag{5.19}$$

Proof. From Lemma 5.5, power mean inequality and exponential type m–convexity of $|Q'|^q$, we have

$$\begin{aligned}
&\left|\frac{Q(kd_1)+Q(d_2)}{2} - \frac{1}{d_2-kd_1}\int_{kd_1}^{d_2} Q(\theta)\,d\theta\right| \\
&\leqslant \left(\frac{d_2-mkd_1}{2}\right)\left\{\int_0^1 |2\theta-1|\,\left|Q'(\theta d_2 + mk(1-\theta)d_1)\right| d\theta\right\} \\
&\leqslant \left(\frac{d_2-mkd_1}{2}\right)\left(\int_0^1 |2\theta-1|\,d\theta\right)^{1-\frac{1}{q}} \\
&\times \left\{\int_0^1 |2\theta-1|\,\left|Q'(\theta d_2 + mk(1-\theta)d_1)\right|^q d\theta\right\}^{\frac{1}{q}} \\
&\leqslant \left(\frac{d_2-mkd_1}{2}\right)\left(\int_0^1 |2\theta-1|\,d\theta\right)^{1-\frac{1}{q}} \\
&\times \left[\int_0^1 |2\theta-1|\left\{m\left(e^{(1-\theta)}-1\right)\left|Q'(kd_1)\right|^q + \left(e^{\theta}-1\right)\left|Q'(d_2)\right|^q\right\}d\theta\right]^{\frac{1}{q}} \\
&= \left(\frac{d_2-mkd_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times \left\{\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left(m\left|Q'(kd_1)\right|^q + \left|Q'(d_2)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Remark 5.22. If $m = 1$ in Theorem 5.21, we have

$$\left|\frac{Q(kd_1)+Q(d_2)}{2}-\frac{1}{d_2-kd_1}\int_{kd_1}^{d_2}Q(\theta)\,d\theta\right| \leqslant \left(\frac{d_2-kd_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times \left\{\frac{8\sqrt{e}-2e-7}{2}\left(|Q'(kd_1)|^q+|Q'(d_2)|^q\right)\right\}^{\frac{1}{q}}. \tag{5.20}$$

Remark 5.23. If $k = 1$ in Theorem 5.21, we have

$$\left|\frac{Q(md_1)+Q(d_2)}{2}-\frac{1}{d_2-md_1}\int_{md_1}^{d_2}Q(\theta)\,d\theta\right| \leqslant \left(\frac{d_2-d_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times \left\{\frac{8\sqrt{e}-2e-7}{2}\left(m|Q'(d_1)|^q+|Q'(d_2)|^q\right)\right\}^{\frac{1}{q}}. \tag{5.21}$$

Remark 5.24. If $m = k = 1$ in Theorem 5.21, we have

$$\left|\frac{Q(d_1)+Q(d_2)}{2}-\frac{1}{d_2-d_1}\int_{d_1}^{d_2}Q(\theta)\,d\theta\right| \leqslant \left(\frac{d_2-d_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times \left\{\frac{8\sqrt{e}-2e-7}{2}\left(|Q'(d_1)|^q+|Q'(d_2)|^q\right)\right\}^{\frac{1}{q}}. \tag{5.22}$$

6. Applications

In this section, we recall the following special means of two positive numbers d_1, d_2 with $d_1 < d_2$:
(1) The arithmetic mean

$$A = A(d_1, d_2) = \frac{d_1+d_2}{2}, \quad d_1, d_2 \geqslant 0.$$

(2) The geometric mean

$$G = G(d_1, d_2) = \sqrt{d_1 d_2}, \quad d_1, d_2 \geqslant 0.$$

(3) The harmonic mean

$$H = H(d_1, d_2) = \frac{2d_1 d_2}{d_1+d_2}, \quad d_1, d_2 > 0.$$

(4) The logarithmic mean

$$L = L(d_1, d_2) = \begin{cases} \frac{d_2-d_1}{\ln d_2 - \ln d_1}, & d_1 \neq d_2, \\ d_1, & d_1 = d_2; \end{cases} \qquad d_1, d_2 > 0.$$

The following relationship are well-known in the literature.

$$H(d_1, d_2) \leqslant G(d_1, d_2) \leqslant A(d_1, d_2).$$

Proposition 6.1. *Let* $d_1, d_2 \in [0, \infty)$ *with* $d_1 < d_2$ *and* $r \in (-\infty, 0) \bigcup [1, \infty) \setminus \{-1\}$. *Then the inequalities*

$$\frac{A^r(d_1, d_2)}{2(\sqrt{e}-1)} \leqslant L_r^r(d_1, d_2) \leqslant 2(e-2)A(d_1^r, d_2^r), \tag{6.1}$$

holds.

Proof. If we put $Q(x) = x^r, \quad x \in [0, \infty)$ and $m = 1$ in the above theorem 4.1, then we can easily obtained the inequality 6.1. □

Proposition 6.2. *Let* $d_1, d_2 \in (0, \infty)$ *with* $d_1 < d_2$. *Then the inequalities*

$$\frac{A^r(d_1, d_2)}{2(\sqrt{e}-1)} \leqslant L^{-1}(d_1, d_2) \leqslant 2(e-2)H^{-1}(d_1, d_2), \tag{6.2}$$

holds.

Proof. If we put $Q(x) = x^{-1}, \quad x \in (0, \infty)$ and $m = 1$ in the above theorem 4.1, then we can easily obtained the inequality 6.2. □

7. Conclusion

In this paper, authors presented new assessment of $(H-H)$ type inequalities for a new type of convexity, called exponential type m–convex function. We have achieved refinements of the $(H-H)$ inequality for functions whose first derivative in absolute value at certain power are exponential type m–convex. As of late, numerous mathematicians put exertion into the hypothesis of inequality to express new dimension to mathematical analysis. Because of widespread perspectives and significance, this hypothesis has become an alluring and engrossing field for researchers. We trust that our novel thoughts and strategies may propel numerous scientists in this intriguing field.

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