



Fixed points of (ψ, ϕ) –contractions and Fredholm type integral equation

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• Received: 12 March 2021 • Accepted: 28 March 2021 • Published Online: 30 March 2021

Abstract

In this paper, we establish a fixed point theorem for controlled rectangular b –metric spaces for mappings that satisfy (ψ, ϕ) –contractive mappings. Also, we give an application of our results as an integral equation.

Keywords: (ψ, ϕ) –contractive mappings, Controlled rectangular b –metric spaces, Fixed point theorems, Integral equations.

2010 MSC: 47H10, 54H25..

1. Introduction

Fixed point theory has been one of the most swiftly developing fields in analysis in the last few years. This theory, with its broad application possibilities, has advanced the research activities in enormous areas. Many researchers have generalized the classical concept of metric space by partially changing the conditions of the metric. The introduction of a rectangular metric spaces concept by Branciari ([1]) resulted in establishing fixed points theorems for numerous contractions on those spaces. In (1969), Boyd and Wong ([2]) characterized a class of contractive mappings called ϕ contractions. Then, in (1997), Alber and Guerre-Delabriere ([3]) generalized this concept by introducing weak ϕ contraction. Various researchers have examined contractions of this kind. The class of (ψ, ϕ) weakly contractive mappings has attracted interest by many researchers (see, e.g, [4, 5, 6, 7, 8, 9]). In this paper, we consider (ψ, ϕ) –contractive mappings on controlled

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rectangular b -metric spaces. We also introduce fixed point theorem for controlled rectangular b -metric spaces for mappings satisfy (ψ, ϕ) -contractive mappings. The paper starts with some preliminaries covering definitions and notations needed throughout the papers. The main result is stated and proved in the third section. The application of our results to integral equations.

2. Preliminaries

The concept of rectangular metric spaces was introduced by Branciari in [1] as follows;

Definition 2.1. [1] (Rectangular (or Branciari) metric spaces)) Let X be a nonempty set. A mapping $\Delta : X^2 \rightarrow [0, \infty)$ is called a rectangular metric on X if for any $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$, it satisfies the following conditions:

- (R₁) $x = y$ if and only if $\Delta(x, y) = 0$;
- (R₂) $\Delta(x, y) = \Delta(y, x)$;
- (R₃) $\Delta(x, y) \leq \Delta(x, u) + \Delta(u, v) + \Delta(v, y)$.

In this case, the pair (X, Δ) is called a rectangular metric space.

In [10], George et al introduced the concept of b -rectangular metric spaces as follows.

Definition 2.2. [10] (Rectangular b -metric spaces)) Let X be a nonempty set. A mapping $B : X^2 \rightarrow [0, \infty)$ is called a rectangular b -metric on X if there exists a constant $\alpha \geq 1$ such that for any $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$, it satisfies the following conditions:

- (R_{b1}) $x = y$ if and only if $B(x, y) = 0$;
- (R_{b2}) $B(x, y) = B(y, x)$;
- (R_{b3}) $B(x, y) \leq \alpha[B(x, u) + B(u, v) + B(v, y)]$.

In this case, the pair (X, B) is called a rectangular metric space.

As a generalization of rectangular b -metric spaces, Abdeljawad et al. in [11], introduced the concept of extended Branciari b -distance spaces as follows:

Definition 2.3. [11] For a non-empty set S and a mapping $\omega : S \times S \rightarrow [1, \infty)$, we say that a function $B_{\text{dist}} : S \times S \rightarrow [0, \infty)$ is called an extended Branciari b -distance if it satisfies

- (i) $B_{\text{dist}}(x, y) = 0$ if and only if $x = y$;
- (ii) $B_{\text{dist}}(x, y) = B_{\text{dist}}(y, x)$;
- (iii) $B_{\text{dist}}(x, y) \leq \omega(x, y)[B_{\text{dist}}(x, u) + B_{\text{dist}}(u, v) + B_{\text{dist}}(v, y)]$,

for all $x, y \in S$ and all distinct $u, v \in S \setminus \{x, y\}$. The couple of the symbols (S, B_{dist}) denotes an extended Branciari b -distance space (shortly, B_{dist} -metric space).

