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## A note on some Ostrowski type inequalities via generalized exponentially convex functions

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### Abstract

In this paper, we define and investigate a new generalized exponential type convex function called exponentially  $s$ -convex function, which embraces the exponential type convex function given by İşcan in [1]. To support this newly introduced idea, we attained some of its algebraic properties and presented some examples. Furthermore, to enhance the paper we have explored and attained several novel refinements of Ostrowski type inequalities in the frame of simple calculus whose first derivatives in absolute value are exponentially  $s$ -convex. Several special cases of the obtained results are deduced as well. Finally, in the last section, few consequences of the results including application to special means are pointed out.

Keywords: Ostrowski type inequality, Hölder's inequality, Power mean inequality, Exponentially  $s$ -convex function.

2010 MSC: 26A51, 26A33, 26D07, 26D10, 26D15.

### 1. Introduction

During the last few decades, the theory of convexity has played crucial and consequential role in the extension and generalizations of theory of inequalities. This theory presents an active and amazing field of research and also played significant and magnificent role in many areas, such as optimization, management science, economics, finance, engineering, game theory and mathematical inequalities. In the field of inequalities, many famously and well-known results in the support of this theory are investigated. The terms convexity and inequalities have a wide range of applications in the growth of variants branches of pure and applied mathematics as well as in different areas of pure and applied sciences. The initiative works and significant contributions of these two terminologies can be found

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in the literature, see the references [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In 1938, Ostrowski [18], developed a significant integral inequality connected with differentiable functions. Mathematically this inequality ( see [19], page:468) is given by Let  $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $J^\circ$ , the interior of the interval  $J$ , such that  $\varphi \in L[\kappa_1, \kappa_2]$ , where  $\kappa_1, \kappa_2 \in J$  with  $\kappa_2 > \kappa_1$ . If  $|\varphi'(z)| \leq \mathbb{M}$ , for all  $z \in [\kappa_1, \kappa_2]$ , then the following inequality holds:

$$\left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq \mathbb{M} (\kappa_2 - \kappa_1) \left[ \frac{1}{4} + \frac{(z - \frac{\kappa_1 + \kappa_2}{2})^2}{(\kappa_2 - \kappa_1)^2} \right] \tag{1.1}$$

holds.

This integral inequality has elegant and effective importance in numerical integration, optimization theory, integral operator theory, information, probability, statistics and stochastic process. During the last few decades, many mathematicians and research scholars concentrated their great contributions and attentions on the study of this inequality. In 1997, this inequality was explored by Dragomir and Wang [20] in terms of the lower and upper bounds of the first derivative. In 2001, Barnett et al. [21] investigated this inequality for twice differentiable functions in terms of the upper and lower bounds. For the current results and their associated and connected variants, extensions and generalizations concerning Ostrowski inequality (see [22, 23, 24, 25, 26] ).

The main objective and aim of this work is to define and investigate a new family of exponentially convex functions namely exponentially  $s$ -convex function. In support of this newly introduced idea, we attain the algebraic properties and examples of this function, and furthermore, in the frame of simple calculus, we explore and attain the novel refinements of Ostrowski type inequalities. To the best of our knowledge, a comprehensive investigation of exponentially convex functions in the manner of newly introduced definition namely exponentially  $s$ -convex function in the present paper is new. The growth of research on big data analysis and deep learning has recently increased interest in information theory involving exponentially convex functions.

We construct the paper in the following way, first in Section 2, we will give some necessary known definitions and literature. Second in section 3, we will explore the concept of exponentially  $s$ -convex function. In addition, some algebraic properties and examples in the manner of this newly introduced definition are elaborated. Further, in Section 4, we investigate some refinements of the Ostrowski type inequality. Finally, we will give some applications and a conclusion.

## 2. Preliminaries

In this section, we recall some known concepts.

**Definition 2.1.** [27] Let  $\varphi : I \rightarrow \mathbb{R}$  be a real valued function. A function  $\varphi$  is said to be convex, if

$$\varphi(\chi\kappa_1 + (1 - \chi)\kappa_2) \leq \chi\varphi(\kappa_1) + (1 - \chi)\varphi(\kappa_2), \tag{2.1}$$

holds for all  $\kappa_1, \kappa_2 \in I$  and  $\chi \in [0, 1]$ .

**Definition 2.2.** [28] Let  $s \in (0, 1]$ , then  $\varphi : [0, +\infty) \rightarrow \mathbb{R}$  is known as  $s$ -convex in the second sense, if

$$\varphi (\chi\kappa_1 + (1 - \chi) \kappa_2) \leq \chi^s \varphi (\kappa_1) + (1 - \chi)^s \varphi (\kappa_2) \tag{2.2}$$

holds  $\forall \kappa_1, \kappa_2 \in [0, +\infty)$  and  $\chi \in [0, 1]$ .

In 1978, W. W. Breckner 1<sup>st</sup> time developed a new family of convex functions which is called  $s$ -convex in the second sense in the published article [29] and discussed variants form and associations related with  $s$ -convexity in the first sense in the published article [30]. Dragomir et al. investigated and explored a novel version of Hadamard’s inequality in the manner of  $s$ -convex functions in the second sense in the published article [31].

