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Thermo-diffusion effect on magnetohydrodynamics flow of fractional Casson fluid with heat generation and first order chemical reaction over a vertical plate

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Abstract

Analytical solution of thermo diffusion effect on magnetohydrodynamics flow of fractionalized Casson fluid over a vertical plate immersed in a porous media is obtained. Moreover, in the model of the problem, additional effects, like a chemical reaction, heat source/sink, and thermal radiation are also considered. The model is solved by three approaches, namely, Atangana-Baleanu, Caputo-Fabrizio, and Caputo fractional derivative of non-integer order γ. The governing dimensionless equations for temperatures, concentrations, and velocities are solved using Laplace transform method and compared graphically. The effects of different parameters like fractional parameter γ, Thermo diffusion Sr, and magnetic parameter M are discussed through numerous graphs. Furthermore, comparisons among ordinary and fractionalized velocity fields are also drawn. It is found that the velocity obtained with Atangana-Baleanu fractional derivative is less than that obtained by Caputo, Caputo-Fabrizio, or ordinary derivatives.

Keywords: Casson fluid, Free convection, Chemical reaction, Heat generation, Thermo diffusion, Caputo(C), Caputo-Fabrizio(CF), Atangana-Baleanu(AB) fractional derivative.

1. Introduction

Convection flow in the presence of porosity has numerous important applications such as flows in soils, solar power collectors, heat transfer correlated with geothermal systems, the heat source in the field of the agricultural storage system, heat transfer in nuclear reactors, heat transfer in aerobic and anaerobic reactions, heat evacuation from nuclear fuel detritus, and heat exchangers for a porous material.

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Convection flow of MHD fluid has many implementations in meteorology, the distillation of gasoline, boundary layer control, energy generators, geophysics, accelerators, the petroleum industry, astrophysics, polymer technology, aerodynamics, and in material processing such as metal forming, glass fiber drawing, extrusion, and casting wire. The flow past through a perpendicular plate with heat and mass transfer is examined by Swamy et al. [1]. The combined effect of heat and mass diffusion on fluid flow through a plate has been observed by Chaudhary and Jain [2]. The analytical solution for magnetohydrodynamics flow through a perpendicular plate in the existence of porosity is obtained by Sivaiah et al. [3]. Das and Jana [4] discussed the solution for MHD flow through a plate in the presence of porous media.

Rajesh [5] highlighted the free convection flow of MHD fluid in the existence of porosity. The convection flow of magnetohydrodynamics fluid immersed in a porous medium has been studied on [6, 7, 8, 9, 10]. Authors in [11] investigated the mathematical modeling and forecasting of COVID-19 in Saudi Arabia. They also discussed the solution for time-dependent concentration and temperature. Convection flow through a surface in the presence of porous media is discussed in [12, 13, 14].

Toki [15] presented the phenomenon of convection flow over a perpendicular plate. Rajesh and Varma [16] analyzed the MHD flow through an exponentially accelerated plate. MHD flow through an accelerated surface in the existence of porous media is discussed by Chaudhary and Jan [17]. The authors also analyzed the solution of the velocity field graphically. Ramzan et al. [18] analyzed the solution of magnetohydrodynamics of convection flow. Pal and Talukdar [19] obtained the solution of a viscous fluid with thermal radiation on magnetohydrodynamics flow, whereas, the solution for convection flow with non-uniform temperature through a moving plate is obtained by Seth et al. [20]. Khan et al. [21] have investigated the problem of hall current effects on convection flow.

This phenomenon plays a vital role in the cooling of the nuclear reactor, tabular reactor, chemical industry, mixture of terracotta material, petroleum industry, and decomposition of rigid materials. Seddeek et al. [22] obtained the solution of MHD fluid flow through porous media with thermal radiation and chemical reaction. An intensive study of first order chemical reactions with heat generation/absorption has been analyzed by Shah et al. [23]. Alharbi et al. [24] investigated the MHD visco-elastic fluid flow over a stretching sheet. The study of fluid has attained great importance due to its practical significance in numerous fields of industry and applied engineering. Experimental and theoretical investigations on fluid have been made in [25, 26, 27, 28, 29, 30, 31, 32, 33].

The aim of this paper is to study the comparison of Caputo-Fabrizio, Caputo, and Atangana-Baleanu fractional derivative with Soret effects on Magnetohydrodynamics free convection flow of Casson fluid over a plate immersed in a porous media with a heat source and thermal radiation. An analytical solution is obtained via Laplace transform and using a special function. Various graphs are plotted and discussed for different parameters, which are used in the model and found that Atangana-Baleanu fractional derivative is the best choice for controlled fluid velocity.

2. Mathematical Formulation of the Problem

A cartesian coordinate system for the study of the MHD viscous flow of a Casson fluid with mass diffusion and heat source over an infinite vertical plate is considered. The plate lies along the x-axis and the z-axis is taken outward normal to it. The fluid flows in the upward direction along the x-axis. A uniform magnetic field is applied in the direction perpendicular to the flow. The length of the plate is infinite, so all physical quantities depend on time t' and the y-axis. At time $t' = 0$, both fluid and plate are at the same concentration C'_∞ and temperature t'_∞ for all the points in the flow. At time $t' = 0^+$, the plate moves in xz-plane with velocity $f(t')$ in x-direction. Under these assumptions and using Boussinesq's approximation, a set of equations for Casson fluid are as follows:

$$\frac{\partial u_0(z,t')}{\partial t'} = \nu(1+\frac{1}{\beta})\frac{\partial^2 u_0(z,t')}{\partial z^2} - \frac{\sigma\beta_0^2}{\rho}u_0(z,t') - \frac{\nu\phi}{k_1}u_0(z,t') + g\beta_T(T'-T'_\infty) + g\beta_{C'}(C'-C'_\infty), \tag{2.1}$$

$$\frac{\partial T'(z,t')}{\partial t'} = \frac{K}{\rho C_p}\frac{\partial^2 T'(z,t')}{\partial z^2} + Q_0(T'-T'_\infty), \tag{2.2}$$

$$\frac{\partial C'(z,t')}{\partial t'} = D_0\frac{\partial^2 C'(z,t')}{\partial z^2} + \frac{D_m K_{T'}}{T_r}\frac{\partial^2 C(y,t)}{\partial z^2} - R_0(C'-C'_\infty), \tag{2.3}$$

The initial and boundary conditions of the flow model are

$$u_0(z,0) = 0,\ \ T'(z,0) = T'_\infty,\ \ C'(z,0) = C'_\infty,\ \ z \geqslant 0, \tag{2.4}$$

$$u_0(0,t') = U_1 f(t'),\ \ T'(0,t') = T'_w,\ \ C'(0,t') = C'_w,\ \ t' \geqslant 0, \tag{2.5}$$

$$u_0(z,t') \to 0,\ \ T'(z,t') \to 0,\ \ C'(z,t') \to 0,\ \ \ t' > 0. \tag{2.6}$$

We introduced the following dimensionless variables and parameters for the flow model

$$\begin{aligned} w^* &= \frac{u_0}{U_1},\quad z^* = \frac{z}{U_1 t_0},\quad t^* = \frac{t'}{t_0},\quad \vartheta^* = \frac{T'-T'_\infty}{T'_w - T'_\infty},\quad Pr^* = \frac{\rho C_p}{k}, \\ &Gr^* = \frac{\nu g\beta_T(T'_w - T'_\infty)}{U_1^3},\quad \varpi^* = \frac{C'-C'_\infty}{C'_w - C'_\infty}, \\ &Gm^* = \frac{\nu g\beta_{C'}(C'_w - C'_\infty)}{U_1^3},\quad M^* = \frac{\beta_0^2\nu\sigma}{\rho U_0^2},\quad Sr^* = \frac{D_m K_T(T'_w - T'_\infty)}{T_r\nu(C'_w - C'_\infty)}, \\ &\frac{1}{K}^* = \frac{\nu\phi}{K_1 U_1^2},\quad Q^* = \frac{Q_0\nu^2}{kU_1^2},\quad R^* = \frac{R_0\nu}{U_1^2},\quad Sc^* = \frac{\nu}{D_0}. \end{aligned} \tag{2.7}$$

Using non-dimensional variables of Eq. (2.7) into Eq. (2.1-2.6), we obtain the following problem (dropping stars)

$$\frac{\partial w(z,t)}{\partial t} = A\frac{\partial^2 w(z,t)}{\partial z^2} - (M+\frac{1}{K})w(z,t) + Gr\vartheta(z,t) + Gm\varpi(z,t), \tag{2.8}$$

$$\frac{\partial \vartheta(z,t)}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \vartheta(z,t)}{\partial z^2} + Q\vartheta(z,t), \tag{2.9}$$

$$\frac{\partial \varpi(z,t)}{\partial t} = \frac{1}{Sc}\frac{\partial^2 \varpi(z,t)}{\partial z^2} - R\varpi(z,t) + Sr\frac{\partial^2 \vartheta(z,t)}{\partial z^2}, \tag{2.10}$$

$$w(z,0) = 0, \;\; \vartheta(z,0) = 0, \;\; \varpi(z,0) = 0, \;\; z \geqslant 0, \tag{2.11}$$

$$w(0,t) = f(t), \;\; \vartheta(0,t) = 1, \;\; \varpi(0,t) = 1, \;\; t \geqslant 0, \tag{2.12}$$

$$w(\infty,t) \to 0, \;\; \vartheta(\infty,t) \to 0, \;\; \varpi(\infty,t) \to 0, \quad t > 0, \tag{2.13}$$

where $A = (1 + \frac{1}{\beta})$, and $Pr, Sc, K, M, Sr, Gm, Gr, Q, R$, and γ represent the Prandtl number, Schmidt number, Porosity, magnetic field, Sorret number, mass Grashof number, thermal Grashof number, heat source, chemical reaction, and fraction parameter respectively.

3. Generalization of Local Model

The local model defined in equations (2.8-2.13) has been generalized by converting ordinary derivatives with Caputo-Fabrizio, Caputo, and Atangana-Baleanu fractional derivative of order γ.

$$D_t^{\gamma} w(z,t) = A\frac{\partial^2 w(z,t)}{\partial z^2} - Mw(y,t) - \frac{1}{k}w(z,t) + Gr\vartheta(z,t) + Gm\varpi(z,t), \tag{3.1}$$

$$D_t^{\gamma} \vartheta(z,t) = \frac{1}{Pr}\frac{\partial^2 \vartheta(z,t)}{\partial z^2} + Q\vartheta(z,t), \tag{3.2}$$

$$D_t^{\gamma} \varpi(z,t) = \frac{1}{Sc}\frac{\partial^2 \varpi(z,t)}{\partial z^2} + Sr\frac{\partial^2 \vartheta(z,t)}{\partial z^2} - R\varpi(z,t)\,, \tag{3.3}$$

$$w(z,0) = 0, \;\; \vartheta(z,0) = 0, \;\; \varpi(z,0) = 0, \;\; z \geqslant 0, \tag{3.4}$$

$$w(0,t) = f(t), \;\; \vartheta(0,t) = 1, \;\; \varpi(0,t) = 1, \;\; t \geqslant 0, \tag{3.5}$$

$$w(\infty,t) \to 0, \;\; \vartheta(\infty,t) \to 0, \;\; \varpi(\infty,t) \to 0, \quad t > 0, \tag{3.6}$$

where $D_t^{\gamma} w(z,t)$ represents the Caputo time-fractional derivative of $w(z,t)$

$$D_t^{\gamma} w(z,t) = \begin{cases} \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{1}{(t-p)^{\gamma}}\frac{\partial w(z,p)}{\partial p}dp, & 0 \leqslant \gamma < 1; \\ \frac{\partial w(z,t)}{\partial t}, & \gamma = 1. \end{cases} \tag{3.7}$$

Now Caputo-Fabrizio fractional derivative is defined as

$$D_t^{\gamma} w(z,t) = \frac{1}{(1-\gamma)}\int_0^t e^{\frac{-\gamma(t-p)}{1-\gamma}}\frac{\partial w(z,p)}{\partial p}dp, \quad 0 \leqslant \gamma \leqslant 1. \tag{3.8}$$

Also Atangana Baleanu time-fractional derivative is given as

$$D_t^{\gamma} w(z,t) = \frac{M(\gamma)}{(1-\gamma)}\int_0^t E_{\gamma}\left(-\gamma\frac{(t-p)^{\gamma}}{1-\gamma}\right)\frac{\partial w(z,p)}{\partial p}dp. \tag{3.9}$$

4. Solution of Problem

Now we solve the flow model by applying Laplace transform technique. We can solve Eq. (3.2) for the temperature profile, Eq. (3.3) for the concentration profile, and Eq. (3.1) for the velocity profile respectively.