Mlaiki et. al. in [12], introduced the concept of controlled rectangular b -metric spaces, which an extension of the rectangular metric spaces.

Definition 2.4. [12] Let X be a non empty set, a function $\zeta : X^4 \rightarrow [1, \infty)$ and $d_\zeta : X^2 \rightarrow [0, \infty)$. We say that (X, d_ζ) is a controlled rectangular b -metric space if all distinct $a, b, u, v \in X$ we have:

1. $d_\zeta(a, b) = 0$ if and only if $a = b$;

2. $d_\zeta(a, b) = d_\zeta(b, a)$;
3. $d_\zeta(a, b) \leq \zeta(a, b, u, v)[d_\zeta(a, u) + d_\zeta(u, v) + d_\zeta(v, b)]$.

Next, we present the topology of controlled rectangular b-metric spaces.

Definition 2.5. [12] Let (X, d_ζ) be a controlled rectangular b-metric space,

1. A sequence $\{a_n\}$ is called d_ζ -convergent in a controlled rectangular b-metric space (X, d_ζ) , if there exists $a \in X$ such that $\lim_{n \rightarrow \infty} d_\zeta(a_n, a) = d_\zeta(a, a) = 0$.
2. A sequence $\{a_n\}$ is called d_ζ -Cauchy if and only if $\lim_{n, m \rightarrow \infty} d_\zeta(a_n, a_m)$ exists and finite.
3. A controlled rectangular b-metric space (X, D_ζ) is called d_ζ -complete if for every d_ζ -Cauchy sequence $\{a_n\}$ in X , there exists $v \in X$, such that $\lim_{n \rightarrow \infty} d_\zeta(a_n, v) = \lim_{n, m \rightarrow \infty} d_\zeta(a_n, a_m) = d_\zeta(v, v) = 0$.
4. Let $a \in X$ define an open ball in a controlled rectangular b-metric space (X, d_ζ) by $B_\zeta(a, \eta) = \{b \in X \mid d_\zeta(a, b) < \eta\}$.

Notice that, rectangular metric spaces and rectangular b-metric spaces are controlled rectangular b-metric spaces, but the converse is not always true. In the following example, we present a controlled rectangular b-metric space which is not a rectangular metric space.

Example 2.6. [12] Let $X = Y \cup Z$ where $Y = \{\frac{1}{m} \mid m \text{ is a natural number}\}$ and Z be the set of positive integers. We define $d_\zeta : X^2 \rightarrow [0, \infty)$ by

$$d_\zeta(a, b) = \begin{cases} 0, & \text{if and only if } a = b \\ 2\beta, & \text{if } a, b \in Y \\ \frac{\beta}{2}, & \text{otherwise,} \end{cases}$$

where, β is a constant bigger than 0. Now, define $\zeta : X^4 \rightarrow [1, \infty)$ by $\zeta(a, b, u, v) = \max\{a, b, u, v\} + 2\beta$. It is not difficult to check that (X, d_ζ) is a controlled rectangular b-metric space. However, (X, d_ζ) is not a rectangular metric space, for instance notice that $d_\zeta(\frac{1}{2}, \frac{1}{3}) = 2\beta > d_\zeta(\frac{1}{2}, 2) + d_\zeta(2, 3) + d_\zeta(3, \frac{1}{3}) = \frac{3\beta}{2}$.

In the next section, we present our main result.

3. Main Results

First of all, before we present our first theorem we give the definition of the following class of functions;

Definition 3.1. Suppose that there exist control functions Ψ and ϕ so that $\psi : [0, +\infty) \rightarrow [0, +\infty)$ with $\psi(0) = 0$ and $\psi(t) = 0$ if and only if $t = 0$, the function ψ is a continuous monotone- decreasing function. and

$\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$ where $L > 0$, the function ψ is nondecreasing.

Moreover, throughout this paper we use the following notations;

$$M(x, y) = \max\{d_\zeta(x, y), d_\zeta(x, Tx), d_\zeta(y, Ty)\};$$

$$m(x, y) = \min\{d_\zeta(x, Tx), d_\zeta(y, Ty), d_\zeta(x, Ty), d_\zeta(y, Tx)\}.$$

Theorem 3.2. Let (X, d_ζ) be a Hausdorff and complete controlled rectangular b-metric space and let $T : X \rightarrow X$ be a self-map satisfying

$$\psi(d_\zeta(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)) + Lm(x, y), \tag{3.1}$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, where $L > 0$.

