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# Positive solution for a class of Caputo-type fractional differential equations

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## Abstract

In this paper, we investigate the existence and uniqueness of positive solutions for a class of Caputo-type fractional differential equations with nonlocal integral boundary conditions. Our analysis based on constructing the upper and lower control functions of the nonlinear terms without having any monotone conditions except the continuity, Green function, and Schauder's (Banach's) fixed point technique on a cone. Finally, some examples are given to substantiate our main results.

Keywords: Fractional differential equations, Caputo fractional derivative, upper and lower solutions, Fixed point theorem.
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## 1. Introduction

Fractional differential equations (FDEs) can be broadly applied to different teaches like physics, chemistry, mechanics, and engineering, see [1, 2]. Consequently, lately, FDEs have been of extraordinary interest and there have been numerous outcomes on the existence and uniqueness of solutions (positive solutions) of FDE, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Particularly, Zhang [14] employed the upper and lower solution method along with the fixed point theorem (FPT) of a cone to investigate the existence and uniqueness of a positive solution for

$$\begin{cases} D^{\theta}_{0+}\omega(\varkappa) = f(\varkappa, \omega(\varkappa)),\ 0 < \varkappa < 1, \\ \omega(0) = 0,\ \ 0 < \theta \leqslant 1. \end{cases} \tag{1.1}$$

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Yao [15] studied the existence of a positive solution to (1.1) controlled by the power function using the cone Krasnosel'skii's FPT.

In [16], Wang et al., obtained the existence and uniqueness of positive solutions of the following integral BVP by using the upper and lower solution method and fixed point theorem on cone.

$$D^{\theta}_{0+}\omega(\varkappa) + f(\varkappa, \omega(\varkappa)) = 0, \quad 0 < \varkappa < 1,$$

$$\omega(0) = 0, \quad \omega(1) = \int_0^1 \omega(s)ds, \; 1 < \theta \leqslant 2.$$

Abdo et al. [19] proved the existence and uniqueness of positive solutions of the following problem

$$D^{\theta}_{0+}\omega(\varkappa) = f(\varkappa, \omega(\varkappa)), \quad 0 \leqslant \varkappa \leqslant 1,$$

$$\omega(0) = \lambda \int_0^1 \omega(s)ds + d, \; 0 < \theta \leqslant 1.$$

by using the upper and lower solution method and FPT on cone. More recently, Wahash et al. [17, 18] discussed the existence of positive solutions to the BVPs for generalized FDEs via the upper and lower solutions method along with a cone Schauder's FPT.

In this article, we investigate the existence and uniqueness of positive solution of integral BVP for a nonlinear Caputo-type FDE:

$$\begin{cases} {}^{C}D^{\theta}_{0+}\omega(\varkappa) + f(\varkappa, \omega(\varkappa)) = 0, & 0 \leqslant \varkappa \leqslant 1, \\ \omega(0) = \omega'(0) + g(\omega), \\ \omega(1) = \int_0^1 \omega(s)ds, \end{cases} \tag{1.2}$$

where $1 < \theta \leqslant 2$, ${}^{C}D^{\theta}_{0+}$ is the Caputo fractional derivative of order $\theta$, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $g : C[0,1] \to \mathbb{R}^+$.

However, in the previous works, the nonlinear term has to satisfy the monotone condition. Indeed, the FDEs with non-monotone function can respond better to weaken the monotone condition. In this regard, we mainly investigate the FDE (1.2) without any monotone requirement on a nonlinear term by structuring the upper and lower control function and utilizing the upper and lower solutions method along with Schauder's FPT.

The rest of the article is organized as follows. In Section 2, we give some definitions and facts preliminary which are prerequisites in the sequel. Section 3 is devoted to proofing our main results and some explicatory examples.

## 2. Preliminary results

The given section appoints some important definitions and lemmas related to the fractional calculus and Green's function.

**Definition 2.1.** [1]The RL fractional integral of order $\theta > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^{\theta}_{0+}f(\varkappa) = \frac{1}{\Gamma(\theta)} \int_0^{\varkappa} \frac{f(s)}{(\varkappa - s)^{1-\theta}} ds.$$

provided that the right side is point wise defined on $(0, \infty)$.

**Definition 2.2.** [1] The RL fractional derivative of order $\theta > 0$ of a function $f : (0,\infty) \to R$ is given by

$$D^{\theta}_{0+}f(\varkappa) = \frac{1}{\Gamma(n-\theta)}\left(\frac{d}{d\varkappa}\right)^{n}\int_0^{\varkappa}\frac{f(s)}{(\varkappa-s)^{1-n+\theta}}ds,$$

where $n-1 < \theta \leqslant n$, $n = [\theta]+1$.