### 3. Exponentially $s$ -Convex Function and its Properties

Next, due to afore-mentioned research activities we are able and capable to introduce the generalized form of exponential type convexity which is called exponentially  $s$ -convex functions. Further we will try to discuss and explore its properties.

**Definition 3.1.** Let  $s \in [\ln 2.4, 1]$ . Then  $\varphi : J \subset \mathbb{R} \rightarrow \mathbb{R}$  is known to be exponentially  $s$ -convex function, if

$$\varphi (\chi\kappa_1 + (1 - \chi) \kappa_2) \leq (e^{s\chi} - 1) \varphi (\kappa_1) + \left( e^{s(1-\chi)} - 1 \right) \varphi (\kappa_2), \tag{3.1}$$

holds  $\forall \kappa_1, \kappa_2 \in J$  and  $\chi \in [0, 1]$ .

*Remark 3.2.* In above Definition 3.1, if  $s = 1$ , then we get exponential type convexity given by İşcan in [1].

*Remark 3.3.* The range of the exponentially  $s$ -convex functions for some fixed  $s \in [\ln 2.4, 1]$  is  $[0, +\infty)$ .

**Lemma 3.4.** For all  $\chi \in [0, 1]$  and for some fixed  $s \in [\ln 2.4, 1]$  the following inequalities  $(e^{s\chi} - 1) \geq \chi$  and  $(e^{s(1-\chi)} - 1) \geq (1 - \chi)$  hold.

*Proof.* The rest of the proof is clearly seen. □

**Lemma 3.5.** For all  $\chi \in [0, 1]$  and for some fixed  $s \in [\ln 2.4, 1]$  the following inequalities  $(e^{s\chi} - 1) \geq \chi^s$  and  $(e^{s(1-\chi)} - 1) \geq (1 - \chi)^s$  hold.

*Proof.* The rest of the proof is clearly seen. □

The above lemmas clearly show that, the new class of exponentially  $s$ -convex function is very larger with respect to the known class of functions like convex and  $s$ -convex. This is an advantage of the newly proposed definition 3.1.

**Proposition 3.6.** Every nonnegative convex function is an exponentially  $s$ -convex function for  $s \in [\ln 2.4, 1]$ .

*Proof.* Applying Lemma 3.4 and  $s \in [\ln 2.4, 1]$ , we have

$$\begin{aligned} \varphi(\chi\kappa_1 + (1-\chi)\kappa_2) &\leq \chi\varphi(\kappa_1) + (1-\chi)\varphi(\kappa_2) \\ &\leq (e^{s\chi} - 1)\varphi(\kappa_1) + (e^{(1-\chi)s} - 1)\varphi(\kappa_2). \end{aligned}$$

□

**Proposition 3.7.** Every nonnegative  $s$ -convex function is an exponentially  $s$ -convex function for  $s \in [\ln 2.4, 1]$ .

*Proof.* Applying Lemma 3.5 and  $s \in [\ln 2.4, 1]$ , we have

$$\begin{aligned} \varphi(\chi\kappa_1 + (1-\chi)\kappa_2) &\leq \chi^s\varphi(\kappa_1) + (1-\chi)^s\varphi(\kappa_2) \\ &\leq (e^{s\chi} - 1)\varphi(\kappa_1) + (e^{(1-\chi)s} - 1)\varphi(\kappa_2). \end{aligned}$$

□

Now we will makes some examples in the support of newly introduced definition namely exponentially  $s$ -convex function.

**Example 3.8.** If  $\varphi(x) = e^x$ ,  $x > 0$  is a nonnegative convex function. By using Proposition 3.6, it is also exponentially  $s$ -convex function for  $s \in [\ln 2.4, 1]$ .

**Example 3.9.** If  $\varphi(x) = c$  is a nonnegative convex function on  $\mathbb{R}$  for any  $c \geq 0$ . By using Proposition 3.6, it is also exponentially  $s$ -convex function for  $s \in [\ln 2.4, 1]$ .

**Example 3.10.** If  $\varphi(x) = \frac{1}{x}$  for all  $x > 0$ , is a nonnegative convex function. By using Proposition 3.6, it is also exponentially  $s$ -convex function for  $s \in [\ln 2.4, 1]$ .

**Example 3.11.**  $\varphi(x) = \frac{q}{s+q}x^{\frac{s}{q}+1}$  for  $s > 1$ , is a nonnegative convex function. By using Proposition 3.6, it is also exponentially  $s$ -convex function for  $s \in [\ln 2.4, 1]$ .

**Example 3.12.** Great mathematician Dragomir clearly investigated and proved that in published article [31], the function  $\varphi(x) = x^{1/s}$ ,  $x > 0$  is  $s$ -convex function, for the all mentioned conditions  $s \in (0,1)$  and  $1 \leq \iota \leq \frac{1}{s}$ . But using Proposition 3.7, it is also exponentially  $s$ -convex function for  $s \in [\ln 2.4, 1]$ .