Suppose

$$\lim_{n, m \rightarrow \infty} \zeta(x_n, x_{n+1}, x_{n+2}, x_m) \leq \frac{1}{q}, \text{ for some } 0 < q < 1.$$

Then T has a unique fixed point in X .

Proof. Let x_0 be an arbitrary point in X . Define the sequence $\{x_n\} \subset X$ by $x_n = Tx_{n-1}$, $n = 1, 2, 3, \dots$.

First, we prove that $d_\zeta(x_{n-1}, x_n) \rightarrow 0$ as $n \rightarrow +\infty$, assume that $x_{n-1} \neq x_n$ substitute $x = x_{n-1}, y = x_n$ in (3.1)

$$\begin{aligned} \psi(d_\zeta(Tx_{n-1}, Tx_n)) &= \psi(d_\zeta(x_n, x_{n+1})) \leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)) + Lm(x_{n-1}, x_n) \\ &= \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)). \end{aligned}$$

Where,

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{d_\zeta(x_{n-1}, Tx_{n-1}), d_\zeta(x_n, Tx_n), d_\zeta(x_{n-1}, Tx_n), d_\zeta(x_n, Tx_{n-1})\} \\ &= \min\{d_\zeta(x_{n-1}, x_n), d_\zeta(x_n, x_{n+1}), d_\zeta(x_{n-1}, x_{n+1}), d_\zeta(x_n, x_n)\} \\ &= 0 \end{aligned}$$

$$M(x_{n-1}, x_n) = \max\{d_\zeta(x_{n-1}, x_n), d_\zeta(x_{n-1}, Tx_{n-1}), d_\zeta(x_n, Tx_n)\}.$$

First, assume that

$$M(x_{n-1}, x_n) = d_\zeta(x_n, x_{n+1}).$$

Hence, we have

$$\psi(d_\zeta(x_n, x_{n+1})) \leq \psi(d_\zeta(x_n, x_{n+1})) - \phi(d_\zeta(x_n, x_{n+1})),$$

which implies $\phi(d_\zeta(x_n, x_{n+1})) = 0$, and hence $d_\zeta(x_n, x_{n+1}) = 0$. Thus,

$$x_n = x_{n+1} = Tx_n,$$

and that is x_n is a fixed point of T and we are done. So, without loss of generality we may assume that $M(x_{n-1}, x_n) = d_\zeta(x_{n-1}, x_n)$, that is

$$\psi(d_\zeta(x_n, x_{n+1})) \leq \psi(d_\zeta(x_{n-1}, x_n)) - \phi(d_\zeta(x_{n-1}, x_n)) \leq \psi(d_\zeta(x_{n-1}, x_n)). \tag{3.2}$$

Since ψ is nondecreasing, we deduce that $d_\zeta(x_n, x_{n+1}) \leq d_\zeta(x_{n-1}, x_n)$ for all $n \geq 1$. Therefore, the sequence $\{d_\zeta(x_n, x_{n+1})\}$ is monotone decreasing and consequently, there exists $r \geq 0$ such that

$$d_\zeta(x_{n-1}, x_n) \rightarrow r. \tag{3.3}$$

Letting $n \rightarrow +\infty$ in (3.2) and using the continuity of ψ and the continuity of ϕ , we obtain

$$\psi(r) \leq \psi(r) - \phi(r)$$

which implies that $\phi(r) = 0$, and then $r = 0$. Thus we have proved

$$d(x_{n-1}, x_n) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Similarly, it is not difficult to see that

$$d(x_{n-2}, x_n) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

For all $n \geq 1$, we have two cases.