**Definition 2.3.** [1] The Caputo fractional derivative of order $\theta$ $(n-1 < \theta \leqslant n)$ for an at least n-times differentiable function $f : (0,\infty) \to \mathbb{R}$ is given by

$$^{c}D^{\theta}_{0+}f(\varkappa) = \frac{1}{\Gamma(n-\theta)}\int_0^{\varkappa}\frac{f^{(n)}(s)}{(\varkappa-s)^{1-n+\theta}}ds,$$

where $f^{(n)}(s) = \frac{d^n}{d\varkappa^n}f(s)$. In particular, if $1 < \theta \leqslant 2$, we have

$$^{C}D^{\theta}_{0+}f(\varkappa) = \frac{1}{\Gamma(2-\theta)}\int_0^{\varkappa}\frac{f^{(2)}(s)}{(\varkappa-s)^{\theta-1}}ds,$$

**Lemma 2.4.** *Let $\theta > 0$ and $\omega \in C(0,1)\cap L(0,1)$. Then the FDE*

$$^{C}D^{\theta}_{0+}\omega(\varkappa) = 0$$

*has a unique solution*

$$\omega(\varkappa) = d_0 + d_1\varkappa + d_2\varkappa^2 + ... + d_{n-1}\varkappa^{n-1},$$

*where $d_i \in \mathbb{R}(i = 0,1,...,n-1)$ and $n = [\theta]+1$.*

**Lemma 2.5.** *Let $\theta > 0$ and $\omega, ^{c}D^{\theta}_{0+}\omega \in C(0,1)\cap L(0,1)$. Then*

$$I^{\theta}_{0+}\ ^{C}D^{\theta}_{0+}\omega(\varkappa) = \omega(\varkappa) + d_0 + d_1\varkappa + d_2\varkappa^2 + ... + d_{n-1}\varkappa^{n-1},$$

*where $d_i \in \mathbb{R}(i = 0,1,...,n-1)$ and $n = [\theta]+1$.*

Here, we will only refer to Banach's FPT[20] and Schauder's FPT[20].

## 3. Main results

Let $C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the norm defined by

$$\|\omega\| = \sup\{|\omega(\varkappa)|;\ \varkappa \in [0,1]\}$$

Define the cone

$$K = \{\omega \in C[0,1] : \omega(\varkappa) \geqslant 0, \qquad \varkappa \in [0,1]\}.$$

The positive solution we mean in this work is that $\omega(\varkappa) \geqslant 0$, $0 < \varkappa \leqslant 1$, $\omega \in C[0,1]$.

**Lemma 3.1.** *Let* $1 < \theta \leqslant 2$ *and* $h(\varkappa) \in C[0,1]$. *Then the BVP*

$$\begin{cases} {}^{C}D_{0+}^{\theta}\omega(\varkappa) + h(\varkappa) = 0, \quad 0 \leqslant \varkappa \leqslant 1, \\ \omega(0) = \omega'(0) + g(\omega), \\ \omega(1) = \int_0^1 \omega(s)ds, \end{cases} \tag{3.1}$$

*has a unique solution*

$$\omega(\varkappa) = g(\omega) + \int_0^1 G(\varkappa, s)h(s)ds. \tag{3.2}$$

*where*

$$G(\varkappa, s) = \begin{cases} \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1} - \theta(\varkappa-s)^{\theta-1}}{\Gamma(\theta+1)}, & 0 \leqslant s \leqslant \varkappa \leqslant 1 \\ \frac{2(\varkappa+1)s(1-s)^{\theta-1}}{\Gamma(\theta+1)}, & 0 \leqslant \varkappa \leqslant s \leqslant 1 \end{cases} \tag{3.3}$$

*Here* $G(\varkappa, s)$ *is called Green function of BVP (3.1).*

*Proof.* Applying $I_{0+}^{\theta}$ on both sides of (3.1), then using Lemma 2.4, we can reduce the equation $-{}^{C}D_{0+}^{\theta}\omega(\varkappa) = h(\varkappa)$, into its equivalent integral equation as

$$\omega(\varkappa) = \omega(0) + \omega'(0)\varkappa - \int_0^{\varkappa} \frac{(\varkappa - s)^{\theta-1}}{\Gamma(\theta)} h(s)ds. \tag{3.4}$$

Hence

$$\omega(1) = \omega(0) + \omega'(0) - \int_0^1 \frac{(1 - s)^{\theta-1}}{\Gamma(\theta)} h(s)ds. \tag{3.5}$$

By the boundary condition $\omega(1) = \int_0^1 \omega(s)ds$ and (3.4), we have

$$\omega(1) = \omega(0) + \frac{\omega'(0)}{2} - \int_0^1 \int_0^{\varkappa} \frac{(\varkappa - s)^{\theta-1}}{\Gamma(\theta)} h(s)dsd\varkappa \tag{3.6}$$

Comparing (3.5) and (3.6) with the help of Fubini's theorem, we get

$$\omega'(0) = 2\int_0^1 \frac{(\theta + s - 1)(1 - s)^{\theta-1}}{\Gamma(\theta + 1)} h(s)ds \tag{3.7}$$

In view of nonlocal condition $\omega(0) = \omega'(0) + g(\omega)$, we obtain

$$\omega(0) = g(\omega) + 2\int_0^1 \frac{(\theta + s - 1)(1 - s)^{\theta-1}}{\Gamma(\theta + 1)} h(s)ds \tag{3.8}$$

From (3.7), (3.8) and (3.4), we get

$$\omega(\varkappa) = g(\omega) + 2(\varkappa + 1)\int_0^1 \frac{(\theta + s - 1)(1 - s)^{\theta-1}}{\Gamma(\theta + 1)} h(s)ds - \int_0^{\varkappa} \frac{(\varkappa - s)^{\theta-1}}{\Gamma(\theta)} h(s)ds.$$