**Theorem 3.13.** Let  $\psi, \varphi : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ . If  $\psi$  and  $\varphi$  are exponentially  $s$ -convex functions and  $s \in [\ln 2.4, 1]$  then

- (i)  $\psi + \varphi$  is exponentially  $s$ -convex function.
- (ii) For nonnegative real number  $c$ ,  $c\psi$  is exponentially  $s$ -convex function.

*Proof.* (i) Let  $\psi$  and  $\varphi$  be exponentially  $s$ -convex functions, then

$$\begin{aligned} & (\psi + \varphi) \left[ (\chi\kappa_1 + (1 - \chi)\kappa_2) \right] \\ &= \psi (\chi\kappa_1 + (1 - \chi)\kappa_2) + \varphi (\chi\kappa_1 + (1 - \chi)\kappa_2) \\ &\leq (e^{s\chi} - 1) \psi (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) \psi (\kappa_2) \\ &+ (e^{s\chi} - 1) \varphi (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) \varphi (\kappa_2) \\ &= (e^{s\chi} - 1) [\psi (\kappa_1) + \varphi (\kappa_1)] + \left( e^{(1-\chi)s} - 1 \right) [\psi (\kappa_2) + \varphi (\kappa_2)] \\ &= (e^{s\chi} - 1) (\psi + \varphi) (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) (\psi + \varphi) (\kappa_2). \end{aligned}$$

(ii) Let  $\psi$  be exponentially  $s$ -convex function, then

$$\begin{aligned} & (c\psi) \left[ (\chi\kappa_1 + (1 - \chi)\kappa_2) \right] \\ &= c \left[ \psi (\chi\kappa_1 + (1 - \chi)\kappa_2) \right] \\ &\leq c \left[ (e^{s\chi} - 1) \psi (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) \psi (\kappa_2) \right] \\ &= (e^{s\chi} - 1) c\psi (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) c\psi (\kappa_2) \\ &= (e^{s\chi} - 1) (c\psi) (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right)^i (c\psi) (\kappa_2). \end{aligned}$$

□

*Remark 3.14.* If we choose  $s = 1$  in above Theorem (3.13), then we get Theorem (2.1) in [1].

**Theorem 3.15.** Let  $\varphi_i : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$  be an arbitrary family of exponentially  $s$ -convex functions for the same fixed  $s \in [\ln 2.4, 1]$  and let  $\varphi(\kappa) = \sup_i \varphi_i(\kappa)$ . If  $P = \{\kappa \in [\kappa_1, \kappa_2] : \varphi(\kappa) < +\infty\} \neq \emptyset$ , then  $P$  is an interval and  $\varphi$  is exponentially  $s$ -convex function on  $P$ .

*Proof.* For all  $\kappa_1, \kappa_2 \in P$  and  $\chi \in [0, 1]$ , and for the same fixed numbers  $s \in [\ln 2.4, 1]$ , we have

$$\begin{aligned} & \varphi (\chi\kappa_1 + (1 - \varphi)\kappa_2) = \sup_i \varphi_i (\chi\kappa_1 + (1 - \chi)\kappa_2) \\ &\leq \sup_i \left[ (e^{s\chi} - 1) \varphi_i (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) \varphi_i (\kappa_2) \right] \\ &\leq (e^{s\chi} - 1) \sup_i \varphi_i (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) \sup_i \varphi_i (\kappa_2) \\ &= (e^{s\chi} - 1) \varphi (\kappa_1) + \left( e^{(1-\chi)s} - 1 \right) \varphi (\kappa_2) < +\infty. \end{aligned}$$

This shows simultaneously that  $P$  is an interval, since it contains every point between any two of its points, and that  $\varphi$  is exponentially  $s$ -convex function on  $P$ . □

*Remark 3.16.* If we choose  $s = 1$  in above Theorem (3.15), then we get Theorem (2.3) in [1].

**Theorem 3.17.** *If the function  $\varphi : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$  is exponentially  $s$ -convex for some fixed  $s \in [\ln 2.4, 1]$ , then  $\varphi$  is bounded on  $[\kappa_1, \kappa_2]$ .*

*Proof.* Let  $L = \max \{ \varphi(\kappa_1), \varphi(\kappa_2) \}$  and  $x \in [\kappa_1, \kappa_2]$  be an arbitrary point. Then there exists  $\chi \in [0, 1]$  such that  $x = \chi\kappa_1 + (1 - \chi)\kappa_2$ . Thus, since  $e^{s\chi} \leq e^s$  and  $e^{(1-\chi)s} \leq e^s$  for some fixed  $s \in [\ln 2.4, 1]$ , we have

$$\begin{aligned} \varphi(x) &= \varphi(\chi\kappa_1 + (1 - \chi)\kappa_2) \leq (e^{s\chi} - 1) \varphi(\kappa_1) + (e^{(1-\chi)s} - 1) \varphi(\kappa_2) \\ &\leq (e^s - 1) L + (e^s - 1) L = L(e^s - 1) = \mathbb{M}. \end{aligned}$$

We have shown that  $\varphi$  is bounded above from real number  $\mathbb{M}$ . Interested reader can also prove the fact that  $\varphi$  is bounded below using the same idea as in Theorem 2.4 in [1].  $\square$

*Remark 3.18.* If we choose  $s = 1$  in above Theorem (3.15), then we get Theorem (2.4) in [1].