Case 1: Let $x_n = x_m$ for some integers $n \neq m$. So, if for $m > n$ we have $T^{m-n}(x_n) = x_n$. Choose $y = x_n$ and $p = m - n$. Then $T^p y = y$, and that is, y is a periodic point of T . Thus, $d_\zeta(y, Ty) = d_\zeta(T^p y, T^{p+1} y) \leq k^p d_\zeta(y, Ty)$. Since $k \in (0, 1)$, we get $d_\zeta(y, Ty) = 0$, so $y = Ty$, that is, y is a fixed point of T .

Case 2: Suppose that $T^n x \neq T^m x$ for all integers $n \neq m$. Let $n < m$ be two natural numbers, to show that $\{x_n\}$ is a d_ζ -Cauchy sequence, we need to consider two subcases: Subcase 1: Assume that $m = n + 2p + 1$. By property (3) of the controlled rectangular b-metric spaces we have,

$$\begin{aligned} d_\zeta(x_n, x_{n+2p+1}) &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) [d_\zeta(x_n, x_{n+1}) \\ &\quad + d_\zeta(x_{n+1}, x_{n+2}) + d_\zeta(x_{n+2}, x_{n+2p+1})] \\ &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) d_\zeta(x_n, x_{n+1}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) d_\zeta(x_{n+1}, x_{n+2}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) [d_\zeta(x_{n+2}, x_{n+3}) \\ &\quad + d_\zeta(x_{n+3}, x_{n+4}) + d_\zeta(x_{n+4}, x_{n+2p+1})] \\ &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) d_\zeta(x_n, x_{n+1}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) d_\zeta(x_{n+1}, x_{n+2}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) d_\zeta(x_{n+2}, x_{n+3}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) d_\zeta(x_{n+3}, x_{n+4}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) d_\zeta(x_{n+4}, x_{n+2p+1}) \\ &\leq \dots \\ &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) d_\zeta(x_n, x_{n+1}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) d_\zeta(x_{n+1}, x_{n+2}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) d_\zeta(x_{n+2}, x_{n+3}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) d_\zeta(x_{n+3}, x_{n+4}) \\ &\quad + \dots + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p+1}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p+1}) \\ &\quad \dots \zeta(x_{n+2p-2}, x_{n+2p-1}, x_{n+2p}, x_{n+2p+1}) d_\zeta(x_{n+2p}, x_{n+2p+1}) \end{aligned}$$

Now, using the fact that

$$d(x_{n-1}, x_n) \rightarrow 0 \text{ as } n \rightarrow +\infty \text{ and } \sup_{m>1} \lim_{n \rightarrow \infty} \zeta(x_n, x_{n+1}, x_{n+2}, x_m) \leq \frac{1}{q}$$

we deduce,

$$\lim_{n,p \rightarrow \infty} d_{\zeta}(x_n, x_{n+2p+1}) = 0.$$

Subcase 2: $m = n + 2p$

$$\begin{aligned} d_{\zeta}(x_n, x_{n+2p}) &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) [d_{\zeta}(x_n, x_{n+1}) \\ &\quad + d_{\zeta}(x_{n+1}, x_{n+2}) + d_{\zeta}(x_{n+2}, x_{n+2p})] \\ &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) d_{\zeta}(x_n, x_{n+1}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) d_{\zeta}(x_{n+1}, x_{n+2}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p}) [d_{\zeta}(x_{n+2}, x_{n+3}) \\ &\quad + d_{\zeta}(x_{n+3}, x_{n+4}) + d_{\zeta}(x_{n+4}, x_{n+2p})] \\ &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) d_{\zeta}(x_n, x_{n+1}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) d_{\zeta}(x_{n+1}, x_{n+2}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p}) d_{\zeta}(x_{n+2}, x_{n+3}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p}) d_{\zeta}(x_{n+3}, x_{n+4}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p}) d_{\zeta}(x_{n+4}, x_{n+2p}) \\ &\leq \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) d_{\zeta}(x_n, x_{n+1}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) d_{\zeta}(x_{n+1}, x_{n+2}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p}) d_{\zeta}(x_{n+2}, x_{n+3}) \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p}) d_{\zeta}(x_{n+3}, x_{n+4}) \\ &\quad + \dots \\ &\quad + \zeta(x_n, x_{n+1}, x_{n+2}, x_{n+2p}) \zeta(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+2p}) \times \dots \\ &\quad \times \dots \zeta(x_{n+2p-3}, x_{n+2p-2}, x_{n+2p-1}, x_{n+2p}) d_{\zeta}(x_{n+2p-3}, x_{n+2p-2}) \\ &\quad + \prod_{i=0}^{2p-2} \zeta(x_{n+2i}, x_{n+2i+1}, x_{n+2i+1}, x_{n+2p}) d_{\zeta}(x_{n+2p-2}, x_{n+2p}) \end{aligned}$$

