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On a novel n-tuple variable order q-fractional derivative with respect to ψ function: hybrid difference equation and the well-posedness of the solution

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Abstract

This paper introduces a novel generalized fractional operator: the n-tuple variable-order q-fractional derivative with respect to a ψ function. This operator provides a unified framework that extends several existing q-fractional models, including the Riemann–Liouville, Caputo, and Hilfer types, along with their variable-order and ψ-dependent variants. Fundamental properties and results are established to analyze the well-posedness of a class of hybrid fractional difference equations. The existence of the solution is established via Krasnoselskii's fixed point theorem, while uniqueness is demonstrated using the Banach contraction principle. Furthermore, the Ulam–Hyers stability of the solution is rigorously investigated. An illustrative example is provided to demonstrate the applicability of the theoretical results. This work concludes by identifying future research trajectories, specifically the extension of n-tuple operators to non-local boundary value problems and the exploration of (q, ψ)-difference systems with higher-order derivatives ($n > 1$), offering a definitive foundation for subsequent developments in quantum fractional calculus.

Keywords: (q, ψ)-fractional calculus, variable-order derivative, n-tuple fractional operator, hybrid fractional difference equation, well-posedness.
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1. Introduction

Fractional calculus is initially a mathematical area involving variants of derivatives and integral operators with the order of operation lying between $n-1 < \alpha < n$ where $n \in \mathbb{N} \setminus \{0\}$. The variations can be extended further. In some studies, authors explore fractional operators with non-constant order, known as variable order (see [1, 2, 3]), such that $n-1 < \alpha(t) < n$. In other works, the rate of change is rescaled so that the operator

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respects a function $\psi(t)$; this leads to the so-called ψ-fractional operators (see [4, 5]). For example, the Riemann-Liouville fractional integral of variable order is defined by

$$ {}_aI_t^{\alpha(t)}f(t) = \frac{1}{\Gamma(\alpha(t))}\int_a^t (t-s)^{\alpha(t)-1}f(s)ds, \quad 0 < \alpha(t) < 1. $$

And, the ψ-Riemann-Liouville fractional integral is defined by

$$ {}_aI_\psi^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (\psi(t)-\psi(s))^{\alpha-1}f(s)\psi'(s)ds, \quad 0 < \alpha < 1. $$

The classical variations of fractional derivative, such as the Caputo and Riemann-Liouville derivatives, can be obtained by combining fractional integral operators and integer-order derivatives in different positions (see [1]).

Later, Hilfer [6] introduced a definition of fractional derivative that interpolates between the Riemann–Liouville and Caputo types. The Hilfer fractional derivative is defined as follows:

$$ {}_aD_t^{\alpha,\beta}f(t) = {}_aI_t^{\beta(n-\alpha)}D^n\,{}_aI_t^{(1-\beta)(n-\alpha)}f(t). $$

In a sense, it can be said that these variations represent generalizations of fractional operators in different forms.

In 1909, q-calculus was introduced by Jackson [7] where q-derivative is

$$ D_qf(t) = \frac{f(qt)-f(t)}{qt-t}, $$

and q-integral is

$$ I_qf(t) = \int_0^t f(s)d_qs = (1-q)\sum_{n=0}^{\infty} tq^nf(tq^n). $$

Moreover, q-fractional calculus has also been widely studied (see [8, 9, 10, 11, 12, 13, 14, 15]). The following timeline summarizes the author's works related to the development of the present study.