This is

$$\begin{aligned}
\omega(\varkappa) &= g(\omega) + \int_0^1 \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)} h(s)ds - \int_0^{\varkappa} \frac{(\varkappa-s)^{\theta-1}}{\Gamma(\theta)} h(s)ds \\
&= g(\omega) + \int_0^{\varkappa} \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)} h(s)ds \\
&\quad + \int_{\varkappa}^1 \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)} h(s)ds - \int_0^{\varkappa} \frac{(\varkappa-s)^{\theta-1}}{\Gamma(\theta)} h(s)ds \\
&= g(\omega) + \int_0^{\varkappa} \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1} - \theta(\varkappa-s)^{\theta-1}}{\Gamma(\theta+1)} h(s)ds \\
&\quad + \int_{\varkappa}^1 \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)} h(s)ds \\
&= g(\omega) + \int_0^1 G(\varkappa,s)h(s)ds,
\end{aligned}$$

where $G(\varkappa, s)$ is defined by (3.3). □

**Lemma 3.2.** *For all* $\theta \in (1,2]$. *The Green function given by (3.3) satisfies the following properties:*

**(i)** $G(\varkappa, s)$ *is continuous on* $[0,1] \times [0,1]$.

**(ii)** $G(\varkappa, s) \geqslant 0, \quad \varkappa, s \in (0,1)$.

**(iii)** $G(1, s) > 0$, *for all* $s \in (0,1)$.

**(iv)** $G(\varkappa, s) \leqslant G(s, s)$ *for all* $\varkappa, s \in [0,1]$.

*Proof.* Let us assume

$$G_1(\varkappa, s) = \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1} - \theta(\varkappa-s)^{\theta-1}}{\Gamma(\theta+1)}, \quad 0 \leqslant s \leqslant \varkappa \leqslant 1,$$

$$G_2(\varkappa, s) = \frac{2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)}, \quad 0 \leqslant \varkappa \leqslant s \leqslant 1.$$

One can check easily that $G_1(\varkappa, s)$ and $G_2(\varkappa, s)$ are continuous on $[0,1] \times [0,1]$, thus (i) is true. Now, we prove that (ii) holds. Obviously, $G_2(\varkappa, s)$ is positive for all $0 \leqslant \varkappa \leqslant s \leqslant 1$. To show that $G_1(\varkappa, s)$ is positive. It is clear that

$$(1-s)^{\theta-1} \geqslant (\varkappa-s)^{\theta-1}, \;\; 0 < s \leqslant \varkappa < 1,$$

and

$$(\theta+s-1) > \theta, \;\; n-1 < \theta \leqslant n, \;\; 0 < s < 1.$$

Hence

$$2(\varkappa+1)(\theta+s-1)(1-s)^{\theta-1} \geqslant \theta(\varkappa-s)^{\theta-1}.$$

Next, we show that (iii) is satisfied. It is clear that $G_2(1,s) > 0$ for $0 < s < 1$, and by the same argument in (ii) one can show that $G_1(1,s) > 0$ for $0 < s < 1$.

Finally, the condition (iv) holds. Indeed, $G_2(\varkappa,s)$ is increasing with respect to $\varkappa$ on $[0,s]$ and we prove that $G_1(\varkappa,s)$ is decreasing in $[s,1]$. Define

$$\begin{aligned} g_1(\varkappa,s) &= 2\varkappa^{1-\theta}(\theta+s-1)(1-s)^{\theta-1} \\ &\quad +2\varkappa^{-\theta}(\theta+s-1)(1-s)^{\theta-1} - \theta(1-\frac{s}{\varkappa})^{\theta-1}. \end{aligned}$$

Then $G_1(\varkappa,s) = \frac{\varkappa^{\theta-1}}{\Gamma(\theta+1)} g_1(\varkappa,s)$. Since

$$\begin{aligned} \frac{\partial g_1(\varkappa,s)}{\partial \varkappa} &= 2(1-\theta)\varkappa^{-\theta}(\theta+s-1)(1-s)^{\theta-1} \\ &\quad -2\theta\varkappa^{-\theta-1}(\theta+s-1)(1-s)^{\theta-1} - \theta(\theta-1)\left(\frac{s}{\varkappa^2}\right)(1-\frac{s}{\varkappa})^{\theta-2} \\ &< 0, \end{aligned}$$

$g_1(\varkappa,s)$ is decreasing on $[s,1]$ with respect to $\varkappa$. Therefore, $G(\varkappa,s)$ is decreasing with respect to $\varkappa$ for $\varkappa \geqslant s$ and $G(\varkappa,s)$ is increasing with respect to $\varkappa$ for $\varkappa \leqslant s$. So, $G(\varkappa,s) \leqslant G(s,s)$ for $s,\varkappa \in [0,1]$. □

**Definition 3.3.** Let $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Take $a, b \in \mathbb{R}^+$, and $b > a$. For any $\omega \in [a,b]$, we define the upper-control function $\overline{f}(\varkappa,\omega) = \sup_{a \leqslant \eta \leqslant \omega} f(\varkappa,\eta)$, and lower-control function $\underline{f}(\varkappa,\omega) = \inf_{\omega \leqslant \eta \leqslant b} f(\varkappa,\eta)$. Clearly, $\overline{f}(\varkappa,\omega)$ and $\underline{f}(\varkappa,\omega)$ are monotonous non-decreasing on $\omega$ and

$$\underline{f}(\varkappa,\omega) \leqslant f(\varkappa,\omega) \leqslant \overline{f}(\varkappa,\omega).$$

*Remark* 3.4. The previous definition is vaild on the function $g(\omega)$, that is

$$\underline{g}(\omega) \leqslant g(\omega) \leqslant \overline{g}(\omega),$$

where $\overline{g}(\omega) = \sup_{a \leqslant \eta \leqslant \omega} g(\eta)$ and $\underline{g}(\omega) = \inf_{\omega \leqslant \eta \leqslant b} g(\eta)$.