**Theorem 3.19.** *Suppose  $s \in [\ln 2.4, 1]$ ,  $\alpha \in (0, 1]$ ,  $\kappa_2 > \kappa_1$  and  $\varphi : J = [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$  is exponentially  $s$ -convex function such that  $\varphi \in L[\kappa_1, \kappa_2]$ . Then one has*

$$\begin{aligned} \frac{1}{2(e^{\frac{s}{2}} - 1)} \varphi\left(\frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \\ &\leq \left(\frac{e^s - s - 1}{s}\right) \left[\varphi(\kappa_1) + \varphi(\kappa_2)\right]. \end{aligned} \tag{3.2}$$

#### 4. Refinements of Ostrowski type inequality via exponentially $s$ -convex functions

The main objective of this section is to attain some refinements of Ostrowski type inequality in the manner of exponentially  $s$ -convex functions. In order to attain the results, we need to state the following Lemma as given in [32].

**Lemma 4.1.** *Suppose a mapping  $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is differentiable on  $J^\circ$ , where  $\kappa_1, \kappa_2 \in J$  with  $\kappa_1 < \kappa_2$ . If  $\varphi' \in L[\kappa_1, \kappa_2]$ , then the following equality holds:*

$$\begin{aligned} \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(\chi) \, d\chi \\ = \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi \varphi'(\chi z + (1 - \chi)\kappa_1) \, d\chi - \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi \varphi'(\chi z + (1 - \chi)\kappa_2) \, d\chi, \end{aligned} \tag{4.1}$$

for each  $z \in [\kappa_1, \kappa_2]$ .

**Theorem 4.2.** *Suppose a mapping  $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is differentiable on  $J^\circ$ , where  $\kappa_1, \kappa_2 \in J$  with  $\kappa_1 < \kappa_2$ . If  $|\varphi'|$  is exponentially  $s$ -convex on  $[\kappa_1, \kappa_2]$  for some  $s \in [\ln 2.4, 1]$ ,  $\varphi' \in L[\kappa_1, \kappa_2]$  and  $|\varphi'(z)| \leq \mathbb{M}$ , for all  $z \in [\kappa_1, \kappa_2]$ , then the following inequality holds:*

$$\begin{aligned}
 & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(\chi) \, d\chi \right| \\
 & \leq \frac{\mathbb{M}}{(\kappa_2 - \kappa_1)} \left[ (z - \kappa_1)^2 \left\{ \left( \frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \right. \\
 & \quad \left. + (\kappa_2 - z)^2 \left\{ \left( \frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \right], \tag{4.2}
 \end{aligned}$$

for each  $z \in [\kappa_1, \kappa_2]$ .

*Proof.* From Lemma 4.1 and  $|\varphi'|$  is exponentially  $s$ -convexity and  $|\varphi'| \leq \mathbb{M}$ , we have

$$\begin{aligned}
 & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(\chi) \, d\chi \right| \\
 & \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_1)| \, d\chi + \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_2)| \, d\chi \\
 & \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi \left\{ (e^{s\chi} - 1) |\varphi'(z)| + (e^{s(1-\chi)} - 1) |\varphi'(\kappa_1)| \right\} \, d\chi \\
 & \quad + \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi \left\{ (e^{s\chi} - 1) |\varphi'(z)| + (e^{s(1-\chi)} - 1) |\varphi'(\kappa_2)| \right\} \, d\chi \\
 & \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \left[ |\varphi'(z)| \int_0^1 \chi (e^{s\chi} - 1) \, d\chi + |\varphi'(\kappa_1)| \int_0^1 \chi (e^{s(1-\chi)} - 1) \, d\chi \right] \\
 & \quad + \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \left[ |\varphi'(z)| \int_0^1 \chi (e^{s\chi} - 1) \, d\chi + |\varphi'(\kappa_2)| \int_0^1 \chi (e^{s(1-\chi)} - 1) \, d\chi \right] \\
 & \leq \frac{\mathbb{M}(z - \kappa_1)^2}{(\kappa_2 - \kappa_1)} \left\{ \left( \frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \\
 & \quad + \frac{\mathbb{M}(\kappa_2 - z)^2}{(\kappa_2 - \kappa_1)} \left\{ \left( \frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \\
 & = \frac{\mathbb{M}}{(\kappa_2 - \kappa_1)} \left[ (z - \kappa_1)^2 \left\{ \left( \frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \right. \\
 & \quad \left. + (\kappa_2 - z)^2 \left\{ \left( \frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \right].
 \end{aligned}$$