4.1. Calculation of Temperature With Caputo

By taking the Laplace transform on Eq. (3.2), we obtain

$$\mathrm{Pr} p^{\gamma}\bar{\vartheta}(z,q) = \frac{\partial^2 \bar{\vartheta}(z,q)}{\partial z^2} + \mathrm{Pr}Q\bar{\vartheta}(z,q), \tag{4.1}$$

Boundary conditions are

$$\bar{\vartheta}(0,q) = \frac{1}{q}, \quad \bar{\vartheta}(z,q) \to 0, \ \ z \to \infty. \tag{4.2}$$

The solution of partial differential Eq. (4.1), by using conditions given in Eq. (4.2) is

$$\bar{\vartheta}(z,q) = \frac{1}{q} e^{-z\sqrt{\mathrm{Pr}(q^{\gamma}-Q)}}. \tag{4.3}$$

Eq. (4.3) can be written in a suitable form as

$$\bar{\vartheta}(z,q) = \left[\frac{q^{\gamma}-Q}{q}\right] \frac{e^{-z\sqrt{\mathrm{Pr}}\sqrt{(q^{\gamma}-Q)}}}{q^{\gamma}-Q}. \tag{4.4}$$

Taking the inverse Laplace transform of Eq. (4.4), we have

$$\vartheta(z,t) = \int_0^t F_1(z,t-p)\left[\frac{p^{-\gamma}}{\Gamma(1-\gamma)} - Q\right]dp, \tag{4.5}$$

where

$$F_1(z,t) = \int_0^{\infty} e^{Qw}\mathrm{Erfc}\left(\frac{z\sqrt{\mathrm{Pr}}}{2\sqrt{w}}\right)t^{-1}\left(0,-\alpha,-wt^{-\alpha}\right)dw. \tag{4.6}$$

4.2. Nusselt Number

In order to find the Nusselt number, we use Eq. (4.4) in the following relation

$$\mathrm{Nu} = -\frac{\partial \vartheta}{\partial z}\bigg|_{z=0} = -L^{-1}\left\{\frac{\partial \bar{\vartheta}}{\partial z}\bigg|_{z=0}\right\} = \sqrt{\mathrm{Pr}}\int_0^t \left(\frac{(t-p)^{-\gamma}}{\Gamma(1-\gamma)} - Q\right)p^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(Qp^{\gamma})dp \tag{4.7}$$

4.3. Calculation of Temperature With Caputo-Fabrizio

By taking the Laplace transform on Eq. (3.2), we obtain

$$\frac{\mathrm{Pr}q}{(1-\gamma)q+\gamma}\bar{\vartheta}(z,q)=\frac{\partial^2\bar{\vartheta}(z,q)}{\partial z^2}+\mathrm{PrQ}\bar{\vartheta}(z,q), \tag{4.8}$$

The solution of partial differential Eq. (4.8), by using conditions given in Eq. (4.2) is

$$\bar{\vartheta}(z,q)=\frac{1}{q}e^{-z\sqrt{\frac{(q-a_3)}{(q+a_1)}a_2}}. \tag{4.9}$$

The inverse Laplace transform of Eq. (4.9) is

$$\vartheta(z,t)=\phi_1(z,t), \tag{4.10}$$

where

$$\phi_1(z,t)=e^{-z\sqrt{a_2}}-\frac{z\sqrt{a_2}\sqrt{-a_3-a_1}}{2\sqrt{\pi}}\int_0^\infty\int_0^t\frac{1}{\sqrt{t}}e^{(-a_1t-\frac{z^2a_2}{4w}-w)}\times I_1(2\sqrt{(-a_3-a_1)wt})\,dt\,dw. \tag{4.11}$$

4.4. Nusselt Number

From Eq. (4.9), the Nu can be calculated in the same way as in Eq. (4.7) and is given by

$$\mathrm{Nu}=\frac{\sqrt{-a_2a_3a_1}}{a_1}. \tag{4.12}$$

4.5. Calculation of Temperature With Atangana-Baleanu

By taking the Laplace transform on Eq. (3.2), we find

$$\frac{\mathrm{Pr}q^\gamma}{(1-\gamma)q^\gamma+\gamma}\bar{\vartheta}(z,q)=\frac{\partial^2\bar{\vartheta}(z,q)}{\partial z^2}+\mathrm{PrQ}\bar{\vartheta}(z,q), \tag{4.13}$$

The solution of partial differential Eq. (4.13), by using initial and boundary conditions are

$$\bar{\vartheta}(z,q)=\frac{1}{q}e^{-z\sqrt{\frac{(q^\gamma-a_3)}{(q^\gamma+a_1)}a_2}}. \tag{4.14}$$

Eq. (4.14) can also be written as

$$\bar{\vartheta}(z,q)=\frac{1}{q^{1-\gamma}}\frac{1}{q^\gamma}e^{-z\sqrt{a_2}\sqrt{\frac{(q^\gamma-a_3)}{(q^\gamma+a_1)}}}. \tag{4.15}$$

Taking the inverse Laplace transform on Eq. (4.15), we develop the following form of solution

$$\vartheta(z,t)=\int_0^t\varphi_1(z,t-p)\frac{p^{-\gamma}}{\Gamma(1-\gamma)}dp, \tag{4.16}$$

where

$$\varphi_1(z,t)=\int_0^\infty\left[e^{-z\sqrt{a_2}}-\frac{z\sqrt{a_2}\sqrt{-a_3-a_1}}{2\sqrt{\pi}}\int_0^\infty\int_0^t\frac{1}{\sqrt{t}}e^{(-a_1t-\frac{z^2a_2}{4w}-w)}\times I_1(2\sqrt{(-a_3-a_1)wt})\,dt\,dw\right]t^{-1}\psi(0,-\gamma,-xt^{-\gamma})\,dx. \tag{4.17}$$

4.6. Nusselt Number

From Eq. (4.15), the Nu can be calculated in the same way as in Eq. (4.7) and is given by

$$Nu = \frac{\sqrt{-a_2 a_3 a_1}}{a_1}. \tag{4.18}$$

4.7. Calculation of Concentration With Caputo

By taking the Laplace transform on Eq. (3.3), we obtain

$$Scq^{\gamma}\bar{\varpi}(z,q) = \frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2} + SrSc\frac{\partial^2\bar{\vartheta}(z,q)}{\partial z^2} - RSc\bar{\varpi}(z,q). \tag{4.19}$$

The boundary conditions satisfying Eq. (4.19) are

$$\frac{\partial\bar{\varpi}(0,q)}{\partial z} = \frac{1}{q}, \quad \bar{\varpi}(z,q) \to 0, \ \ z \to \infty. \tag{4.20}$$

The solution of partial differential Eq. (4.19), by using conditions given in Eq. (4.20) is

$$\bar{\varpi}(z,q) = \frac{1}{q}e^{-z\sqrt{Sc(q^{\gamma}+R)}} - \frac{a_4(q^{\gamma}-Q)}{q[q^{\gamma}-a_5]}\left(e^{-z\sqrt{Sc(q^{\gamma}+R)}} - e^{-z\sqrt{Pr(q^{\gamma}-Q)}}\right). \tag{4.21}$$

Eq. (4.21) can also be written in the following form

$$\begin{aligned}\bar{\varpi}(z,q) \ &= \ \left(1 + a_4 + \frac{a_6}{(q^{\gamma}-a_5)}\right)\frac{(q^{\gamma}+R)}{q}\frac{e^{-z\sqrt{Sc}\sqrt{(q^{\gamma}+R)}}}{q^{\gamma}+R} - \left(a_4 + \frac{a_6}{(q^{\gamma}-a_5)}\right) \times \\ &\quad \frac{(q^{\gamma}-Q)}{q}\frac{e^{-z\sqrt{Pr}\sqrt{(q^{\gamma}-Q)}}}{q^{\gamma}-Q}.\end{aligned} \tag{4.22}$$

Taking the inverse Laplace transform of Eq. (4.22), we obtain

$$\begin{aligned}\varpi(z,t) \ &= \ \int_0^t F_2(z,t-p)\left((1+a_4)\left(\frac{p^{-\gamma}}{\Gamma(1-\gamma)} + R\right) + a_6E_{\gamma}(a_5p^{\gamma})\right. \\ &\quad \left. - \frac{Ra_6(1-E_{\gamma}(a_5p^{\gamma}))}{(a_5)}\right)dp - \\ &\quad \int_0^t F_1(z,t-p)\left(a_4\left(\frac{p^{-\gamma}}{\Gamma(1-\gamma)} - Q\right) + a_6E_{\gamma}(a_5p^{\gamma})\right. \\ &\quad \left. + \frac{-Qa_6}{(a_5)}(1-E_{\gamma}(a_5p^{\gamma}))\right)dp,\end{aligned} \tag{4.23}$$

where

$$F_2(z,t) = \int_0^{\infty} e^{-Rw}Erfc\left(\frac{z\sqrt{Sc}}{2\sqrt{w}}\right)t^{-1}\left(0,-\alpha,-wt^{-\alpha}\right)dw. \tag{4.24}$$

4.8. Sherwood Number

In order to find the rate of mass transfer, we use Eq. (4.22) in the following relation

$$\begin{aligned} Sh &= -\frac{\partial \varpi}{\partial z}\Big|_{z=0} = -L^{-1}\left\{\frac{\partial \bar{\varpi}}{\partial z}\Big|_{z=0}\right\} = \int_0^t \left((1+a_4)\frac{(t-p)^{-\gamma}}{\Gamma(1-\gamma)} + R + a_6 E^1_{\gamma,0}(a_5(t-p)^{\gamma}) + \right. \\ & \left. a_6 R E_{\gamma,\gamma}(a_5(t-p)^{\gamma})\right) p^{\frac{\gamma}{2}-1} E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(-Rp^{\gamma})dp - \int_0^t \left(a_4 \frac{(t-p)^{-\gamma}}{\Gamma(1-\gamma)} - Q + \right. \\ & \left. a_6 E^1_{\gamma,0}(a_5(t-p)^{\gamma}) - a_6 Q E_{\gamma,\gamma}(a_5(t-p)^{\gamma})\right) p^{\frac{\gamma}{2}-1} E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(Qp^{\gamma})dp. \end{aligned} \tag{4.25}$$

4.9. Calculation of Concentration With Caputo-Fabrizio

By taking the Laplace transform on Eq. (3.3), we obtain

$$\frac{Scq}{(1-\gamma)q+\gamma}\bar{\varpi}(z,q) = \frac{\partial^2 \bar{\varpi}(z,q)}{\partial z^2} + SrSc\frac{\partial^2 \bar{\vartheta}(z,q)}{\partial z^2} + ScR\bar{\varpi}(z,q). \tag{4.26}$$

The solution of Eq. (4.26), by using conditions of Eq. (4.20) is

$$\begin{aligned} \bar{\varpi}(z,q) = \frac{1}{q} e^{-z\sqrt{\frac{(q+a_8)}{(q+a_1)}a_7}} + \frac{S_r Sc(q-a_3)a_2}{q\Big[(q-a_3)a_2 - (q-a_8)a_7\Big]} \\ \left(e^{-z\sqrt{\frac{(q+a_8)}{(q+a_1)}a_7}} - e^{-z\sqrt{\frac{(q-a_3)}{(q+a_1)}a_2}}\right). \end{aligned} \tag{4.27}$$

Eq. (4.27) can also be written as

$$\bar{\varpi}(z,q) = \left[\frac{(1+a_{11})}{q} + \frac{a_{12}}{q-a_{10}}\right] e^{-z\sqrt{\frac{(q+a_8)}{(q+a_1)}a_7}} - \left[\frac{a_{11}}{q} + \frac{a_{12}}{q-a_{10}}\right] e^{-z\sqrt{\frac{(q-a_3)}{(q+a_1)}a_2}}. \tag{4.28}$$