**Definition 3.5.** Let $\overline{\omega}(\varkappa), \underline{\omega}(\varkappa) \in K$ and $a \leqslant \underline{\omega}(\varkappa) \leqslant \overline{\omega}(\varkappa) \leqslant b$ satisfy

$$\begin{cases} -{}^{C}D^{\theta}_{0+}\,\overline{\omega}(\varkappa) \geqslant \overline{f}(\varkappa,\overline{\omega}(\varkappa)), & 0 \leqslant \varkappa \leqslant 1, \\ \overline{\omega}(0) \geqslant \overline{\omega}'(0) + \overline{g}(\overline{\omega}), & \\ \overline{\omega}(1) \geqslant \int_0^1 \overline{\omega}(\varkappa)d\varkappa, & \end{cases} \tag{3.9}$$

or

$$\begin{aligned} \overline{\omega}(\varkappa) &\geqslant \overline{g}(\overline{\omega}) + 2(\varkappa+1)\int_0^1 \frac{(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)}\overline{f}(\varkappa,\overline{\omega}(\varkappa))ds \\ &\quad -\int_0^{\varkappa} \frac{(\varkappa-s)^{\theta-1}}{\Gamma(\theta)}\overline{f}(\varkappa,\overline{\omega}(\varkappa))ds, \quad \varkappa \in [0,1] \end{aligned}$$

and

$$\begin{cases} -{}^{C}D^{\theta}_{0+}\,\underline{\omega}(\varkappa) \leqslant \underline{f}(\varkappa,\underline{\omega}(\varkappa)), & 0 \leqslant \varkappa \leqslant 1, \\ \underline{\omega}(0) \leqslant \underline{\omega}'(0) + \underline{g}(\underline{\omega}), & \\ \underline{\omega}(1) \leqslant \int_0^1 \underline{\omega}(\varkappa)d\varkappa, & \end{cases}$$

or

$$\begin{aligned}\underline{\omega}(\varkappa) &\geqslant g(\underline{\omega}) + 2(\varkappa+1)\int_0^1 \frac{(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)} f(\varkappa,\underline{\omega}(\varkappa))ds \\ &\quad - \int_0^{\varkappa} \frac{(\varkappa-s)^{\theta-1}}{\Gamma(\theta)} f(\varkappa,\underline{\omega}(\varkappa))ds, \quad \varkappa \in [0,1].\end{aligned}$$

Thus, $\overline{\varpi}(\varkappa)$ and $\underline{\omega}(\varkappa)$ are the upper and lower solutions, respectively for problem (1.2).

Now, we give the following hypotheses:

$(H_1)$ There exist two constants $L_f, L_g > 0$ such that

$$|f(\varkappa,\omega_1) - f(\varkappa,\omega_2)| \leqslant L_f |\omega_1 - \omega_2|, \ \forall \varkappa \in [0,1], \ \omega_1,\omega_2 \in \mathbb{R}^+$$

$$|g(\omega_1) - g(\omega_2)| \leqslant L_g |\omega_1 - \omega_2|, \ \forall \ \omega_1,\omega_2 \in C([0,1],\mathbb{R}^+).$$

$(H_2)$ There exist nonnegative function $\rho \in L^1[0,1]$ and positive constants $c_1 \in \mathbb{R}^+, c_2, c_3 > 0$ such that

$$|f(\varkappa,\omega)| \leqslant \rho(\varkappa) + c_1 |\omega|, \ .$$

$$|g(\omega)| \leqslant c_2 + c_3 |\omega|, \quad \omega \in C([0,1],\mathbb{R}^+).$$

**Theorem 3.6.** *Assume that* $f : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ *is continuous. Let* $(H_2)$ *holds, and* $\overline{\varpi}(\varkappa), \underline{\omega}(\varkappa)$ *are upper and lower solutions of problem (1.2). If*

$$\left(\frac{5\theta+1}{\Gamma(\theta+2)}c_1 + c_3\right) < 1. \tag{3.10}$$

*Then the Caputo-FDFs (1.2) has a solution* $\omega \in C[0,1]$. *Moreover,*

$$\underline{\omega}(\varkappa) \leqslant \omega(\varkappa) \leqslant \overline{\varpi}(\varkappa), \quad \varkappa \in [0,1].$$

*Proof.* Define $Q : K \to K$ by

$$(Q\omega)(\varkappa) = g(\omega) + \int_0^1 G(\varkappa,s)f(s,\omega(s))ds, \quad \varkappa \in [0,1]. \tag{3.11}$$