This is the required proof. □

**Corollary 4.3.** Under the similar consideration in Theorem 4.2, by choosing  $s = 1$ , we obtain

$$\begin{aligned}
 & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \\
 & \leq \frac{\mathbb{M}}{(\kappa_2 - \kappa_1)} \left[ (z - \kappa_1)^2 \left\{ \left( \frac{1}{2} \right) + \left( \frac{2e - 5}{2} \right) \right\} \right. \\
 & \quad \left. + (\kappa_2 - z)^2 \left\{ \left( \frac{1}{2} \right) + \left( \frac{2e - 5}{2} \right) \right\} \right].
 \end{aligned}$$

Also we have

1. If we choose  $z = \frac{\kappa_1 + \kappa_2}{2}$  in Corollary 4.3, then we have the following mid-point inequality

$$\left| \varphi \left( \frac{\kappa_1 + \kappa_2}{2} \right) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq \mathbb{M} \frac{(\kappa_2 - \kappa_1)}{2} \left\{ \left( \frac{1}{2} \right) + \left( \frac{2e - 5}{2} \right) \right\}.$$

2. If we choose  $z = \kappa_1$  in Corollary 4.3, then we have the following inequality

$$\left| \varphi(\kappa_1) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq \mathbb{M} (\kappa_2 - \kappa_1) \left\{ \left( \frac{1}{2} \right) + \left( \frac{2e - 5}{2} \right) \right\}.$$

3. If we choose  $z = \kappa_2$  in Corollary 4.3, then we have the following inequality

$$\left| \varphi(\kappa_2) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq \mathbb{M} (\kappa_2 - \kappa_1) \left\{ \left( \frac{1}{2} \right) + \left( \frac{2e - 5}{2} \right) \right\}.$$

**Theorem 4.4.** Suppose a mapping  $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is differentiable on  $J^\circ$ , where  $\kappa_1, \kappa_2 \in J$  with  $\kappa_1 < \kappa_2$ . If  $|\varphi'|^q$  is exponentially  $s$ -convex on  $[\kappa_1, \kappa_2]$  for some  $s \in [\ln 2.4, 1]$ ,  $q > 1$ ,  $q^{-1} = 1 - p^{-1}$ ,  $\varphi' \in L[\kappa_1, \kappa_2]$  and  $|\varphi'(z)| \leq \mathbb{M}$ , for all  $z \in [\kappa_1, \kappa_2]$ , then the following inequality holds:

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(\chi) \, d\chi \right| \\ & \leq \frac{2^{\frac{1}{q}} \mathbb{M}}{(\kappa_2 - \kappa_1)} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ (z - \kappa_1)^2 \left\{ \left( \frac{e^s - s - 1}{s} \right) \right\}^{\frac{1}{q}} + (\kappa_2 - z)^2 \left\{ \left( \frac{e^s - s - 1}{s} \right) \right\}^{\frac{1}{q}} \right], \end{aligned} \tag{4.3}$$

for each  $z \in [\kappa_1, \kappa_2]$ .

*Proof.* From Lemma 4.1 and famous Hölder’s inequality, we have

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(\chi) \, d\chi \right| \\ & \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi \, |\varphi'(\chi z + (1 - \chi) \kappa_1)| \, d\chi + \frac{(\kappa_2 - z)^2}{\kappa_1 - \kappa_1} \int_0^1 \chi \, |\varphi'(\chi z + (1 - \chi) \kappa_2)| \, d\chi \\ & \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \left( \int_0^1 \chi^p \, d\chi \right)^{\frac{1}{p}} \left( \int_0^1 |\varphi'(\chi z + (1 - \chi) \kappa_1)|^q \, d\chi \right)^{\frac{1}{q}} \\ & \quad + \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \left( \int_0^1 \chi^p \, d\chi \right)^{\frac{1}{p}} \left( \int_0^1 |\varphi'(\chi z + (1 - \chi) \kappa_2)|^q \, d\chi \right)^{\frac{1}{q}}. \end{aligned} \tag{4.4}$$

Since  $|\varphi'|^q$  is exponentially  $s$ -convexity and  $|\varphi'(z)| \leq \mathbb{M}$ , we obtain

$$\begin{aligned} \int_0^1 |\varphi'(\chi z + (1-\chi)\kappa_1)|^q d\chi &= \int_0^1 \left\{ (e^{s\chi} - 1) |\varphi'(z)|^q + (e^{s(1-\chi)} - 1) |\varphi'(\kappa_1)|^q \right\} d\chi \\ &\leq \mathbb{M}^q \left( \frac{e^s - s - 1}{s} \right) + \mathbb{M}^q \left( \frac{e^s - s - 1}{s} \right) \\ &\leq 2\mathbb{M}^q \left( \frac{e^s - s - 1}{s} \right) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \int_0^1 |\varphi'(\chi z + (1-\chi)\kappa_2)|^q d\chi &= \int_0^1 \left\{ (e^{s\chi} - 1) |\varphi'(z)|^q + (e^{s(1-\chi)} - 1) |\varphi'(\kappa_2)|^q \right\} d\chi \\ &\leq \mathbb{M}^q \left( \frac{e^s - s - 1}{s} \right) + \mathbb{M}^q \left( \frac{e^s - s - 1}{s} \right) \\ &\leq 2\mathbb{M}^q \left( \frac{e^s - s - 1}{s} \right). \end{aligned} \tag{4.6}$$