Now, using the fact that

$$d(x_{n-1}, x_n) \rightarrow 0 \text{ as } n \rightarrow +\infty; \quad d(x_{n-2}, x_n) \rightarrow 0 \text{ as } n \rightarrow +\infty$$

$$\text{and } \sup_{m>1} \lim_{n \rightarrow \infty} \zeta(x_n, x_{n+1}, x_{n+2}, x_m) \leq \frac{1}{q}$$

we deduce,

$$\lim_{n,p \rightarrow \infty} d_{\zeta}(x_n, x_{n+2p}) = 0.$$

Therefore, by subcase 1 and subcase 2, we deduce that the sequence $\{x_n\}$ is a d_{ζ} -Cauchy sequence. Since (X, d_{ζ}) is complete controlled rectangular metric, then $\{x_n\}$ converges to a limit, there exists $x \in X$ such that $x_n \rightarrow x$. Applying inequality (3.1) with $x = x_n$ and $y = x$, we obtain

$$\begin{aligned} \psi(d_{\zeta}(Tx_n, Tx)) &\leq \psi(M(x_n, x)) - \phi(M(x_n, x)) + Lm(x_n, x) \\ &= \psi(M(x_n, x)) - \phi(M(x_n, x)) \leq \psi(d_{\zeta}(x_n, x)) \end{aligned}$$

which implies that $d_\zeta(Tx_n, Tx) \leq d_\zeta(x_n, x)$

Since $x_n \rightarrow x$, letting $n \rightarrow +\infty$ in the above inequality, we obtain that $x_{n+1} = Tx_n \rightarrow Tx$. As (X, d_ζ) is Hausdorff, we have $x = Tx$, a contradiction with the assumption that T has no periodic point. Consequently T admits a periodic point, that is, there exists $a \in X$ such that $a = T^p a$ for some $p \geq 1$.

we prove the Existence of a fixed point. If $p = 1$, then $a = Ta$, that is, a is a fixed point of T . Suppose now that $p > 1$. We will prove that $u = T^{p-1}a$ is a fixed point of T . Suppose that this is not the case, that is, $T^{p-1}a \neq Ta$. Then $d_\zeta(T^{p-1}a, Ta) > 0$ and $\phi(d_\zeta(T^{p-1}a, Ta)) > 0$. Now, using inequality (3.1), we obtain

$$\begin{aligned} \psi(d_\zeta(a, Ta)) &= \psi(d_\zeta(T^p a, T^{p+1} a)) \\ &= \psi(d_\zeta(T(T^{p-1} a), T(T^p a))) \\ &\leq \psi(M(T^{p-1} a, T^p a)) - \phi(M(T^{p-1} a, T^p a)) + Lm((T^{p-1} a, T^p a)) \\ &< \psi(d_\zeta(T^{p-1} a, T^p a)). \end{aligned}$$

Since,

$$\begin{aligned} M(T^{p-1} a, T^p a) &= \max\{d_\zeta(T^{p-1} a, T^p a), d_\zeta(T^{p-1} a, Ta), d_\zeta(T^p a, T^{p+1} a)\}, \\ m(T^{p-1} a, T^p a) &= \min\{d_\zeta(T^{p-1} a, T^p a), d_\zeta(T^p a, T^{p+1} a), d_\zeta(T^{p-1} a, T^{p+1} a), d_\zeta(T^p a, Ta)\} \\ &= 0 \end{aligned}$$