By Lemma 3.1, fixed points of Q are solutions of the problem (1.2). From continuity of $f, g$ and $G(\varkappa,s)$, the operator Q is continuous. Define a closed ball

$$\mathcal{B}_r = \{\omega \in K : \|\omega\| \leqslant r, \ \ \varkappa \in [0,1]\},$$

with

$$r \geqslant \frac{c_2 + \frac{5\theta+1}{\Gamma(\theta+2)}\|\rho\|_{L^1}}{1 - \left(\frac{5\theta+1}{\Gamma(\theta+2)}c_1 + c_3\right)}.$$

Then we can show that $Q : K \to K$. Indeed, for any $\omega \in \mathcal{B}_r$ and by $(H_2)$. Then

$$\begin{aligned}
|(Q\omega)(\varkappa)| &\leqslant |g(\omega)| + \int_0^1 |G(\varkappa,s)|\,|f(s,\omega(s))|\,ds \\
&\leqslant c_2 + c_3\,|\omega(s)| + \int_0^1 |G(\varkappa,s)|\,[\rho(s) + c_1\,|\omega(s)|]\,ds \\
&\leqslant c_2 + c_3 r + (\|\rho\|_{L^1} + c_1 r)\int_0^1 |G(\varkappa,s)|\,ds.
\end{aligned}$$

Also, we have

$$\int_0^1 |G(\varkappa,s)|\,ds \leqslant \frac{5\theta+1}{\Gamma(\theta+2)}. \tag{3.12}$$

Hence

$$\|(Q\omega)\| \leqslant c_2 + \left(\frac{5\theta+1}{\Gamma(\theta+2)}c_1 + c_3\right) r + \frac{5\theta+1}{\Gamma(\theta+2)}\|\rho\|_{L^1} \leqslant r.$$

This show that Q maps K into K . Next, we show that Q is completely continuous.

First, the operator $Q : K \to K$ is continuous in light of the assumptions of nonnegativeness and continuity of $f(\varkappa,\omega)$, $g(\omega)$ and $G(\varkappa,s)$. Next, $Q : \mathcal{B}_r \to \mathcal{B}_r$ is uniformly bounded, due to Q maps K into K.

Finally, we prove that Q is equicontinuous. Set $M := \max_{(\varkappa,\omega)\in[0,1]\times[0,r]} f(\varkappa,\omega(\varkappa)) + 1$. For each $\omega \in \mathcal{B}_r$. Then for $\varkappa_1, \varkappa_2 \in [0,1]$ with $\varkappa_1 < \varkappa_2$, we have

$$\begin{aligned}
|(Q\omega)(\varkappa_2) - (Q\omega)(\varkappa_1)| &= \left|\int_0^1 G(\varkappa_2,s)f(s,\omega(s))ds - \int_0^1 G(\varkappa_1,s)f(s,\omega(s))ds\right| \\
&\leqslant 2(\varkappa_2-\varkappa_1)\int_0^1 \frac{(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)}|f(s,\omega(s))|\,ds \\
&\quad + \frac{1}{\Gamma(\theta)}\int_0^{\varkappa_1}\left|(\varkappa_1-s)^{\theta-1} - (\varkappa_2-s)^{\theta-1}\right||f(s,\omega(s))|\,ds \\
&\quad + \frac{1}{\Gamma(\theta)}\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-s)^{\theta-1}|f(s,\omega(s))|\,ds \\
&\leqslant \frac{2M\theta}{\Gamma(\theta+2)}(\varkappa_2-\varkappa_1) + \frac{M}{\Gamma(\theta+1)}\left(\varkappa_1^\theta - \varkappa_2^\theta + 2(\varkappa_2-\varkappa_1)^\theta\right) \\
&\leqslant \frac{2M\theta}{\Gamma(\theta+2)}(\varkappa_2-\varkappa_1) + \frac{2M}{\Gamma(\theta+1)}(\varkappa_2-\varkappa_1)^\theta.
\end{aligned} \tag{3.13}$$

The right-hand side of the inequality (3.13) tends to zero as As $\varkappa_2 - \varkappa_1 \to 0$, which means that $(Q\mathcal{B}_r)$ is equicontinuous. So Q is relatively compact on $\mathcal{B}_r$, as consequence of the Arzela-Ascoli theorem, we conclude that Q is completely continuous.

To apply Schauder's fixed point theorem, we need only to prove $Q : \Lambda \to \Lambda$, where

$$\Lambda = \{v(\varkappa) : v(\varkappa) \in K,\ \underline{\omega}(\varkappa) \leqslant v(\varkappa) \leqslant \overline{\omega}(\varkappa),\ \varkappa \in [0,1]\},$$

endowed with norm $\|v(\varkappa)\| = \max_{\varkappa\in[0,1]} |v(\varkappa)| \leqslant b$. Hence $\Lambda$ is a convex, closed and bounded subset of the Banach space $C([0,1],\mathbb{R}^+)$. For any $v(\varkappa) \in \Lambda$, then $\underline{\omega}(\varkappa) \leqslant v(\varkappa) \leqslant \overline{\omega}(\varkappa)$ it follows from the definitions 3.3, 3.5 that