Taking the inverse Laplace transform of Eq. (4.28), we find the following expression

$$\varpi(z,t) = (1+a_{11})\phi_2(z,t) + a_{12}\phi_3(z,t) - (a_{11})\phi_1(z,t) + a_{12}\phi_4(z,t), \tag{4.29}$$

where

$$\begin{aligned} \phi_2(z,t) = e^{-z\sqrt{a_7}} - \frac{z\sqrt{a_7}\sqrt{a_8-a_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{1}{\sqrt{t}} e^{(-a_1 t - \frac{z^2 a_7}{4w} - w)} \\ I_1(2\sqrt{(a_8-a_1)wt})dtdw, \end{aligned} \tag{4.30}$$

$$\begin{aligned} \phi_3(z,t) &= e^{a_{10}t} e^{-z\sqrt{a_7}} - \frac{z\sqrt{a_7}\sqrt{a_8-a_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{a_{10}t}}{\sqrt{t}} e^{(-a_{10}t - a_1 t - \frac{z^2 a_7}{4w} - w)} \times \\ & I_1(2\sqrt{(a_8-a_1)wt})dtdw, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \phi_4(z,t) &= e^{a_{10}t} e^{-z\sqrt{a_2}} - \frac{z\sqrt{a_2}\sqrt{-a_3-a_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{a_{10}t}}{\sqrt{t}} e^{(-a_{10}t - a_1 t - \frac{z^2 a_2}{4w} - w)} \times \\ & I_1(2\sqrt{(-a_3-a_1)wt})dtdw. \end{aligned} \tag{4.32}$$

4.10. Sherwood Number

From Eq. (4.28), Sh can be calculated in the same way as in Eq. (4.25) and is given by

$$Sh = (1 + a_{11})e_1 + a_{12}e_2 e^{a_{10}t} - a_{11}e_3 - a_{12}e_4 e^{a_{10}t}. \tag{4.33}$$

4.11. Calculation of Concentration With Atangaba-Baleanu

By taking the Laplace transform on Eq. (3.3), we obtain

$$\frac{Scq^{\gamma}}{(1-\gamma)q^{\gamma}+\gamma}\bar{\varpi}(z,q) = \frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2} + SrSc\frac{\partial^2\bar{\vartheta}(z,q)}{\partial z^2} + ScR\bar{\varpi}(z,q). \tag{4.34}$$

The solution of Eq. (4.34), by using conditions given in Eq. (4.20), we have

$$\bar{\varpi}(z,q) = \frac{1}{q}e^{-z\sqrt{\frac{(q^{\gamma}+a_8)}{(q^{\gamma}+a_1)}a_7}} + \frac{S_r Sc(q^{\gamma}-a_3)a_2}{q\left[(q^{\gamma}-a_3)a_2-(q^{\gamma}-a_8)a_7\right]} \left(e^{-z\sqrt{\frac{(q^{\gamma}+a_8)}{(q^{\gamma}+a_1)}a_7}} - e^{-z\sqrt{\frac{(q^{\gamma}-a_3)}{(q^{\gamma}+a_1)}a_2}}\right). \tag{4.35}$$

Eq. (4.35) can also be written as

$$\bar{\varpi}(z,q) = \left[\frac{(1+a_9)q^{\gamma}}{q} + \frac{a_{13}q^{\gamma}}{q(q^{\gamma}-a_{10})}\right]\frac{e^{-z\sqrt{\frac{(q^{\gamma}+a_8)}{(q^{\gamma}+a_1)}a_7}}}{q^{\gamma}} - \left[\frac{a_9 q^{\gamma}}{q} + \frac{a_{13}q^{\gamma}}{q(q^{\gamma}-a_{10})}\right]\frac{e^{-z\sqrt{\frac{(q^{\gamma}-a_3)}{(q^{\gamma}+a_1)}a_2}}}{q^{\gamma}}. \tag{4.36}$$

Taking the inverse Laplace transform of Eq. (4.36), we obtain

$$\varpi(z,t) = \int_0^t \varphi_2(z,t-p)\left(\frac{(1+a_9)p^{-\gamma}}{\Gamma(1-\gamma)} + a_{13}E_{\gamma}(a_{10}p^{\gamma})\right)dp - \int_0^t \varphi_1(z,t-p)\left(\frac{a_9 p^{-\gamma}}{\Gamma(1-\gamma)} + a_{13}E_{\gamma}(a_{10}p^{\gamma})\right)dp, \tag{4.37}$$

where

$$\varphi_2(z,t) = \int_0^{\infty}\left[e^{-z\sqrt{a_7}} - \frac{z\sqrt{a_7}\sqrt{a_8-a_1}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t \frac{1}{\sqrt{t}}e^{(-a_1 t - \frac{z^2 a_7}{4w} - w)} \times I_1(2\sqrt{(a_8-a_1)wt})dtdw\right]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz. \tag{4.38}$$

4.12. Sherwood Number

From Eq. (4.36), Sh can be calculated in the same way as in Eq. (4.25) and is given by

$$Sh = (1+a_9)e_1 + a_{13}e_2\frac{(1-E_{\gamma}(a_{10}t^{\gamma}))}{a_{10}} - a_9 e_3 - a_{13}e_4\frac{(1-E_{\gamma}(a_{10}t^{\gamma}))}{a_{10}}. \tag{4.39}$$

4.13. Calculation of Velocity With Caputo

By taking Laplace transform on Eq. (3.1), we find

$$q^{\gamma}\bar{w}(z,q)=\frac{\partial^2\bar{w}(z,q)}{\partial z^2}-M\bar{w}(z,q)-\frac{1}{k}\bar{w}(z,q)+Gr\bar{\vartheta}(z,q)+Gm\bar{\varpi}(z,q), \tag{4.40}$$

Boundary conditions satisfying Eq. (4.40) is

$$\bar{w}(0,q)=\frac{1}{q-a},\quad \bar{w}(z,q)\to 0,\ \ z\to\infty. \tag{4.41}$$

The solution Eq. (4.40), by using condition of Eq. (4.41), is

$$\bar{w}(z,q)=\frac{1}{q-a}e^{-z\sqrt{\frac{q^{\gamma}+H}{A}}}+\left(\frac{Gr}{q}-\frac{a_4Gm(q^{\gamma}-Q)}{q(q^{\gamma}-a_5)}\right)\frac{1}{b_1(q^{\gamma}-b_2)}\left(e^{-z\sqrt{\frac{q^{\gamma}+H}{A}}}-\right.$$
$$\left.e^{-z\sqrt{Pr(q^{\gamma}-Q)}}\right)+\left(\frac{Gm}{q}+\frac{a_4Gr(q^{\gamma}-Q)}{q(q^{\gamma}-a_5)}\right)\frac{1}{b_3(q^{\gamma}+b_4)}$$
$$\left(e^{-z\sqrt{\frac{q^{\gamma}+H}{A}}}-e^{-z\sqrt{Pr(q^{\gamma}+R)}}\right). \tag{4.42}$$

Eq. (4.42) can be written in suitable form as

$$\bar{w}(z,q)=\frac{(q^{\gamma}+H)}{q-a}\frac{e^{\frac{-z}{\sqrt{A}}\sqrt{q^{\gamma}+H}}}{q^{\gamma}+H}+\left[\frac{b_{17}}{q}+\frac{b_{18}}{q(q^{\gamma}-b_2)}-\frac{b_{19}}{q}-\frac{b_{20}}{q(q^{\gamma}-a_5)}\right.$$
$$\left.-\frac{b_{21}}{q(q^{\gamma}-b_2)}+\frac{b_{22}}{q}+\frac{b_{23}}{q(q^{\gamma}+b_4)}+\frac{b_{24}}{q}+\frac{b_{25}}{q(q^{\gamma}-a_5)}+\frac{b_{26}}{q(q^{\gamma}+b_4)}\right]\frac{e^{\frac{-z}{\sqrt{A}}\sqrt{q^{\gamma}+H}}}{q^{\gamma}+H}$$
$$-\left[\frac{b_{17}}{q}+\frac{b_{27}}{q(q^{\gamma}-b_2)}-\frac{b_{19}}{q}-\frac{b_{28}}{q(q^{\gamma}-a_5)}-\frac{b_{29}}{q(q^{\gamma}-b_2)}\right]\frac{e^{-z\sqrt{Pr(q^{\gamma}-Q)}}}{(q^{\gamma}-Q)}$$
$$-\left[\frac{b_{22}}{q}+\frac{b_{30}}{q(q^{\gamma}+b_4)}+\frac{b_{24}}{q}+\frac{b_{31}}{q(q^{\gamma}-a_5)}+\frac{b_{32}}{q(q^{\gamma}+b_4)}\right]\frac{e^{-z\sqrt{Sc(q^{\gamma}+R)}}}{(q^{\gamma}+R)}. \tag{4.43}$$

Taking inverse Laplace transform of Eq. (4.43), we have

$$\begin{aligned} w(z,t) \;=\; & \int_0^t\left[F_3(z,t-p)\left[g_1(p)+He^{at}+b_{17}-b_{19}+b_{22}+b_{24}+(b_{18}-b_{21})g_2(p)+\right.\right.\\ & \left.(b_{25}-b_{20})g_3(p)+(b_{23}+b_{26})g_4(p)\right]-F_1(z,t-p)\left[(b_{17}-b_{19})\right.\\ & \left.+(b_{27}-b_{29})g_2(p)-b_{28}g_3(p)\right]-F_2(z,t-p)\left[(b_{22}+b_{24})\right.\\ & \left.\left.+b_{31}g_3(p)+(b_{30}+b_{32})g_4(p)\right]\right]dp, \end{aligned} \tag{4.44}$$

where

$$F_3(z,t)=\int_0^{\infty}e^{-Hw}Erfc\left(\frac{z}{2\sqrt{Aw}}\right)t^{-1}\left(0,-\alpha,-wt^{-\alpha}\right)dw, \tag{4.45}$$

$$g_1(p)=-p^{\gamma}E_{1,1-\gamma}(ap), \tag{4.46}$$

$$g_2(p) = \frac{-1}{b_2}(1 - E_\gamma(b_2 p^\gamma)), \tag{4.47}$$

$$g_3(p) = \frac{-1}{a_5}(1 - E_\gamma(a_5 p^\gamma)), \tag{4.48}$$

$$g_4(p) = \frac{1}{b_4}(1 - E_\gamma(-b_4 p^\gamma)). \tag{4.49}$$

4.14. Skin Friction

In order to find the skin friction, we use Eq. (4.43) in the following relation

$$\begin{aligned} \tau \;=\; & -\frac{\partial w}{\partial z}\bigg|_{z=0} = -L^{-1}\left\{\frac{\partial \bar{w}}{\partial z}\bigg|_{z=0}\right\} = \int_0^t \Big[g_1(p) + He^{at} + b_{17} - b_{19} + b_{22} + b_{24} + \\ & (b_{18} - b_{21})g_2(p) + (b_{25} - b_{20})g_3(p) + (b_{23} + b_{26})g_4(p) \\ & \Big]\frac{1}{\sqrt{A}}(t-p)^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(-H(t-p)^\gamma) - \\ & \Big[(b_{17} - b_{19}) + (b_{27} - b_{29})g_2(p) - b_{28}g_3(p)\Big]\sqrt{Pr}(t-p)^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(Q(t-p)^\gamma) - \\ & \Big[(b_{22} + b_{24}) + b_{31}g_3(p) + (b_{30} + b_{32})g_4(p)\Big]\sqrt{Sc}(t-p)^{\frac{\gamma}{2}-1} \\ & E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(-R(t-p)^\gamma)\Big]dp. \end{aligned} \tag{4.50}$$

4.15. Calculation of Velocity With Caputo-Fabrizio

By taking the Laplace transform on Eq. (3.1), we obtain

$$\begin{aligned} \frac{q}{(1-\gamma)q+\gamma}\bar{w}(z,q) \;=\; & \frac{\partial^2 \bar{w}(z,q)}{\partial z^2} - M\bar{w}(z,q) - \frac{1}{k}\bar{w}(z,q) \\ & + Gr\bar{\vartheta}(z,q) + Gm\bar{\varpi}(z,q), \end{aligned} \tag{4.51}$$

with boundary conditions

$$\bar{w}(0,q) = \frac{1}{q-a}, \quad \bar{w}(z,q) \to 0, \;\; z \to \infty. \tag{4.52}$$