For $M(T^{p-1} a, T^p a) = d_\zeta(T^p a, T^{p+1} a)$,

$$\psi(a, Ta) = \psi(d_\zeta(T^p a, T^{p+1} a)) \leq \psi(d_\zeta(T^{p-1} a, T^p a)) - \phi(d_\zeta(T^{p-1} a, T^p a)) \tag{3.4}$$

$$< \psi(d_\zeta(T^{p-1} a, T^p a)), \tag{3.5}$$

and taking into account the fact that ψ is nondecreasing, we deduce

$$d_\zeta(a, Ta) < d_\zeta(T^{p-1} a, T^p a).$$

Again, using inequality (3.1), $x = T^{p-2}a$ and $y = T^{p-1}a$ we have

$$\begin{aligned} \psi(d_\zeta(a, Ta)) &= \psi(d_\zeta(T^{p-1} a, T^p a)) \\ &\leq \psi(M(T^{p-2} a, T^{p-1} a)) - \phi(M(T^{p-2} a, T^{p-1} a)) + Lm((T^{p-2} a, T^{p-1} a)). \end{aligned}$$

where

$$\begin{aligned} M(T^{p-2} a, T^{p-1} a) &= \max\{d_\zeta(T^{p-2} a, T^{p-1} a), d_\zeta(T^{p-2} a, T^p a), d_\zeta(T^{p-1} a, T^p a)\}, \\ m(T^{p-2} a, T^{p-1} a) &= \min\{d_\zeta(T^{p-2} a, T^{p-1} a), d_\zeta(T^{p-1} a, T^p a), d_\zeta(T^{p-2} a, T^p a), d_\zeta(T^{p-1} a, T^{p-1} a)\} \\ &= 0. \end{aligned}$$

Since, $\phi(d_\zeta(T^{p-2}a, T^{p-1}a)) = 0$ and $d_\zeta(T^{p-2}a, T^{p-1}a) = 0$,

$$M(T^{p-2}a, T^{p-1}a) = d_\zeta(T^{p-2}a, T^{p-1}a)$$

Using the monotone property of the ψ -function, we deduce that

$$\begin{aligned} \psi(d_\zeta(T^{p-1}a, T^p a)) &\leq \psi(d_\zeta(T^{p-2}a, T^{p-1}a)) - \phi(d_\zeta(T^{p-2}a, T^{p-1}a)) \\ &\leq \psi(d_\zeta(T^{p-2}a, T^{p-1}a)) \end{aligned}$$

This leads to

$$\begin{aligned} 0 \leq d_\zeta(a, Ta) &= d_\zeta(T^{p-1}a, T^p a) \\ &< \zeta(T^{p-1}a, T^p a, x_n, x_{n+1}) [d_\zeta(T^{p-1}a, x_n) + d_\zeta(x_n, x_{n+1}) + d_\zeta(x_{n+1}, T^p a)] \\ &\leq \zeta(T^{p-1}a, T^p a, x_n, x_{n+1}) [d_\zeta(T^{p-1}a, x_n) + d_\zeta(x_n, x_{n+1}) + d_\zeta(Tx_n, T^p a)] \\ &\leq \dots \leq d_\zeta(a, Ta). \end{aligned}$$

Then we obtain the following contradiction: $d_\zeta(a, Ta) < d_\zeta(a, Ta)$. Hence, the assumption of u is not a fixed point of T is not true and thus $u = T^{p-1}a$ is a fixed point of T . Thus we have proved the existence of a fixed point of T .

Finally, to prove the uniqueness, we assume that T has two distinct fixed points, say z and w . Then letting $x = z$ and $y = w$ in (3.1), we have

$$\begin{aligned} \psi(d_\zeta(a, w)) &= \psi(d_\zeta(Ta, Tw)) \\ &\leq \psi(M(a, w)) - \phi(M(a, w)) + Lm((a, w)). \end{aligned}$$

where

$$\begin{aligned} M(a, w) &= \max\{d_\zeta(a, w), d_\zeta(a, Ta), d_\zeta(w, Tw)\} = d_\zeta(a, w) \\ m(a, w) &= \min\{d_\zeta(a, Ta), d_\zeta(w, Tw), d_\zeta(w, Ta), d_\zeta(a, Tw)\} = 0 \\ \psi(d_\zeta(a, w)) &\leq \psi(d_\zeta(a, w)) - \phi(d_\zeta(a, w)) \end{aligned}$$

implying $\phi(d_\zeta(z, w)) = 0$, and hence $d_\zeta(z, w) = 0$, which completes the proof of the uniqueness. □