$$\begin{aligned}(Q\nu)(\varkappa) &= g(\nu) + \int_0^1 G(\varkappa,s)f(s,\nu(s))ds \\ &\leqslant \overline{g}(\nu) + \int_0^1 G(\varkappa,s)\overline{f}(s,\nu(s))ds \\ &\leqslant \overline{g}(\overline{\omega}) + \int_0^1 G(\varkappa,s)\overline{f}(s,\overline{\omega}(s))ds \\ &\leqslant \overline{\omega}(\varkappa),\end{aligned}$$

and

$$\begin{aligned}Q\nu(\varkappa) &= g(\nu) + \int_0^1 G(\varkappa,s)f(s,\nu(s))ds \\ &\geqslant \underline{g}(\nu) + \int_0^1 G(\varkappa,s)\underline{f}(s,\nu(s))ds \\ &\geqslant \underline{g}(\underline{\omega}) + \int_0^1 G(\varkappa,s)\underline{f}(s,\underline{\omega}(s))ds \\ &\geqslant \underline{\omega}(\varkappa)\end{aligned}$$

Therefore, $\underline{\omega}(\varkappa) \leqslant Q\nu(\varkappa) \leqslant \overline{\omega}(\varkappa)$, $0 \leqslant \varkappa \leqslant 1$ which implies that $Q\nu(\varkappa) \in \Lambda$ for all $\varkappa \in [0,1]$. This proves that $Q : \Lambda \to \Lambda$. As consequence of Schauder's FPT [20], the operator $Q$ has at least one fixed point $\omega(\varkappa) \in \Lambda$, $0 \leqslant \varkappa \leqslant 1$. Therefore, the problem (1.2) has at least one solution $\omega(\varkappa) \in C[0,1]$ and $\underline{\omega}(\varkappa) \leqslant \omega(\varkappa) \leqslant \overline{\omega}(\varkappa)$, $\varkappa \in [0,1]$. □

**Corollary 3.7.** *Assume that* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and there exist* $M_1, M_2, N_1, N_2 > 0$ *such that*

$$M_1 \leqslant f(\varkappa,\sigma_1) \leqslant M_2, \quad (\varkappa,\sigma_1) \in [0,1] \times \mathbb{R}^+, \tag{3.14}$$

$$N_1 \leqslant g(\sigma_2) \leqslant N_2, \quad \sigma_2 \in C([0,1],\mathbb{R}^+). \tag{3.15}$$

*Then there exists at least a solution* $\omega(\varkappa)$ *of the Caputo-FDFs (1.2). Moreover, for* $\varkappa \in [0,1]$,

$$\omega(\varkappa) \leqslant N_2 + 2(\varkappa+1)\left[\left(\frac{\theta-1}{\theta\Gamma(\theta+1)} + \frac{1}{\theta\Gamma(\theta+2)}\right) - \frac{\varkappa^\theta}{\Gamma(\theta+1)}\right]M_2, \tag{3.16}$$

*and*

$$\omega(\varkappa) \geqslant N_1 + 2(\varkappa+1)\left[\left(\frac{\theta-1}{\theta\Gamma(\theta+1)} + \frac{1}{\theta\Gamma(\theta+2)}\right) - \frac{\varkappa^\theta}{\Gamma(\theta+1)}\right]M_1. \tag{3.17}$$

*Proof.* From the Definition 3.3 and the assumptions (3.14), (3.15) we have

$$M_1 \leqslant \underline{f}(\varkappa,\sigma_1) \leqslant \overline{f}(\varkappa,\sigma_1) \leqslant M_2, \quad (\varkappa,\sigma_1) \in [0,1] \times [a,b] \tag{3.18}$$

$$N_1 \leqslant \underline{g}(\sigma_2) \leqslant \overline{g}(\sigma_2) \leqslant N_{2\cdot 2}, \quad \sigma_2 \in C([0,1],[a,b]) \tag{3.19}$$

Now, we consider the following Caputo problem

$$\begin{aligned} -{}^{C}D^{\theta}_{0^+}\,\overline{\omega}(\varkappa) = M_2, \qquad 0 \leqslant \varkappa \leqslant 1, \\ \overline{\omega}(0) = \overline{\omega}'(0) + N_2, \\ \overline{\omega}(1) = \textstyle\int_0^1 \overline{\omega}(\varkappa)d\varkappa, \end{aligned} \tag{3.20}$$

Then, the Caputo problem (3.20) has a positive solution

$$\begin{aligned}\overline{\omega}(\varkappa) &= N_2 + 2(\varkappa+1)\int_0^1 \frac{(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)} M_2 ds \\ &\quad - \int_0^{\varkappa} \frac{(\varkappa-s)^{\theta-1}}{\Gamma(\theta)} M_2 ds \\ &= N_2 + 2(\varkappa+1)\left[\left(\frac{\theta-1}{\theta\Gamma(\theta+1)} + \frac{1}{\theta\Gamma(\theta+2)}\right) - \frac{\varkappa^\theta}{\Gamma(\theta+1)}\right] M_2, \quad \varkappa \in [0,1]\end{aligned}$$

By (3.18) and (3.19), we conclude that

$$\overline{\omega}(\varkappa) = N_2 + \int_0^1 G(\varkappa,s)M_2 ds \geqslant \overline{g}(\overline{\omega}) + \int_0^1 G(\varkappa,s)\overline{f}(\varkappa,\overline{\omega}(\varkappa))ds.$$