The solution of partial differential Eq. (4.51), by using conditions given in Eq. (4.52), we have

$$\begin{aligned} \bar{w}(z,q) \;=\; & \frac{1}{q-a}e^{-z\sqrt{\frac{(q+c_2)c_1}{(q+a_1)A}}} + \left[\frac{Gr}{q} - \frac{Gma_9(q-a_3)}{q(q-a_{10})}\right]\left[\frac{q+a_1}{Aa_2(q-a_3) - c_1(q+c_2)}\right] \times \\ & \left[e^{-z\sqrt{\frac{(q+c_2)c_1}{(q+a_1)A}}} - e^{-z\sqrt{\frac{q-a_3}{(q+a_1)}a_2}}\right] + \left[\frac{Gm}{q} + \frac{Gma_9(q-a_3)}{q(q-a_{10})}\right] \\ & \left[\frac{q+a_1}{Aa_7(q-a_3) - c_1(q+c_2)}\right]\left[e^{-z\sqrt{\frac{(q+c_2)c_1}{(q+a_1)A}}} - e^{-z\sqrt{\frac{q+a_8}{(q+a_1)}a_7}}\right]. \end{aligned} \tag{4.53}$$

The suitable form of Eq. (4.53) is

$$\begin{aligned}\bar{w}(z,q) &= \frac{1}{q-a}e^{-z\sqrt{\frac{(q+c_2)c_1}{(q+a_1)A}}} + \left[\frac{c_7}{q} + \frac{c_8}{q-c_4} - \frac{c_9}{q} - \frac{c_{10}}{q-a_{10}} - \frac{c_{11}}{q-c_4}\right]\left[e^{-z\sqrt{\frac{(q+c_2)c_1}{(q+a_1)A}}} - \right.\\ &\left. e^{-z\sqrt{\frac{q-a_3}{(q+a_1)}a_2}}\right] + \left[\frac{c_{12}}{q} + \frac{c_{13}}{q+c_6} + \frac{c_{14}}{q} + \frac{c_{15}}{q-a_{10}} + \frac{c_{16}}{q+c_6}\right] \times \\ &\left[e^{-z\sqrt{\frac{(q+c_2)c_1}{(q+a_1)A}}} - e^{-z\sqrt{\frac{q+a_8}{(q+a_1)}a_7}}\right]. \end{aligned} \tag{4.54}$$

Taking the inverse Laplace transform of Eq. (4.54), we have

$$\begin{aligned} w(z,t) &= \phi_5(z,t) + (c_7 - c_9 + c_{12} + c_{14})\phi_6(z,t) + (c_8 - c_{11})\phi_7(z,t) \\ &+ (c_{15} - c_{10})\phi_8(z,t) + (c_{13} + c_{16})\phi_9(z,t) \\ &- \Big[(c_7 - c_9)\phi_1(z,t) - c_{10}\phi_4(z,t) + (c_8 - c_{11})\phi_{10}(z,t)\Big] \\ &- \Big[(c_{12} + c_{14})\phi_2(z,t) + c_{15}\phi_3(z,t) + (c_{13} + c_{16})\phi_{11}(z,t)\Big], \end{aligned} \tag{4.55}$$

where

$$\phi_5(z,t) = e^{at}e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}}\int_0^\infty\int_0^t \frac{e^{at}}{\sqrt{t}}e^{\left(-at-a_1t-\frac{z^2c_1}{4Aw}-w\right)} I_1(2\sqrt{(c_2-a_1)wt})\,dt\,dw, \tag{4.56}$$

$$\phi_6(z,t) = e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}}\int_0^\infty\int_0^t \frac{1}{\sqrt{t}}e^{\left(-a_1t-\frac{z^2c_1}{4Aw}-w\right)} I_1(2\sqrt{(c_2-a_1)wt})\,dt\,dw, \tag{4.57}$$

$$\phi_7(z,t) = e^{c_4t}e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}}\int_0^\infty\int_0^t \frac{e^{c_4t}}{\sqrt{t}}e^{\left(-c_4t-a_1t-\frac{z^2c_1}{4Aw}-w\right)} I_1(2\sqrt{(c_2-a_1)wt})\,dt\,dw, \tag{4.58}$$

$$\phi_8(z,t) = e^{b_{10}t}e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}}\int_0^\infty\int_0^t \frac{e^{b_{10}t}}{\sqrt{t}}e^{\left(-b_{10}t-a_1t-\frac{z^2c_1}{4Aw}-w\right)} I_1(2\sqrt{(c_2-a_1)wt})\,dt\,dw, \tag{4.59}$$

$$\phi_9(z,t) = e^{-c_6t}e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}}\int_0^\infty\int_0^t \frac{e^{-c_6t}}{\sqrt{t}}e^{\left(c_6t-a_1t-\frac{z^2c_1}{4Aw}-w\right)} I_1(2\sqrt{(c_2-a_1)wt})\,dt\,dw, \tag{4.60}$$

$$\phi_{10}(z,t) = e^{c_4t}e^{-z\sqrt{a_2}} - \frac{z\sqrt{a_2}\sqrt{-a_3-a_1}}{2\sqrt{\pi}}\int_0^\infty\int_0^t \frac{e^{c_4t}}{\sqrt{t}}e^{\left(-c_4t-a_1t-\frac{z^2a_2}{4w}-w\right)} I_1(2\sqrt{(-a_3-a_1)wt})\,dt\,dw, \tag{4.61}$$

$$\phi_{11}(z,t) = e^{-c_6t}e^{-z\sqrt{a_7}} - \frac{z\sqrt{a_7}\sqrt{a_8-a_1}}{2\sqrt{\pi}}\int_0^\infty\int_0^t \frac{e^{-c_6t}}{\sqrt{t}}e^{\left(c_6t-a_1t-\frac{z^2a_7}{4w}-w\right)} I_1(2\sqrt{(a_8-a_1)wt})\,dt\,dw. \tag{4.62}$$

4.16. Skin Friction

From Eq. (4.54), the τ can be calculated in the same way as in Eq. (4.50) and is given by

$$\begin{aligned}\tau &= e_5 e^{at} + (c_7 - c_9 + c_{12} + c_{14})e_6 + (c_8 - c_{11})e_7 e^{c_4 t} + (c_{15} - c_{10})e_8 e^{a_{10} t} + \\ &(c_{16} - c_{11})e_9 e^{-c_6 t} - (c_7 - c_9)e_3 - (c_8 - c_{11})e_{10} e^{c_4 t} + c_{10} e_4 e^{a_{10} t} - \\ &(c_{12} + c_{14})e_1 - c_{15} e_2 e^{a_{10} t} - (c_{13} + c_{16})e_{11} e^{-c_6 t}. \end{aligned} \tag{4.63}$$

4.17. Calculation of Velocity With Atangana-Baleanu

By taking the Laplace transform on Eq.(3.1), we obtain

$$\begin{aligned}\frac{q^\gamma}{(1-\gamma)q^\gamma + \gamma}\bar{w}(z,q) &= \frac{\partial^2 \bar{w}(z,q)}{\partial z^2} - M\bar{w}(z,q) - \frac{1}{k}\bar{w}(z,q) \\ &+ Gr\bar{\vartheta}(z,q) + Gm\bar{\varpi}(z,q), \end{aligned} \tag{4.64}$$

with boundary conditions

$$\bar{w}(0,q) = \frac{1}{q-a}, \quad \bar{w}(z,q) \to 0, \ z \to \infty. \tag{4.65}$$

The solution of partial differential Eq. (4.64), by using conditions given in Eq. (4.65), we have

$$\begin{aligned}\bar{w}(z,q) &= \frac{1}{q-a} e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} + \left[\frac{Gr}{q} - \frac{Gm a_9 (q^\gamma - a_3)}{q(q^\gamma - a_{10})}\right]\left[\frac{q^\gamma + a_1}{A a_2 (q^\gamma - a_3) - c_1(q^\gamma + c_2)}\right] \times \\ &\left[e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} - e^{-z\sqrt{\frac{q^\gamma - a_3}{(q^\gamma+a_1)} a_2}}\right] + \left[\frac{Gm}{q} + \frac{Gm a_9 (q^\gamma - a_3)}{q(q^\gamma - a_{10})}\right] \times \\ &\left[\frac{q^\gamma + a_1}{A a_7 (q^\gamma - a_3) - c_1 (q^\gamma + c_2)}\right]\left[e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} - e^{-z\sqrt{\frac{q^\gamma + a_8}{(q^\gamma+a_1)} a_7}}\right]. \end{aligned} \tag{4.66}$$

Rearrangement of Eq. (4.66) is

$$\begin{aligned}\bar{w}(z,q) &= \frac{1}{q-a} e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} + \left[\frac{d_1}{q}\left(1 + \frac{a_1 + c_4}{q^\gamma - c_4}\right) + \frac{d_2 a_9}{q}\left(1 + \frac{d_6 (q^\gamma - d_7)}{(q^\gamma - a_{10})(q^\gamma - c_4)}\right)\right] \times \\ &\left[e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} - e^{-z\sqrt{\frac{q^\gamma - a_3}{(q^\gamma+a_1)} a_2}}\right] + \left[\frac{d_3}{q}\left(1 + \frac{a_1 - c_6}{q^\gamma} + c_6\right) + \right. \\ &\left.\frac{d_3 a_9}{q}\left(1 + \frac{d_8 (q^\gamma + d_9)}{(q^\gamma - a_{10})(q^\gamma + c_6)}\right)\right]\left[e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} - e^{-z\sqrt{\frac{q^\gamma + a_8}{(q^\gamma+a_1)} a_7}}\right]. \end{aligned} \tag{4.67}$$

The suitable form of Eq. (4.67) is

$$\begin{aligned}\bar{w}(z,q) &= \frac{q^\gamma}{q-a} \frac{e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}}}{q^\gamma} + \left[\frac{d_{10} q^\gamma}{q^{1+\gamma}} + \frac{d_{11}}{q(q^\gamma - c_4)} - \frac{d_{12}}{q(q^\gamma - a_{10})}\right]\left[e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} - \right. \\ &\left. e^{-z\sqrt{\frac{q^\gamma - a_3}{(q^\gamma+a_1)} a_2}}\right] + \left[\frac{d_{13} q^\gamma}{q^{1+\gamma}} + \frac{d_{14}}{q(q^\gamma + c_6)} + \frac{d_{15}}{q(q^\gamma - a_{10})}\right] \times \\ &\left[e^{-z\sqrt{\frac{(q^\gamma+c_2)c_1}{(q^\gamma+a_1)A}}} - e^{-z\sqrt{\frac{q^\gamma + a_8}{(q^\gamma+a_1)} a_7}}\right]. \end{aligned} \tag{4.68}$$

Taking the inverse Laplace transform of Eq. (4.68), we have

$$\begin{aligned} w(z,t) &= \int_0^t \Big[g_1(p)\varphi_3(z,t-p) + (d_{10}+d_{13})\varphi_3(z,t-p)\frac{p^{-\gamma}}{\Gamma(1-\gamma)} + \varphi_4(z,t-p)d_{11} + \\ &\varphi_5(z,t-p)(d_{15}-d_{12}) + \varphi_6(z,t-p)d_{14} - d_{10}\varphi_1(z,t-p)\frac{p^{-\gamma}}{\Gamma(1-\gamma)} - \\ &\varphi_7(z,t-p)d_{11} + \varphi_8(z,t-p)d_{12} - d_{13}\varphi_2(z,t-p)\frac{p^{-\gamma}}{\Gamma(1-\gamma)} \\ &-\varphi_9(z,t-p)d_{14} - \varphi_{10}(z,t-p)d_{15}\Big] dp, \end{aligned} \tag{4.69}$$

where

$$\begin{aligned} \varphi_3(z,t) &= \int_0^\infty \Big[e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}} \int_0^\infty \int_0^t \frac{1}{\sqrt{t}} e^{(-a_1 t - \frac{z^2 c_1}{4Aw} - w)} \times \\ &I_1(2\sqrt{(c_2-a_1)wt})dtdw \Big] \times t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx. \end{aligned} \tag{4.70}$$

$$\begin{aligned} \varphi_4(z,t) &= \int_0^\infty \Big[e^{c_4 t} e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{c_4 t}}{\sqrt{t}} e^{(-c_4 t - a_1 t - \frac{z^2 c_1}{4Aw} - w)} \times \\ &I_1(2\sqrt{(c_2-a_1)wt})dtdw \Big] t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx, \end{aligned} \tag{4.71}$$

$$\begin{aligned} \varphi_5(z,t) &= \int_0^\infty \Big[e^{a_{10} t} e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{a_{10} t}}{\sqrt{t}} e^{(-a_{10} t - a_1 t - \frac{z^2 c_1}{4Aw} - w)} \times \\ &I_1(2\sqrt{(c_2-a_1)wt})dtdw \Big] t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx, \end{aligned} \tag{4.72}$$