By connecting (4.5) and (4.6) with (4.4), we get (4.3). This is the required proof.  $\square$

**Corollary 4.5.** Under the similar consideration in Theorem 4.4, by choosing  $s = 1$ , we obtain

$$\begin{aligned} &\left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) du \right| \\ &\leq \frac{2^{\frac{1}{q}} \mathbb{M}}{(\kappa_2 - \kappa_1)} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ (z - \kappa_1)^2 \left\{ (e-2) \right\}^{\frac{1}{q}} + (\kappa_2 - z)^2 \left\{ (e-2) \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Also we have

1. If we choose  $z = \frac{\kappa_1 + \kappa_2}{2}$  in Corollary 4.5, then we have the following mid-point inequality

$$\left| \varphi\left(\frac{\kappa_1 + \kappa_2}{2}\right) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) du \right| \leq 2^{\frac{1}{q}-1} \mathbb{M} (\kappa_2 - \kappa_1) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ (e-2) \right\}^{\frac{1}{q}}.$$

2. If we choose  $z = \kappa_1$  in Corollary 4.5, then we have the following inequality

$$\left| \varphi(\kappa_1) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) du \right| \leq 2^{\frac{1}{q}} \mathbb{M} (\kappa_2 - \kappa_1) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ (e-2) \right\}^{\frac{1}{q}}.$$

3. If we choose  $z = \kappa_2$  in Corollary 4.5, then we have the following inequality

$$\left| \varphi(\kappa_2) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) du \right| \leq 2^{\frac{1}{q}} \mathbb{M} (\kappa_2 - \kappa_1) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ (e-2) \right\}^{\frac{1}{q}}.$$

**Theorem 4.6.** Suppose a mapping  $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is differentiable on  $J^\circ$ , where  $\kappa_1, \kappa_2 \in J$  with  $\kappa_1 < \kappa_2$ . If  $|\varphi'|^q$  is exponentially  $s$ -convex on  $[\kappa_1, \kappa_2]$  for some  $s \in [\ln 2.4, 1]$ ,  $q \geq 1$ ,  $q^{-1} = 1 - p^{-1}$ ,  $\varphi' \in L[\kappa_1, \kappa_2]$  and  $|\varphi'(z)| \leq M$ , for all  $z \in [\kappa_1, \kappa_2]$ , then the following inequality holds:

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(\chi) \, d\chi \right| \\ & \leq \frac{M}{(\kappa_2 - \kappa_1) 2^{1-\frac{1}{q}}} \left[ (z - \kappa_1)^2 \left\{ \left( \frac{2 + (2s - 2)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\}^{\frac{1}{q}} \right. \\ & \left. + (\kappa_2 - z)^2 \left\{ \left( \frac{2 + (2s - 2)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\}^{\frac{1}{q}} \right], \end{aligned} \tag{4.7}$$

for each  $z \in [\kappa_1, \kappa_2]$ .

*Proof.* From Lemma 4.1 and power mean inequality, we have

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(\chi) \, d\chi \right| \\ & \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_1)| \, d\chi + \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_2)| \, d\chi \\ & \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \left( \int_0^1 \chi \, d\chi \right)^{1-\frac{1}{q}} \left( \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_1)|^q \, d\chi \right)^{\frac{1}{q}} \\ & \quad + \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \left( \int_0^1 \chi \, d\chi \right)^{1-\frac{1}{q}} \left( \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_2)|^q \, d\chi \right)^{\frac{1}{q}}. \end{aligned} \tag{4.8}$$

Since  $|\varphi'|^q$  is exponentially  $s$ -convexity and  $|\varphi'(z)| \leq M$ , then we obtain

$$\begin{aligned} & \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_1)|^q \, d\chi = \int_0^1 \chi \left\{ (e^{s\chi} - 1) |\varphi'(z)|^q + (e^{s(1-\chi)} - 1) |\varphi'(\kappa_1)|^q \right\} d\chi \\ & \leq M^q \left( \frac{(2s - 2)e^s - s^2 + 2}{2s^2} \right) + M^q \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\kappa_2)|^q \, d\chi = \int_0^1 \chi \left\{ (e^{s\chi} - 1) |\varphi'(z)|^q + (e^{s(1-\chi)} - 1) |\varphi'(\kappa_2)|^q \right\} d\chi \\ & \leq M^q \left( \frac{(2s - 2)e^s - s^2 + 2}{2s^2} \right) + M^q \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right). \end{aligned} \tag{4.10}$$

By connecting (4.9) and (4.10) with (4.8), we get (4.7). This is the required proof.  $\square$