Corollary 3.3. Assume (X, d_ζ) be a Hausdorff and complete controlled rectangular b -metric space and let $T : X \rightarrow X$ be a self-map satisfying

$$\psi(d_\zeta(Tx, Ty)) \leq \psi(M(x, y)) \sim \phi(M(x, y)) \tag{3.6}$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, also, assume that

$$\lim_{n, m \rightarrow \infty} \zeta(x_n, x_{n+1}, x_{n+2}, x_m) \leq \frac{1}{q}, \text{ for some } 0 < q < 1.$$

Then T has a unique fixed point in X .

Proof. For

$$\begin{aligned} \psi(d_\zeta(Tx, Ty)) &\leq \psi(M(x, y)) \sim \phi(M(x, y)) \\ &\leq \psi(M(x, y)) \sim \phi(M(x, y)) + L \min\{d_\zeta(x, Tx), d_\zeta(y, Ty), d_\zeta(x, Ty), d_\zeta(y, Tx)\}. \end{aligned}$$

for some $L > 0$ and by Theorem (3.2), T has a unique fixed point in X . □

4. Fredholm type integral equation

Fixed point theory has many applications, such as fractional differential equations, the importance of these types of equations is their use in modelling in many fields. In this section, we use our results to prove the existence and uniqueness of Fredholm type integral equation. Now, Consider the set $\Upsilon = C([0, 1], (-\infty, \infty))$ and the following Fredholm type integral equation:

$$p'(t) = \int_0^1 S(t, s, p'(t)) ds, \text{ for } t, s \in [0, 1] \tag{4.1}$$

where $S(t, s, p'(t))$ is a continuous function from $[0, 1]^2$ into \mathbb{R} . Now, define

$$d_\zeta : \Upsilon \times \Upsilon \longrightarrow \mathbb{C} \\ (p, q) \mapsto |p'(t) - q(t)|$$

Note that (Υ, d_ζ) is a complete controlled rectangular b-metric space, where

$$\zeta(p, q, u, v) = 2.$$

Theorem 4.1. Assume that for all $p, q \in \Upsilon$

(1) $|S(t, s, p'(t)) - S(t, s, q(t))| \leq \frac{|p'(t) - q(t)|}{2}$, for some $\vartheta \in B$.

(2) $|S(t, s, \int_0^1 S(t, s, p'(t)) ds) - S(t, s, \int_0^1 S(t, s, q(t)) ds)| < |S(t, s, p'(t)) - S(t, s, q(t))|$ for all t, s .

Then the above integral equation has a unique solution.

Proof. Let $\psi : \Upsilon \longrightarrow \Upsilon$ defined by $p'(t) = \int_0^1 S(t, s, p'(t)) ds$, then $d_\zeta(p', q) = |p'(t) - q(t)|$. Now we have

$$\begin{aligned} d_\zeta(p'(t), q(t)) &= |p'(t) - q(t)| \\ &= \left| \int_0^1 S(t, s, p'(t)) ds - \int_0^1 S(t, s, q(t)) ds \right| \\ &\leq |S(t, s, p'(t)) - S(t, s, q(t))| \\ &\leq \frac{|p'(t) - q(t)|}{2} \\ &\leq \frac{1}{2} d_\zeta(p'(t), q(t)) = \psi(M(p'(t), q(t))) - \phi(M(p'(t), q(t))). \end{aligned}$$

where $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$. Also, notice that,

$$\zeta(p, q, u, v) < 3.$$

Therefore, all the hypothesis of Theorem 3.2, are satisfied and hence equation (4.1) has a unique solution as desired. □

5. Conclusion

In this manuscript, we have proved the existence and uniqueness of a fixed point for (ψ, ϕ) -contractions in controlled rectangular b -metric spaces, which generalizes many results in the literature. Also, we presented an application to our results on integral equations. In closing, we would like to bring to the reader's attention the following open question;

Question 5.1. *If we omit the completeness of the controlled rectangular b -metric space in Theorem 3.2, Is there a weaker hypothesis that we can add to get a unique fixed point?*

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