$$\begin{aligned} \varphi_6(z,t) &= \int_0^\infty \Big[e^{-c_6 t} e^{-z\sqrt{\frac{c_1}{A}}} - \frac{z\sqrt{c_1}\sqrt{c_2-a_1}}{2\sqrt{A}\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{-c_6 t}}{\sqrt{t}} e^{(c_6 t - a_1 t - \frac{z^2 c_1}{4Aw} - w)} \times \\ &I_1(2\sqrt{(c_2-a_1)wt})dtdw \Big] t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx, \end{aligned} \tag{4.73}$$

$$\begin{aligned} \varphi_7(z,t) &= \int_0^\infty \Big[e^{c_4 t} e^{-z\sqrt{a_2}} - \frac{z\sqrt{a_2}\sqrt{-a_3-a_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{c_4 t}}{\sqrt{t}} e^{(-c_4 t - a_1 t - \frac{z^2 a_2}{4w} - w)} \times \\ &I_1(2\sqrt{(-a_3-a_1)wt})dtdw \Big] t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx, \end{aligned} \tag{4.74}$$

$$\begin{aligned} \varphi_8(z,t) &= \int_0^\infty \Big[e^{a_{10} t} e^{-z\sqrt{a_2}} - \frac{z\sqrt{a_2}\sqrt{-a_3-a_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{a_{10} t}}{\sqrt{t}} e^{(-a_{10} t - a_1 t - \frac{z^2 a_2}{4w} - w)} \times \\ &I_1(2\sqrt{(-a_3-a_1)wt})dtdw \Big] t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx, \end{aligned} \tag{4.75}$$

$$\varphi_9(z,t) = \int_0^\infty \left[e^{-c_6 t} e^{-z\sqrt{a_7}} - \frac{z\sqrt{a_7}\sqrt{a_8-a_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{-c_6 t}}{\sqrt{t}} e^{(c_6 t - a_1 t - \frac{z^2 a_7}{4w} - w)} \times \right.$$
$$\left. I_1(2\sqrt{(a_8-a_1)wt})\,dt\,dw \right] t^{-1}\psi(0,-\gamma,-xt^{-\gamma})\,dx, \tag{4.76}$$

$$\varphi_{10}(z,t) = \int_0^\infty \left[e^{a_{10} t} e^{-z\sqrt{a_7}} - \frac{z\sqrt{a_7}\sqrt{a_8-a_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{a_{10} t}}{\sqrt{t}} e^{(-a_{10} t - a_1 t - \frac{z^2 a_7}{4w} - w)} \times \right.$$
$$\left. I_1(2\sqrt{(a_8-a_1)wt})\,dt\,dw \right] t^{-1}\psi(0,-\gamma,-xt^{-\gamma})\,dx. \tag{4.77}$$

4.18. Skin Friction

From Eq. (4.68), the τ can be calculated in the same way as in Eq. (4.50) and is given by

$$\tau = e_5 e^{at} + (d_{10}+d_{13})e_6 + d_{11}e_7 \frac{(1-E_\gamma(c_4 t^\gamma))}{-c_4} + (d_{15}-d_{12})e_8 \frac{(1-E_\gamma(a_{10} t^\gamma))}{-a_{10}} +$$
$$d_{14}e_9 \frac{(1-E_\gamma(-c_6 t^\gamma))}{c_6} - d_{10}e_3 - d_{11}e_{10} \frac{(1-E_\gamma(c_4 t^\gamma))}{-c_4} + d_{12}e_2 \frac{(1-E_\gamma(a_{10} t^\gamma))}{-a_{10}} -$$
$$d_{13}e_1 - d_{14}e_2 \frac{(1-E_\gamma(a_{10} t^\gamma))}{-a_{10}} + d_{16}e_{11} \frac{(1-E_\gamma(-c_6 t^\gamma))}{c_6}. \tag{4.78}$$

5. Results and Discussion

The exact solutions for the comparison among fluid models are obtained through Caputo-Fabrizio, Caputo, and Atangana-Baleanu fractional derivatives. MHD Casson fluid flow through a vertical plate with mass diffusion has been discussed in this work. The effect of various flow parameters is plotted graphically.

Fig. 1 represents the effect of Gr on the velocity of the fluid, it is examined that an increasing value of Gr accelerates the fluid velocity. Gr is the ratio of buoyancy forces due to temperature gradient to viscous hydrodynamics forces, therefore growth in buoyancy forces rise the convectional effects. Similarly, growth in buoyancy forces due to concentration gradient Gm also rise the convectional effects as shown in Fig. 2.

Fig. 3 shows the impact of M on velocity fields. From this graph, it is clear that an increasing value of magnetic parameter M slows down the velocity profile due to Lorentz's force. The effect of the permeability of porosity K on the velocity profile is shown in Fig. 4. It is noted that for increasing the velocity profile, we need to increase the value of K due to increased drag force. The effect of Pr on the velocity profile is highlighted in Fig. 5. It is analyzed that increasing the value of Pr decreases the velocity profile. It is due to larger momentum and thermal diffusion, which slows down the fluid motion. Fig. 6 shows the effect of Sr on velocity profiles. Hence, it is depicted that increasing value of Sr enhanced the fluid motion due to growth in the molecular diffusivity. Fig. 7 shows the effect of the Casson parameter on fluid motion. From this Fig. it is found that velocity increases with increasing value of Casson parameter β. Fig. 8 part (a) represents that ordinary fluid moves faster as compared to ordinary and fractionalized Casson fluid. Part (b) of Fig. 8 and part (a) of Fig. 9 gives the comparison of the present work with Olisa [32] by taking $f(q) = 0$. If we take $f(q) = R = Sc = Gm = 0, Sr = 1, K = \beta = \infty$, and $\gamma \to 1$, then observed identical results with Olisa [32] which shows the validation of present work. Part (b) of Fig. 9 and Fig. 10 gives a comparison of the present work with Nehad et al. [33]. If we put $M = R = Q = Sc = Gm = 0, Sr = 1$, and K= Casson parameter $\beta = \infty$, then the fluid velocity with Caputo-Fabrizio and Caputo fractional derivatives are identical of present work and Nehad et al [33] but the fluid velocity with Atangana-Baleanu fractional derivative is slower as compared to Caputo-Fabrizio and Caputo fractional derivatives as shown in part (a) of Fig. 10. If we take fractional parameter $\gamma \to 1$ then velocity profiles become the same as shown in part (b) of Fig. 10.

6. Conclusion

In this paper, the model of Casson's fluid is generalized to the fractional model of order γ. Comparative analysis of the fractional model has been made among CF, C, and AB approaches. The most significant remarks of the study can be summarized as follow:

- Fluid velocity increases with increasing values of Sorret effect Sr.
- Fluid velocity becomes faster with increasing values of Gr, Gm, and K.
- Fluid velocity slows down with increasing values of M,Pr.
- Fluid velocity increases with increasing value of fractional parameter γ.
- Atangana-Baleanu fractional derivative is the best approach to obtain control fluid velocity.
- The Advantage of fractional derivative over usual the derivative is that the usual derivative can not be used for fractional number.

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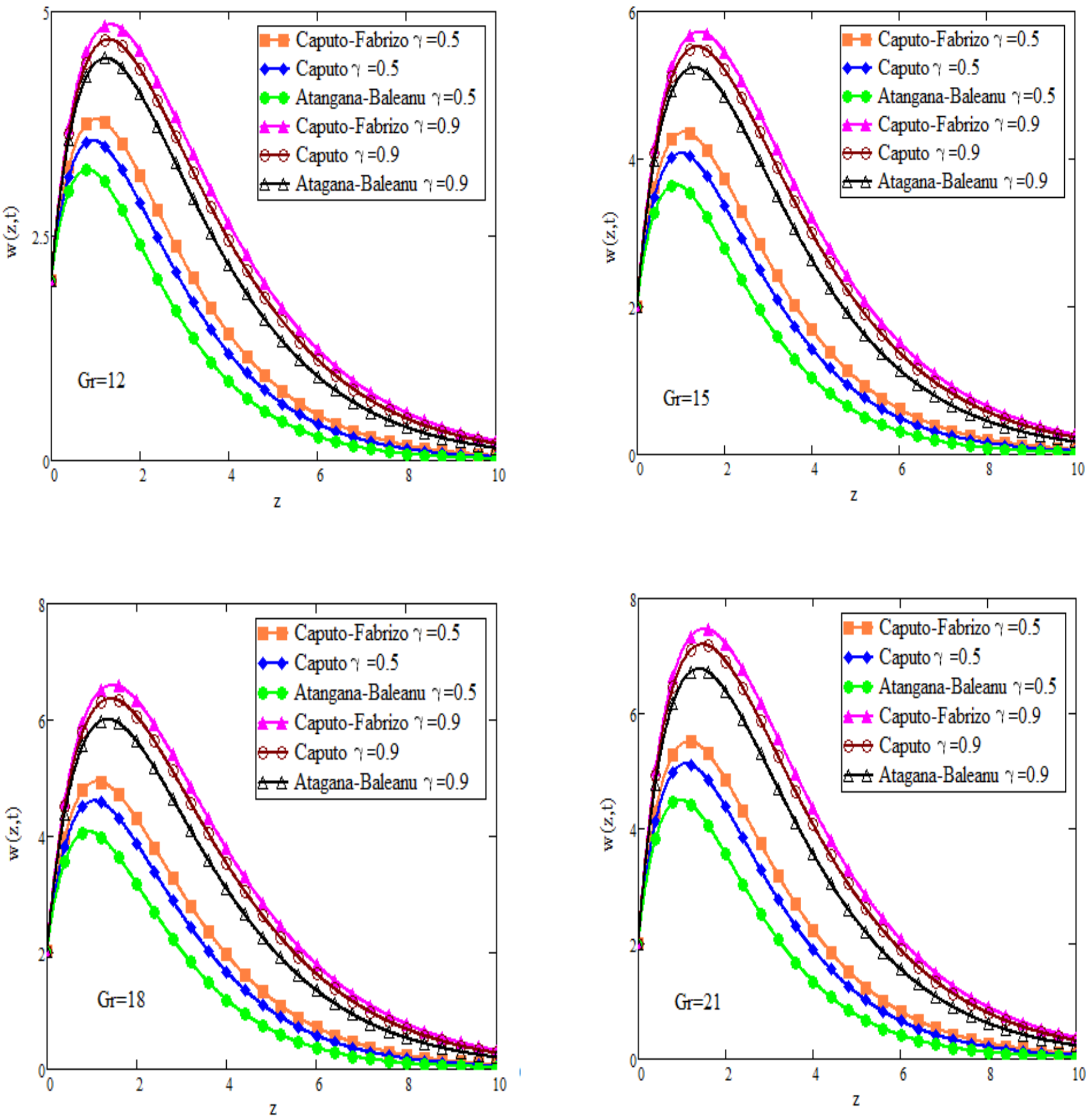

Figure 1: $w(z,t)$ for different valves of Gr where the values of other parameters are $K = 2.4, Gm = 6, \beta = 0.35, Q = 0.4, M = 0.6, Sc = 2.5, \gamma = 0.5, Sr = 0.2, Pr = 2, t = 0.5, R = 0.4$.