Thus, the function $\overline{\omega}(\varkappa)$ is the upper solution of the Caputo problem (1.2).
In the above same way, if the Caputo problem of the type

$$\begin{aligned} -{}^C D_{0+}^{\theta}\, \underline{\omega}(\varkappa) &= M_1, \qquad 0 \leqslant \varkappa \leqslant 1, \\ \underline{\omega}(0) &= \underline{\omega}'(0) + N_1, \\ \underline{\omega}(1) &= \textstyle\int_0^1 \underline{\omega}(\varkappa)d\varkappa, \end{aligned} \tag{3.21}$$

Obviously, the Caputo problem (3.21) has also a positive solution

$$\begin{aligned}\underline{\omega}(\varkappa) &= N_1 + 2(\varkappa+1)\int_0^1 \frac{(\theta+s-1)(1-s)^{\theta-1}}{\Gamma(\theta+1)} M_1 ds \\ &\quad - \int_0^{\varkappa} \frac{(\varkappa-s)^{\theta-1}}{\Gamma(\theta)} M_1 ds \\ &= N_1 + 2(\varkappa+1)\left[\left(\frac{\theta-1}{\theta\Gamma(\theta+1)} + \frac{1}{\theta\Gamma(\theta+2)}\right) - \frac{\varkappa^\theta}{\Gamma(\theta+1)}\right] M_1, \quad \varkappa \in [0,1]\end{aligned}$$

By (3.18) and (3.19), we conclude that

$$\underline{\omega}(\varkappa) = N_1 + \int_0^1 G(\varkappa,s)M_2 ds \leqslant \underline{g}(\underline{\omega}) + \int_0^1 G(\varkappa,s)\underline{f}(\varkappa,\underline{\omega}(\varkappa))ds.$$

Thus, the function $\underline{\omega}(\varkappa)$ is the upper solution of the Caputo problem (1.2).
By Theorem (3.6), we get that the problem (1.2) has at least one positive solution $\omega(\varkappa) \in \Lambda$, which verifies the inequalities (3.16) and (3.17). □

The final result is based on the Banach fixed point theorem.

**Theorem 3.8.** *Assume that* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous. Let* $(H_1)$ *holds, If* $\left(L_g + \frac{5\theta+1}{\Gamma(\theta+2)} L_f\right) < 1$. *Then the problem (1.2) has a unique solution* $\omega(\varkappa) \in C[0,1]$.

*Proof.* Theorem 3.6 shows that the problem (1.2) has at least one positive solution in K given by

$$\omega(\varkappa) = g(\omega) + \int_0^1 G(\varkappa,s)f(s,\omega(s))ds, \quad \varkappa \in [0,1]. \tag{3.22}$$

Hence, we need only to show that Q defined by

$$(Q\omega)(\varkappa) = g(\omega) + \int_0^1 G(\varkappa, s) f(s, \omega(s)) ds, \quad \varkappa \in [0,1] \tag{3.23}$$

is contraction map in $C[0,1]$. Indeed, by $(H_1)$ and (3.12), then for $\omega_1, \omega_2 \in C[0,1]$ and $\varkappa \in [0,1]$, we have

$$\begin{aligned}
& |(Q\omega_1)(\varkappa) - (Q\omega_2)(\varkappa)| \\
\leqslant\ & |g(\omega_1) - g(\omega_2)| + \int_0^1 |G(\varkappa, s)| \, |f(s, \omega_1(s)) - f(s, \omega_2(s))| \, ds \\
\leqslant\ & L_g \, |\omega_1 - \omega_2| + L_f \int_0^1 |G(\varkappa, s)| \, |\omega_1(s) - \omega_2(s)| \, ds \\
\leqslant\ & L_g \, \|\omega_1 - \omega_2\| + L_f \, \|\omega_1 - \omega_2\| \int_0^1 |G(\varkappa, s)| \, ds \\
\leqslant\ & \left( L_g + \frac{5\theta + 1}{\Gamma(\theta + 2)} L_f \right) \|\omega_1 - \omega_2\| .
\end{aligned}$$

Since $\left( L_g + \frac{5\theta+1}{\Gamma(\theta+2)} L_f \right) < 1$, Q is contraction mapping. As consequence of Theorem **??**, we can conclude that Q has a unique fixed point which is the unique positive solution of (1.2) on $[0,1]$. □

## 4. Examples

In this section, we give two examples to illuminate our results.

**Example 4.1.** Consider the FDE with integral boundary condition

$$\begin{cases} -{}^{C}D_{0^+}^{\frac{3}{2}} \omega(\varkappa) = 1 + \frac{\omega(\varkappa)}{6 + \sin(\omega(\varkappa))}, \qquad 0 < \varkappa < 1, \\ \omega(0) = \omega'(0) + \frac{1}{8}\omega(\frac{1}{3}), \\ \omega(1) = \int_0^1 \omega(\varkappa) d\varkappa, \end{cases} \tag{4.1}$$

where $\theta = \frac{3}{2}$, $g(\omega) = \frac{1}{8}\omega(\frac{1}{3})$, and $f(\varkappa, \omega) = 1 + \frac{\omega}{6+\sin(\omega)}$. It is easy to see that f is continuous function and for all $\varkappa \in [0,1]$, we have

$$|f(\varkappa, \omega_1) - f(\varkappa, \omega_2)| \leqslant \frac{1}{6} |\omega_1 - \omega_2|, \text{ for } \omega_1, \omega_2 \in [0, \infty),$$

$$|g(\omega_1) - g(\omega_2)| \leqslant \frac{1}{8} |\omega_1 - \omega_2|, \text{ for } \omega_1, \omega_2 \in C[0,1],$$