**Corollary 4.7.** Under the similar consideration in Theorem 4.6, by choosing  $s = 1$ , we obtain

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \\ & \leq \frac{M}{(\kappa_2 - \kappa_1) 2^{1-\frac{1}{q}}} \\ & \times \left[ (z - \kappa_1)^2 \left\{ \left(\frac{1}{2}\right) + \left(\frac{2e-5}{2}\right) \right\}^{\frac{1}{q}} + (\kappa_2 - z)^2 \left\{ \left(\frac{1}{2}\right) + \left(\frac{2e-5}{2}\right) \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Also we have

1. If we choose  $z = \frac{\kappa_1 + \kappa_2}{2}$  in Corollary 4.7, then we have the following mid-point inequality

$$\left| \varphi\left(\frac{\kappa_1 + \kappa_2}{2}\right) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq \frac{M(\kappa_2 - \kappa_1)}{2 \cdot 2^{1-\frac{1}{q}}} \left\{ \left(\frac{1}{2}\right) + \left(\frac{2e-5}{2}\right) \right\}^{\frac{1}{q}}.$$

2. If we choose  $z = \kappa_1$  in Corollary 4.7, then we have the following inequality

$$\left| \varphi(\kappa_1) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq M \frac{(\kappa_2 - \kappa_1)}{2^{1-\frac{1}{q}}} \left\{ \left(\frac{1}{2}\right) + \left(\frac{2e-5}{2}\right) \right\}^{\frac{1}{q}}.$$

3. If we choose  $z = \kappa_2$  in Corollary 4.7, then we have the following inequality

$$\left| \varphi(\kappa_2) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq M \frac{(\kappa_2 - \kappa_1)}{2^{1-\frac{1}{q}}} \left\{ \left(\frac{1}{2}\right) + \left(\frac{2e-5}{2}\right) \right\}^{\frac{1}{q}}.$$

**Theorem 4.8.** Suppose a mapping  $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is differentiable on  $J^\circ$ , where  $\kappa_1, \kappa_2 \in J$  with  $\kappa_2 > \kappa_1$ . If  $|\varphi'|^q$  is exponentially  $s$ -concave on  $[\kappa_1, \kappa_2]$ , for some  $s \in [\ln 2.4, 1]$ ,  $\varphi' \in L[\kappa_1, \kappa_2]$ ,  $p, q > 1$ ,  $q^{-1} + p^{-1} = 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq \frac{1}{\kappa_2 - \kappa_1} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2(e^{\frac{s}{2}} - 1)}\right)^{\frac{1}{q}} \\ & \times \left\{ (z - \kappa_1)^2 \left| \varphi' \left(\frac{z + \kappa_1}{2}\right) \right| + (\kappa_2 - z)^2 \left| \varphi' \left(\frac{z + \kappa_2}{2}\right) \right| \right\}. \end{aligned} \tag{4.11}$$

*Proof.* Suppose  $p > 1$ , by Lemma 4.1 and using the Hölder inequality, we obtain

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) \, du \right| \leq \frac{(z - \kappa_1)^2}{\kappa_2 - \kappa_1} \left( \int_0^1 \chi^p \, d\chi \right)^{\frac{1}{p}} \left( \int_0^1 |\varphi'(\chi z + (1 - \chi)\kappa_1)|^q \, d\chi \right)^{\frac{1}{q}} \\ & + \frac{(\kappa_2 - z)^2}{\kappa_2 - \kappa_1} \left( \int_0^1 \chi^p \, d\chi \right)^{\frac{1}{p}} \left( \int_0^1 |\varphi'(\chi z + (1 - \chi)\kappa_2)|^q \, d\chi \right)^{\frac{1}{q}} \end{aligned} \tag{4.12}$$

$|\varphi'|^q$  is exponentially  $s$ -concave and using the left hand side of (3.2) inequality, we obtain

$$\int_0^1 |\varphi'(\chi z + (1-\chi)\kappa_1)|^q d\chi \geq \left(\frac{1}{2(e^{\frac{s}{2}}-1)}\right) \left|\varphi'\left(\frac{z+\kappa_1}{2}\right)\right|^q \tag{4.13}$$

and

$$\int_0^1 |\varphi'(\chi z + (1-\chi)\kappa_2)|^q d\chi \geq \left(\frac{1}{2(e^{\frac{s}{2}}-1)}\right) \left|\varphi'\left(\frac{z+\kappa_2}{2}\right)\right|^q. \tag{4.14}$$

By connecting the (4.13), (4.14) with (4.12), we get (4.11). This is the required proof.  $\square$

**Corollary 4.9.** Under the similar consideration in Theorem 4.8, by choosing  $s = 1$ , we obtain

$$\left| \varphi(z) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \varphi(u) du \right| \leq \frac{1}{\kappa_2 - \kappa_1} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2(e^{\frac{1}{2}}-1)}\right)^{\frac{1}{q}} \times \left\{ (z - \kappa_1)^2 \left|\varphi'\left(\frac{z + \kappa_1}{2}\right)\right| + (\kappa_2 - z)^2 \left|\varphi'\left(\frac{z + \kappa_2}{2}\right)\right| \right\}.$$