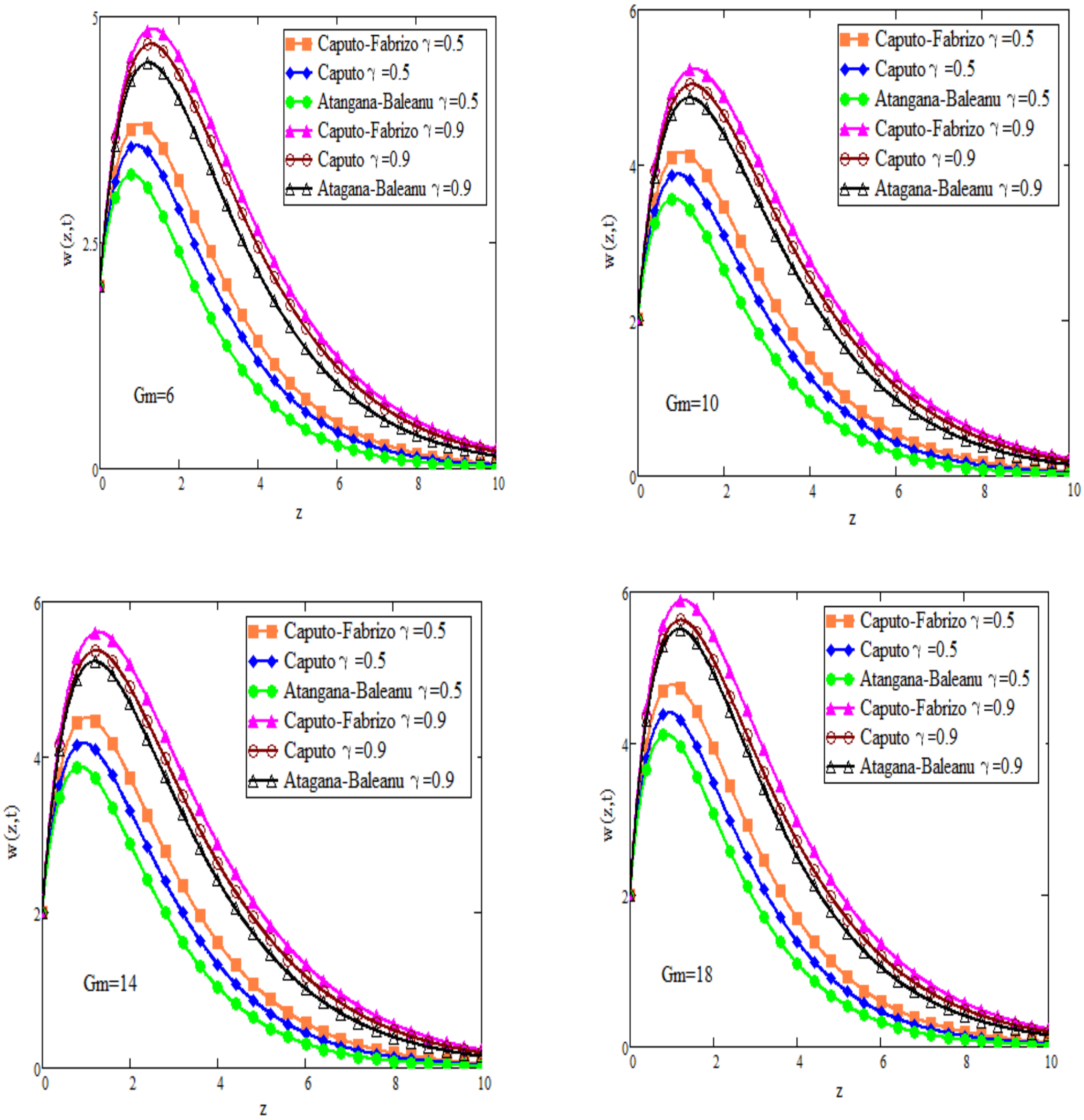

Figure 2: $w(z,t)$ for different valves of Gm where the values of other parameters are $Gr = 12, \beta = 0.35, Q = 0.4, t = 0.5, M = 0.6, K = 2.4, Sc = 2.5, \gamma = 0.5, Sr = 0.2, Pr = 2, R = 0.4$.

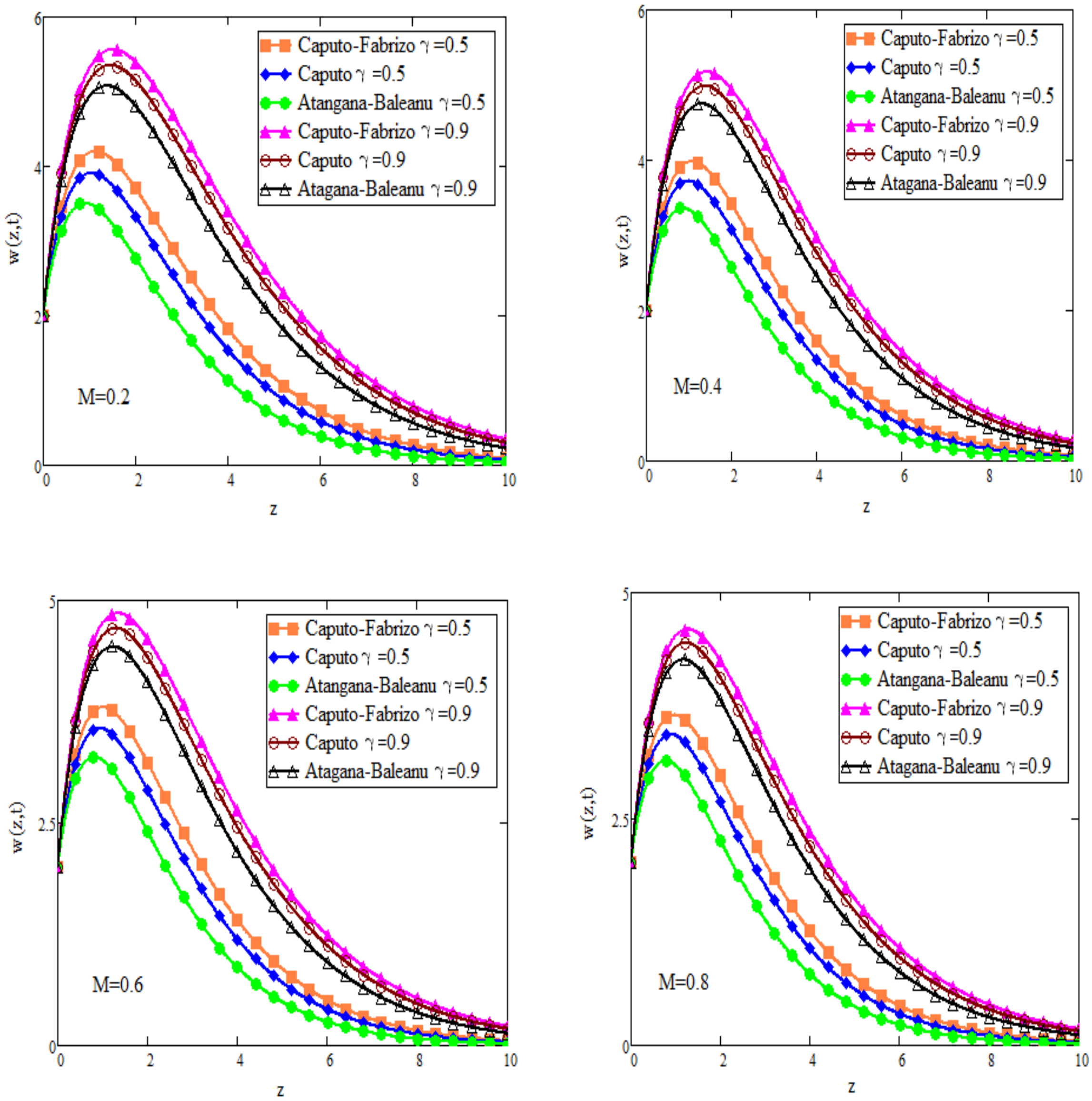

Figure 3: $w(z,t)$ for different valves of M where the values of other parameters are $\mathrm{Gr} = 12, \beta = 0.35, \mathrm{Gm} = 6, \mathrm{Q} = 0.4, \mathrm{t} = 0.5, \mathrm{K} = 2.4, \mathrm{Sc} = 2.5, \gamma = 0.5, \mathrm{Sr} = 0.2, \mathrm{Pr} = 2, \mathrm{R} = 0.4$.

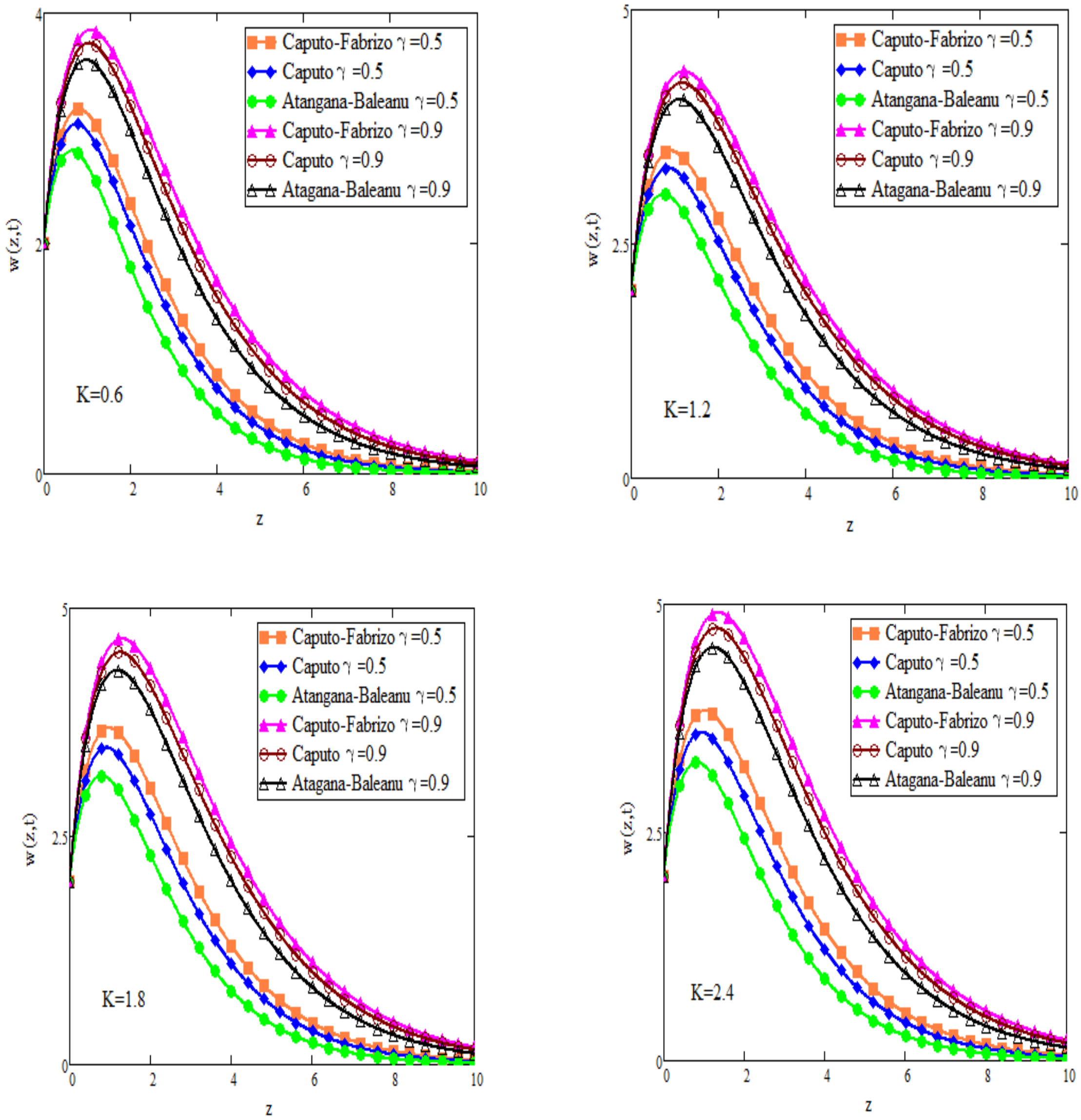

Figure 4: $w(z,t)$ for different valves of K where the values of other parameters are $Gr = 12, \beta = 0.35, Gm = 6, Q = 0.4, t = 0.5, Sc = 2.5, \gamma = 0.5, Sr = 0.2, Pr = 2, R = 0.4$.