Therefore $(H_1)$ holds with $L_f = \frac{1}{6}$ and $L_g = \frac{1}{8}$. Moreover, by some simple calculations, we get

$$L_g + \frac{5\theta + 1}{\Gamma(\theta + 2)} L_f = \frac{1}{8} + \frac{34}{45\sqrt{\pi}} \approx 0.6 < 1.$$

All assumptions of Theorem 3.8 hold. Therefore, Theorem 3.8 guarantees that (4.1) has a unique positive solution $\omega(\varkappa) \in C[0,1]$.

**Example 4.2.** Consider the following FDE

$$\begin{aligned} -{}^{C}D^{\frac{7}{4}}_{0^+}\omega(\varkappa) &= \tfrac{4}{17}\left(\varkappa|\sin\omega(\varkappa)|+1\right), \qquad 0<\varkappa<1,\\ \omega(0) &= \omega'(0)+\tfrac{1}{8}\left(1+\sin(\tfrac{\omega}{1+\omega})\right),\\ \omega(1) &= \int_0^1\omega(\varkappa)d\varkappa, \end{aligned} \tag{4.2}$$

where $\theta=\frac{7}{4}$. Set $f(\varkappa,\omega)=\frac{4}{17}\left(\varkappa|\sin\omega|+1\right)$, $g(\omega)=\frac{1}{8}\left(1+\sin(\frac{\omega}{1+\omega})\right)$. Then for $\varkappa\in[0,1]$ and $\omega\in\mathbb{R}^+$, we have

$$|f(\varkappa,\omega)|=\left|\frac{4}{17}\varkappa\sin\omega+\frac{4}{17}\right|\leqslant\frac{4}{17}\left(|\omega|+1\right),$$

Therefore the condition ($H_2$) holds with $\rho(\varkappa)=\frac{4}{17}$ and $c_1=\frac{4}{17}$. Also, for all $\varkappa\in[0,1]$ and $\omega\in C([0,1],\mathbb{R}^+)$, we have

$$|g(\omega)|\leqslant\frac{1}{8}\left|1+\sin\left(\frac{\omega}{1+\omega}\right)\right|\leqslant\frac{1}{8}+\frac{1}{8}|\omega|.$$

So the condition ($H_4$) holds with $c_2=c_3=\frac{1}{8}$. It is easy to verify that

$$\frac{5\theta+1}{\Gamma(\theta+2)}c_1+c_3=\approx 0.64<1.$$

Therefore, all assumptions of Theorem 3.6 are satisfied. Thus the problem considered (4.2) has a positive solution $\omega$.

On the other hand, since $f,g$ are continuous, then by some simple calculation, we have

$$\frac{4}{17}\leqslant f(\varkappa,\omega)\leqslant\frac{8}{17},\quad (\varkappa,\omega)\in[0,1]\times[0,\infty),$$

and

$$\frac{1}{8}\leqslant g(\omega)\leqslant\frac{1}{4},\quad \omega\in[0,\infty).$$

The functions $f,g$ in the problem (4.2) verifies the assumptions of Corollary 3.7 with $M_1=\frac{8}{17},M_2=\frac{4}{17},N_1=\frac{1}{4},N_2=\frac{1}{8}$. Then the Caputo problem (4.2) has a positive solution which verifies $\underline{\omega}(\varkappa)\leqslant\omega(\varkappa)\leqslant\overline{\omega}(\varkappa)$ where

$$\overline{\omega}(\varkappa)=\frac{1}{8}+\frac{8}{17}(\varkappa+1)\left[\frac{3}{7\Gamma(\frac{11}{4})}+\frac{4}{7\Gamma(\frac{15}{4})}-\frac{\varkappa^{\frac{7}{4}}}{\Gamma(\frac{11}{4})}\right],\quad \varkappa\in[0,1],$$

and

$$\underline{\omega}(\varkappa)=\frac{1}{4}+\frac{16}{17}(\varkappa+1)\left[\frac{3}{7\Gamma(\frac{11}{4})}+\frac{4}{7\Gamma(\frac{15}{4})}-\frac{\varkappa^{\frac{7}{4}}}{\Gamma(\frac{11}{4})}\right],\quad \varkappa\in[0,1],$$

are respectively the upper and lower solutions of Caputo problem (4.2).

## 5. Conclusion

We investigated the existence and uniqueness of positive solutions for a class of Caputo-type FDEs with nonlocal integral boundary conditions. The existence result is established by using the Schauder fixed point technique on a cone, whereas the uniqueness result is obtained via Banach's fixed point theorem.In addition, our analysis is supported by the properties of Green function and upper and lower control function. Some examples are also constructed to substantiate our results. The proposed problem contributes to the growth of FDEs with non-local integrated boundary conditions and with modern and simple techniques.

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