### 5. Applications

Consider the following special means for different positive real numbers  $\kappa_1, \kappa_2$  and  $\kappa_1 < \kappa_2$  as follows:

1. The arithmetic mean:

$$A(\kappa_1, \kappa_2) = \frac{\kappa_1 + \kappa_2}{2}.$$

2. The Harmonic mean:

$$H(\kappa_1, \kappa_2) = \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}, \quad \kappa_1, \kappa_2 > 0.$$

3. The log-mean:

$$L = L(\kappa_1, \kappa_2) = \frac{\kappa_2 - \kappa_1}{\ln \kappa_2 - \ln \kappa_1}, \quad \kappa_1 \neq \kappa_2.$$

4. The generalized log-mean:

$$L_r(\kappa_1, \kappa_2) = \left[ \frac{\kappa_2^{r+1} - \kappa_1^{r+1}}{(r+1)(\kappa_2 - \kappa_1)} \right]^{\frac{1}{r}}; \quad r \in \mathbb{R} \setminus \{-1, 0\}.$$

5. The Identric mean:

$$I(\kappa_1, \kappa_2) = \left\{ \begin{array}{ll} \kappa_1 & \kappa_1 = \kappa_2 \\ \frac{1}{e} \left(\frac{\kappa_2^{\kappa_2}}{\kappa_1^{\kappa_1}}\right)^{\frac{1}{\kappa_2 - \kappa_1}} & \kappa_1 \neq \kappa_2 \end{array} \right\}.$$

Dragomir et al. [31], have proved that for  $s \in (0, 1)$ , where  $1 \leq r \leq \frac{1}{s}$ , the function  $\varphi(z) = z^{rs}$ ,  $z > 0$  is  $s$ -convex function.

Now, using the theory results in section 4, we give some applications to above special means for positive different real numbers.

**Proposition 5.1.** *Let  $0 < \kappa_1 < \kappa_2$ . Then for some fixed  $s \in [\ln 2.4, 1)$ , we obtain*

$$|A^{rs}(\kappa_1, \kappa_2) - L_{rs}^{rs}(\kappa_1, \kappa_2)| \leq (\kappa_2 - \kappa_1) \frac{\mathbb{M}}{2} \left\{ \left( \frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\}. \tag{5.1}$$

*Proof.* The result holds by letting  $z = \frac{\kappa_1 + \kappa_2}{2}$  in Theorem 4.2 with exponentially  $s$ -convexity for  $\varphi : (0, \infty) \rightarrow \mathbb{R}$ ,  $\varphi(z) = z^{rs}$ , we obtain the required result.  $\square$

**Proposition 5.2.** *Let  $0 < \kappa_1 < \kappa_2$  and  $q > 1$ . Then for some fixed  $s \in [\ln 2.4, 1)$ , we obtain*

$$|\ln I(\kappa_1, \kappa_2) - \ln A(\kappa_1, \kappa_2)| \leq 2^{\frac{1}{q}-1} \mathbb{M}(\kappa_2 - \kappa_1) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left( \frac{e^s - s - 1}{s} \right) \right\}^{\frac{1}{q}}. \tag{5.2}$$

*Proof.* The result holds by letting  $z = \frac{\kappa_1 + \kappa_2}{2}$  in Theorem 4.4 with exponentially  $s$ -convexity for  $\varphi : (0, \infty) \rightarrow \mathbb{R}$ ,  $\varphi(z) = -\ln z$ , we obtain the required result.  $\square$

**Proposition 5.3.** *Let  $0 < \kappa_1 < \kappa_2$  and  $q \geq 1$ . Then for some fixed  $s \in [\ln 2.4, 1)$ , we obtain*

$$|H(\kappa_1, \kappa_2) - L^{-1}(\kappa_1, \kappa_2)| \leq (\kappa_2 - \kappa_1) \frac{\mathbb{M}}{2^{2-\frac{1}{q}}} \times \left\{ \left( \frac{2 + (2s-2)e^s - s^2}{2s^2} \right) + \left( \frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\}^{\frac{1}{q}}. \tag{5.3}$$

*Proof.* The result holds by letting  $z = \frac{\kappa_1 + \kappa_2}{2}$  in Theorem 4.6 with exponentially  $s$ -convexity for  $\varphi : (0, \infty) \rightarrow \mathbb{R}$ ,  $\varphi(z) = \frac{1}{z}$ , we obtain the required result.  $\square$

## 6. conclusion

In this article, we introduced and studied the concept of a new family of convex functions, called exponentially  $s$ -convex functions. Ostrowski-type inequalities are especially very important in the estimation of different quadrature rules. Some new Ostrowski type inequalities for exponentially  $s$ -convex function are established. Finally, we derive several new versions of the Ostrowski type inequalities regarding this newly discussed idea. Some applications to special means are attained as well. In the future, we hope the results of this paper and the new idea can be extended in different directions like fractional calculus, quantum calculus, time scale calculus, etc. We hope that our new ideas and techniques may energize and excite many researchers in this fascinating field of Mathematics.

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