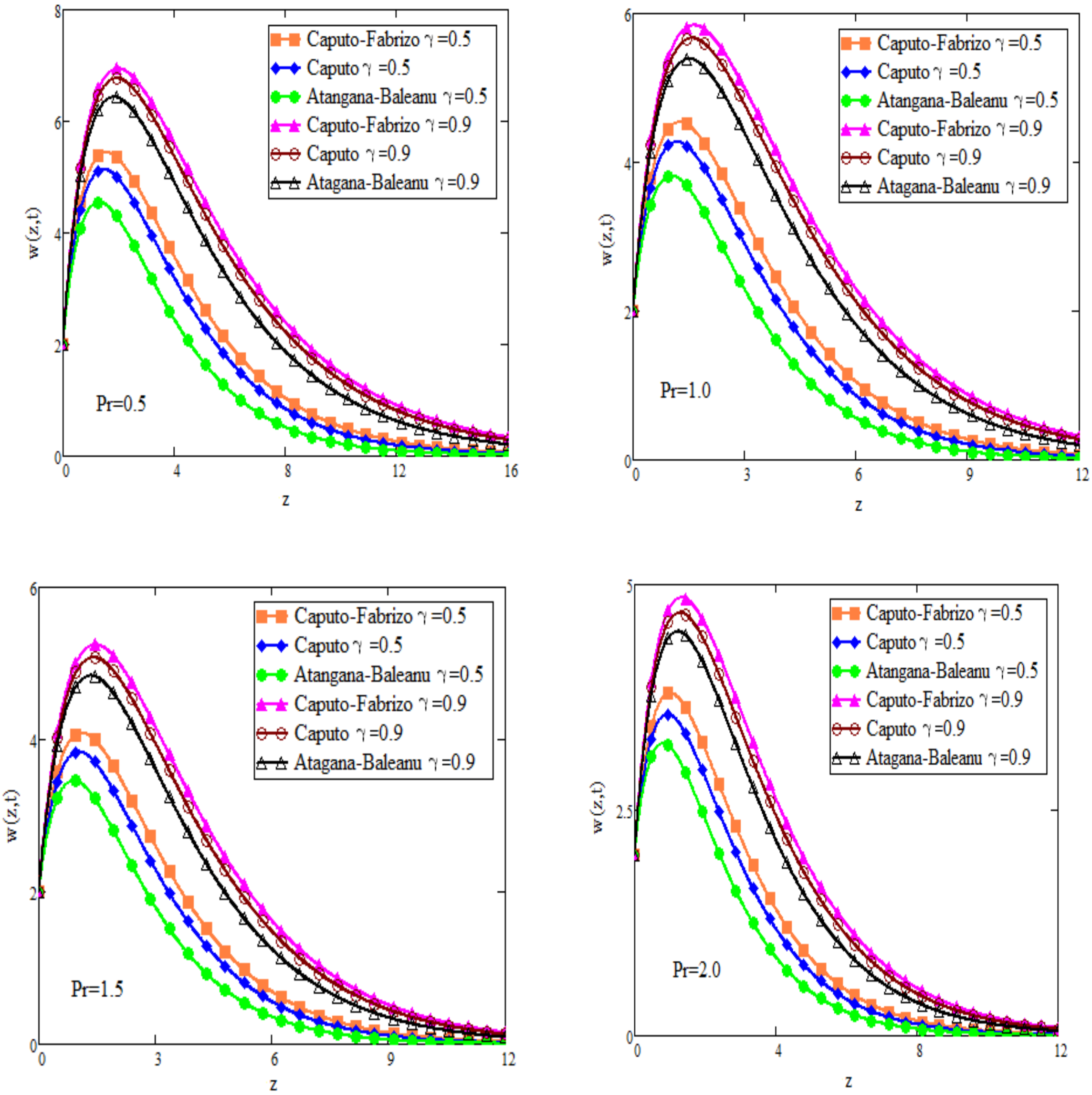

Figure 5: $w(z,t)$ for different valves of Pr where the values of other parameters are $\mathrm{Gr} = 12, \beta = 0.35, \mathrm{Gm} = 6, \mathrm{Q} = 0.4, \mathrm{t} = 0.5, \mathrm{K} = 2.4, \mathrm{Sc} = 2.5, \gamma = 0.5, \mathrm{Sr} = 0.2, \mathrm{R} = 0.4$.

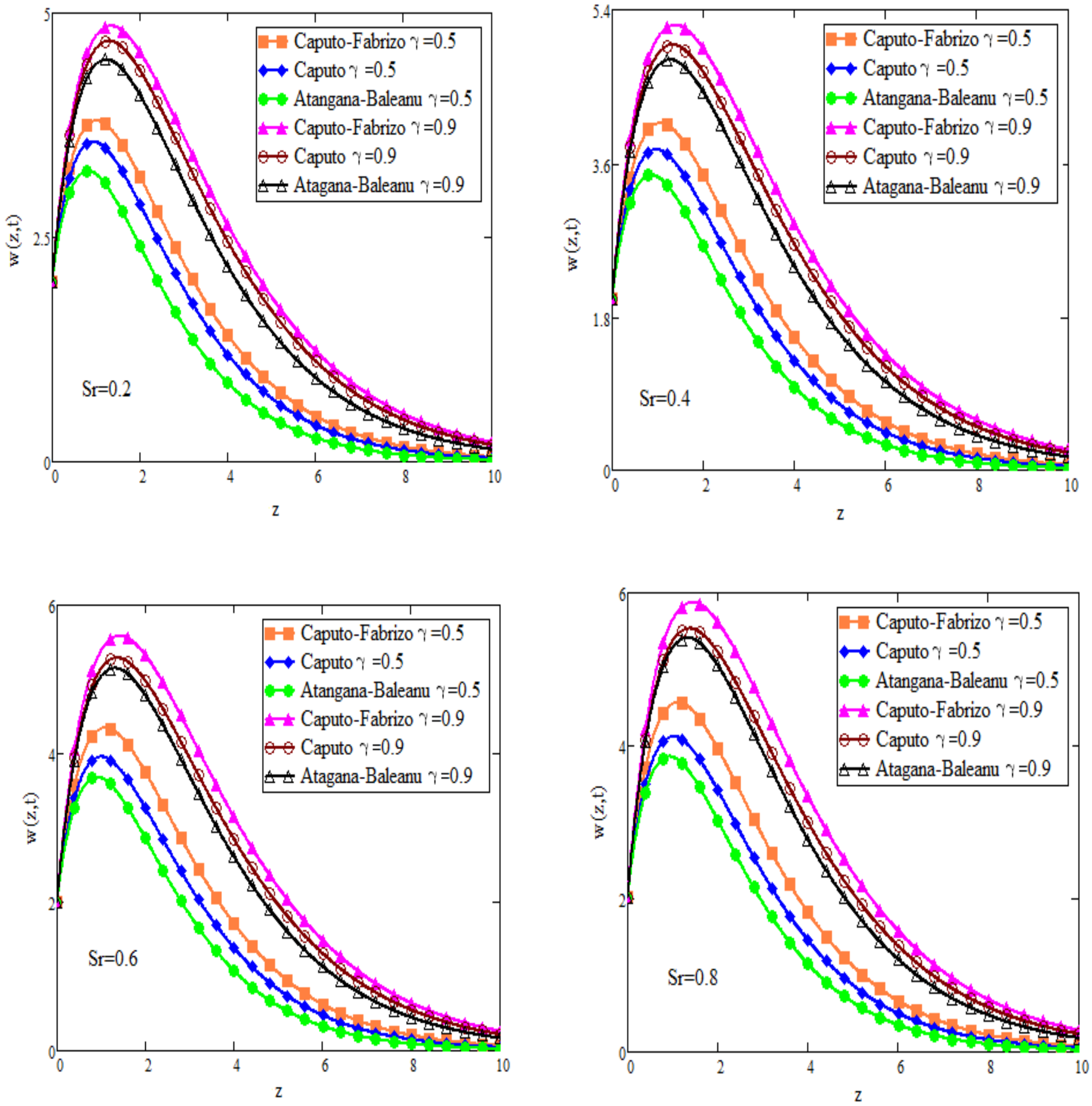

Figure 6: Velocity profile against z due to S_r where the values of other parameters are $Gr = 12, Gm = 6, \beta = 0.35, Q = 0.4, t = 0.5, K = 2.4, Sc = 2.5, \gamma = 0.5, Pr = 2, R = 0.4$.

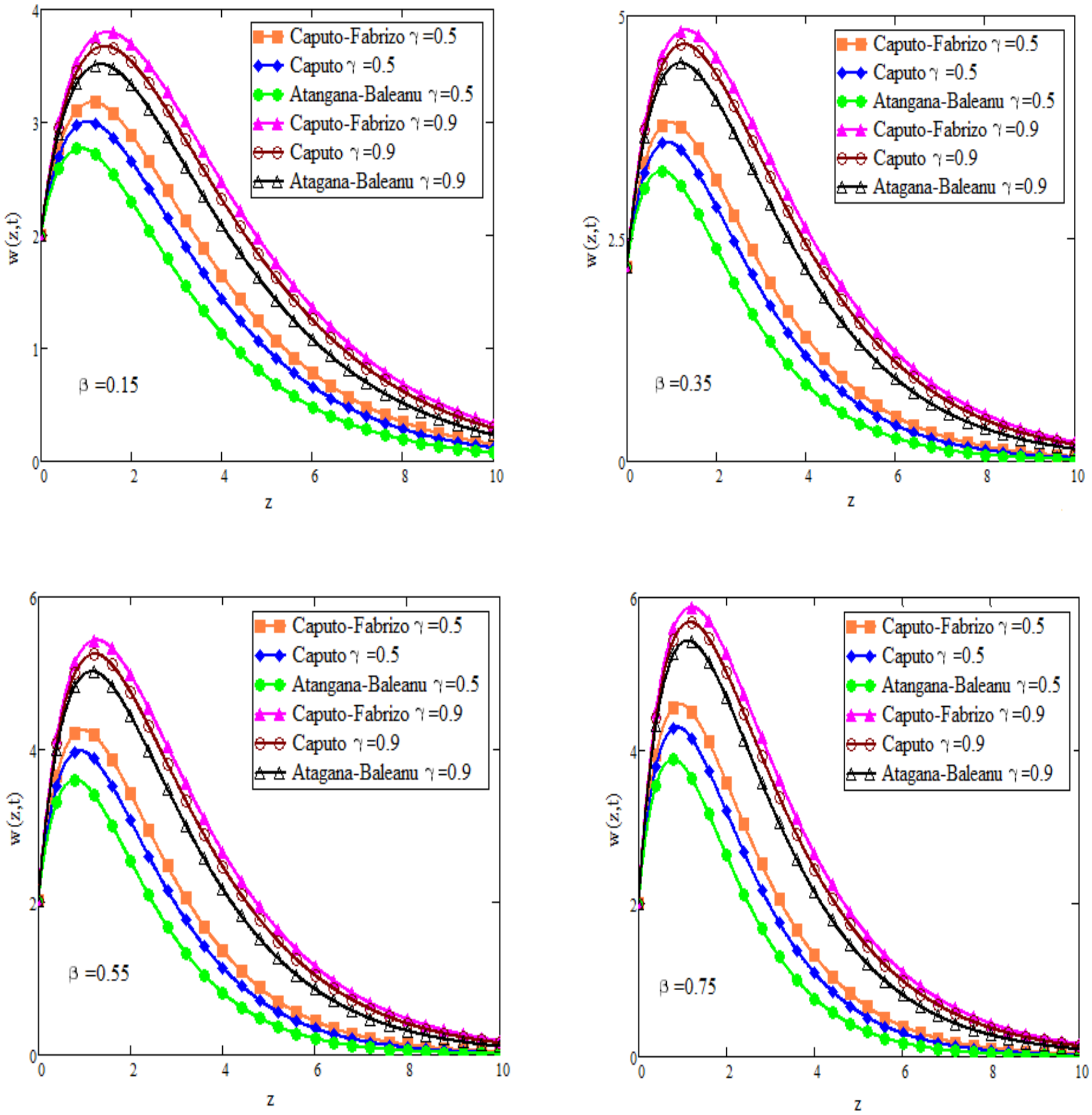

Figure 7: $w(z, t)$ for β where the values of other parameters are $Gr = 12, Gm = 6, Q = 0.4, t = 0.5, K = 2.4, Sc = 2.5, \gamma = 0.5, Sr = 0.2, Pr = 2, R = 0.4$.

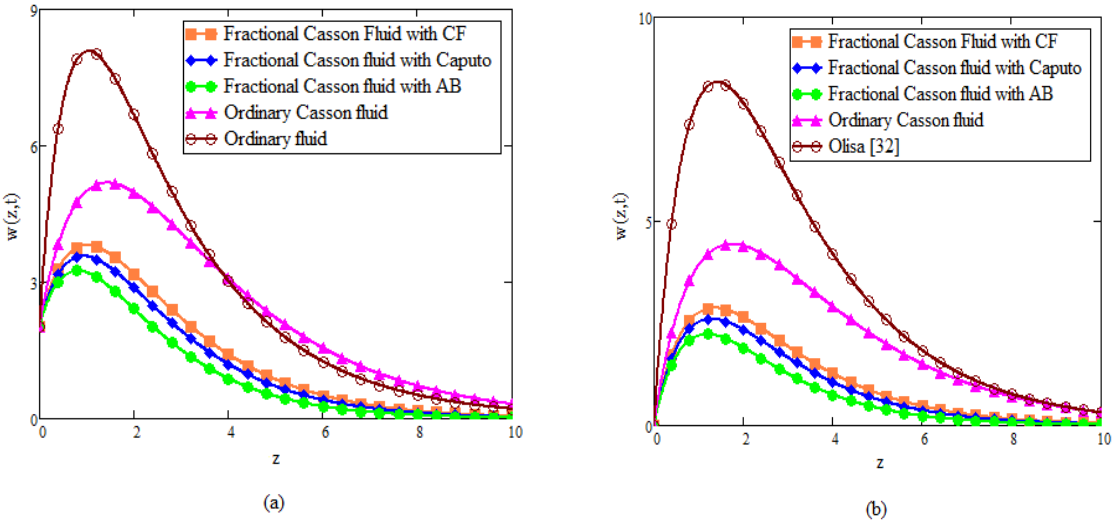

Figure 8: Profile of velocity comparison against z

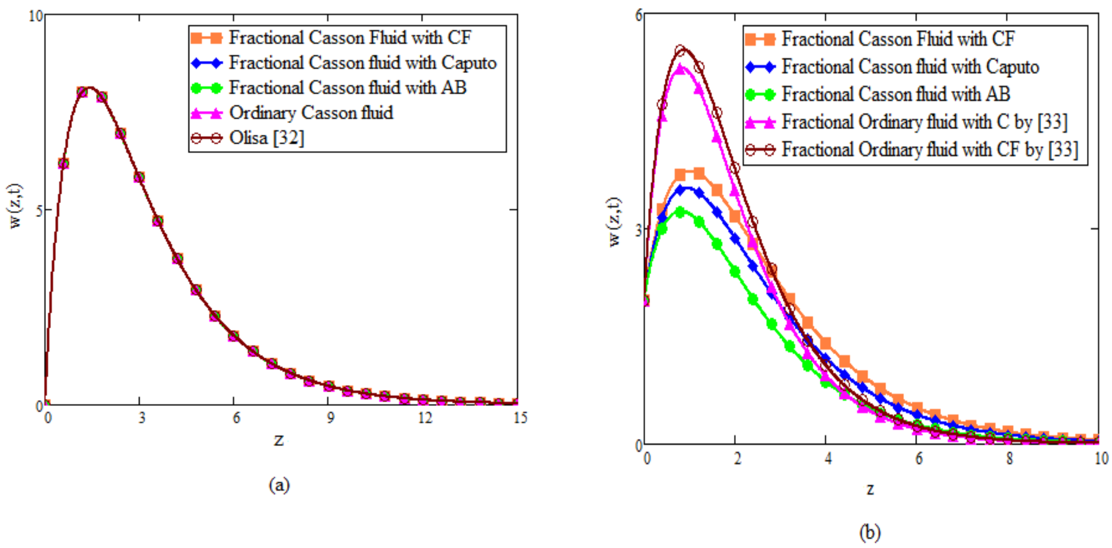

Figure 9: Profile of velocity comparison against z

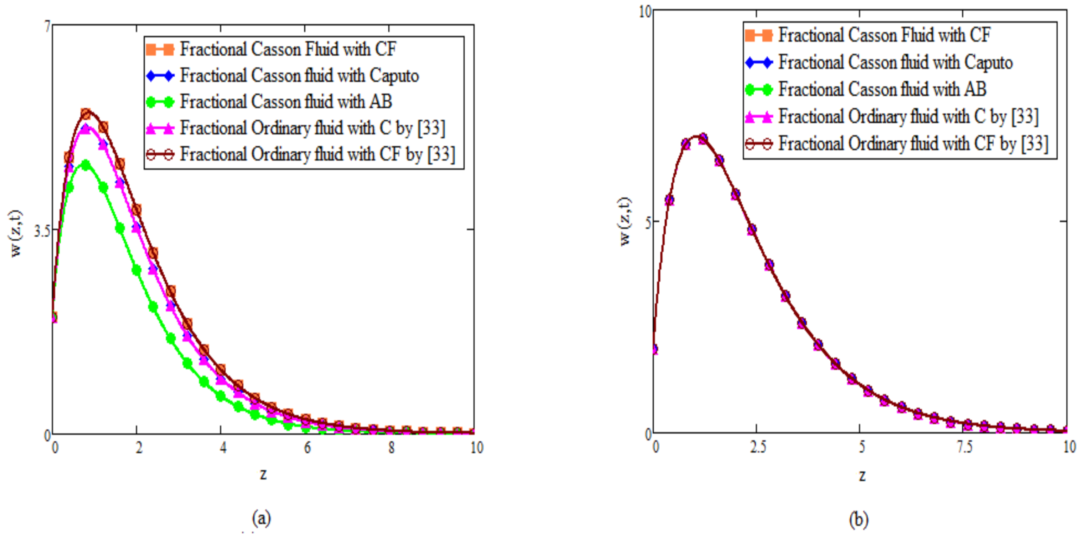

Figure 10: Profile of velocity comparison against z

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7. Appendix

$$a_1 = \frac{\gamma}{(1-\gamma)},\ a_2 = \frac{Pr - PrQ(1-\gamma)}{(1-\gamma)},\ a_3 = \frac{\gamma QPr}{Pr - PrQ(1-\gamma)},\ a_4 = \frac{SrPrSc}{Pr - Sc},$$
$$a_5 = \frac{PrQ + ScR}{Pr - Sc}, a_6 = a_4[a_5 - Q],\ a_7 = \frac{Sc + ScR(1-\gamma)}{(1-\gamma)},\ a_8 = \frac{\gamma RSc}{Sc + ScR(1-\gamma)},$$
$$a_9 = \frac{Sra_2Sc}{a_2 - a_7},\ a_{10} = \frac{a_2a_3 + a_7a_8}{a_2 - a_7}, a_{11} = \frac{a_3a_9}{a_{10}},\ a_{12} = \frac{a_9[a_{10} - a_3]}{a_{10}},$$
$$a_{13} = a_9[a_{10} - a_3],\ b_1 = APr - 1,\ b_2 = \frac{APrQ + H}{APr - 1},\ b_3 = ASc - 1$$
$$b_4 = \frac{AScR - H}{ASc - 1},\ b_5 = b_2 + H,\ b_6 = H - b_4,\ b_7 = H - Q + b_2 + a_5,\ b_8 = HQ + a_5b_2,$$
$$b_9 = H - Q - b_4 + a_5,\ b_{10} = a_5b_4 - HQ,\ b_{11} = b_2 - Q,\ b_{12} = a_5 + b_2 - 2Q,$$
$$b_{13} = Q^2 - a_5b_2,\ b_{14} = R - b_4,\ b_{15} = R - Q + a_5 - b_4,\ b_{16} = a_5b_4 - RQ,$$
$$b_{17} = \frac{Gr}{b_1},\ b_{18} = \frac{Grb_5}{b_1},\ b_{19} = \frac{Gma_4}{b_1},\ b_{20} = \frac{b_{19}[a_5b_7 - b_8]}{a_5 - b_2}, b_{21} = \frac{b_{19}[b_2b_7 - b_8]}{b_2 - a_5},$$
$$b_{22} = \frac{Gm}{b_3},\ b_{23} = \frac{Gmb_6}{b_3},\ b_{24} = \frac{Gma_4}{b_3},\ b_{25} = \frac{b_{24}[a_5b_9 + b_{10}]}{a_5 + b_4},$$
$$b_{26} = \frac{b_{24}[b_4b_9 - b_{10}]}{a_5 + b_4},\ b_{27} = \frac{Grb_{11}}{b_1},\ b_{28} = \frac{b_{19}[a_5b_{12} + b_{13}]}{a_5 - b_2},\ b_{29} = \frac{b_{19}[b_2b_{12} + b_{13}]}{b_2 - b_5},$$
$$b_{30} = \frac{b_{14}Gm}{b_3},\ b_{31} = \frac{b_{24}[a_5b_{15} + b_{16}]}{a_5 + b_4},\ b_{32} = \frac{b_{24}[b_4b_{15} - b_{16}]}{a_5 + b_4},\ c_1 = \frac{1 + H(1-\gamma)}{(1-\gamma)},$$
$$c_2 = \frac{H\gamma}{1 + H(1-\gamma)},\ c_3 = Aa_2 - c_1, c_4 = \frac{Aa_2a_3 + c_1c_2}{c_3},\ c_5 = Aa_7 - c_1,$$
$$c_6 = \frac{Aa_7a_8 - c_1c_2}{c_5},\ c_7 = \frac{a_1Gr}{-c_3c_4},\ c_8 = \frac{Gr(a_1 + c_4)}{c_3c_4},\ c_9 = \frac{-a_1a_3Gm}{a_{10}c_3c_4},$$
$$c_{10} = \frac{Gm(a_{10} - a_3)(a_{10} + a_1)}{a_{10}(a_{10} - c_4)c_3},\ c_{11} = \frac{Gm(c_4 - a_3)(a_1 + c_4)}{c_4(c_4 - a_{10})c_3},\ c_{12} = \frac{a_1Gm}{c_5c_6},$$
$$c_{13} = \frac{Gm(c_6 - a_1)}{c_5c_6}, c_{14} = \frac{a_1a_3Gm}{a_{10}c_5c_6},\ c_{15} = \frac{Gm(a_{10} - a_3)(a_{10} + a_1)}{a_{10}(a_{10} + c_6)c_5},$$
$$c_{16} = \frac{Gm(c_6 + a_3)(c_6 - a_1)}{c_5(c_6 + a_{10})c_6},\ d_1 = \frac{Gr}{c_3},\ d_2 = \frac{Gm}{c_3},$$
$$d_3 = \frac{Gm}{c_5},\ d_4 = d_1(a_1 + c_4),\ d_5 = d_3(a_1 - c_6),\ d_6 = a_1 - a_3 + a_{10} + c_4,$$
$$d_7 = \frac{a_1a_3 + a_{10}c_4}{d_6}, d_8 = a_1 - a_3 + a_{10} - c_6,\ d_9 = \frac{a_{10}c_6 - a_1a_3}{d_8},$$
$$d_{10} = d_1 - d_2a_9,\ d_{11} = d_4 - \frac{(d_2d_6a_9)(c_4 - d_7)}{c_4 - a_{10}}, d_{12} = \frac{d_2d_6a_9(a_{10} - d_7)}{a_{10} - c_4},$$

$$d_{13} = d_3 + a_9 d_3,\ d_{14} = \frac{d_3 d_8 a_9 (a_{10} + d_9)}{a_{10} + c_6},\ d_{15} = d_5 + \frac{d_3 d_8 a_9 (c_6 - d_9)}{a_{10} + c_6},$$

$$e_1 = \frac{\sqrt{a_7 a_8}}{a_1},\ e_2 = \frac{\sqrt{a_7 (a_{10} + a_8)}}{\sqrt{a_{10} + a_1}},\ e_3 = \frac{\sqrt{-a_2 a_3}}{a_1},$$

$$e_4 = \frac{\sqrt{a_2 (a_{10} - a_3)}}{\sqrt{a_{10} + a_1}},\ e_5 = \frac{\sqrt{c_1 (a + c_2)}}{a + a_1},$$

$$e_6 = \frac{\sqrt{c_1 c_2}}{a_1},\ e_7 = \frac{\sqrt{c_1 (c_4 + c_2)}}{\sqrt{c_4 + a_1}},\ e_8 = \frac{\sqrt{c_1 (a_{10} + c_2)}}{\sqrt{a_{10} + a_1}},\ e_9 = \frac{\sqrt{c_1 (-c_6 + c_2)}}{\sqrt{-c_6 + a_1}},$$

$$e_{10} = \frac{\sqrt{a_2 (c_4 - a_3)}}{\sqrt{c_4 + a_1}},\ e_{11} = \frac{\sqrt{a_7 (-c_6 + a_8)}}{\sqrt{-c_6 + a_1}}.$$

Nomenclature

K: Thermal conductivity of fluid
C_∞: Fluid concentration level far away from the plate
Sr: Non-Dimensional Soret number
Gm: Grashof number,$[\beta T_w]$
ρ: Density of fluid
K_1: Permeability of fluid
$\acute{C}$: Concentration of fluid
ξ: Fluid temperature
ϖ: Fluid concentration
T_w: Wall temperature at the wall
γ: Fractional parameter
Q :Non-Dimensional heat source
M: Magnetic parameter
ν: Kinematic viscosity
β : Casson parameter