In 2021, inspired by the Hilfer fractional derivative, the author and collaborators [9] introduced q-Hilfer fractional derivative such that

$$ {}_qD_t^{\alpha,\beta}f(t) = {}_qI_t^{\beta(1-\alpha)}D_q\,{}_qI_t^{(1-\beta)(1-\alpha)}f(t) = {}_qI_t^{\gamma-\alpha}\,{}_qD_t^{\gamma}f(t), \quad \gamma = \alpha+\beta-\alpha\beta, $$

and the q-Hilfer fractional derivative of variable order such that

$$ {}_qD_t^{\alpha(t),\beta}f(t) = {}_qI_t^{\beta(1-\alpha(t))}D_q\,{}_qI_t^{(1-\beta)(1-\alpha(t))}f(t) $$

when $0 < \alpha < 1$, $0 < \alpha(t) < 1$ and $0 \leqslant \beta \leqslant 1$. The uniqueness of the continuous solution of a fractional hybrid integro-difference equation was examined in that work. The q-Hilfer fractional derivative caught a certain amount of attention from mathematicians. There are further studies (see [13, 15]), including the operator with a time interval shifted such that initial time $t_0 = a$ (see [12]).

In the same year — although the paper [11] (hereinafter the previous paper) was published several years later — the author and collaborators derived q-fractional operators with respect to the ψ function. However, the proof was not clearly provided. The process of deriving such an operator was inspired by the traditional concept [16] of iterated integral in q-calculus fashion follows

$$_qI_t^n f(t) = \int_0^t \int_0^{x_{n-1}} \dots \int_0^{x_1} f(s) d_q s d_q x_1 \dots d_q x_{n-1} = \frac{1}{\Gamma_q(n)} \int_0^t (t - qs)_q^{(n-1)} f(s) d_q s.$$

Nonetheless, almost four years later, I encountered a paper written by Kamsrisuk et al. [14] on q-calculus that respects another function, revisited the previous paper, and discovered that the result concerning the derivation of the n-iteration of the q-integral in the previous paper contained errors, making it valid only under very limited conditions (when ψ is homogeneous). However, the conditions for uniqueness and Ulam–Hyers stability of the solution — which are the main results — remain correct. Therefore, I would like to dedicate one section of this paper to revisiting the proof techniques that I did not clearly present in my earlier work, explaining how and why certain operators and theorems were obtained, and clarifying what are the mistakes. Nowadays, q-fractional calculus that respects another function has already started to catch mathematicians' attention (see [18, 19]).

In 2022 — through a stroke of inspiration — the author independently discovered a further interpolation beyond the q-Hilfer derivative where for $0 < \alpha_1 < 1$ and $\alpha_2, \alpha_3, \dots, \alpha_n \in [0, 1]$ the q-fractional derivative with n order of operations is introduced such that

$$_qD_t^{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n} f(t) = {_qI_t^{\left(1 - \prod_{i=2}^n (1-\alpha_i)\right)(1-\alpha_1)}} D_q \, {_qI_t^{\prod_{i=1}^n (1-\alpha_i)}} f(t).$$

The operator is referred to as the n-tuple q-fractional derivative (see [10]).

Recent advancements in the field have highlighted the necessity of more flexible operators to describe complex hereditary properties. For instance, the study of existence theorems for hybrid fractional differential equations has been expanded through ψ-weighted Caputo–Fabrizio derivatives, which offer a non-singular kernel approach to ψ-fractional systems [21]. Similarly, the investigation of ψ-fractional hybrid systems with periodic boundary conditions [22] and the development of existence results for nonlinear neutral generalized Caputo fractional equations [23] demonstrate a clear trend toward high-degree interpolation between classical derivative types. While these works provide robust results in the continuous and ψ-weighted domains, a gap remains in the q-calculus analog. The n-tuple variable-order (q, ψ)-fractional operator introduced in this paper fills this gap by unifying these diverse approaches into a single, discrete-time scale framework.

In this paper, which will serve as the author's final contribution to the mathematical field, I combine all the aforementioned generalizations — that is, I define the q-fractional derivative with respect to ψ, incorporating n-tuple orders of operation, where each order is variable order. I focus on a hybrid difference equation, similar to that of my first publication in this field [20], in order to investigate the existence, uniqueness, and Ulam-Hyers stability (altogether is the well-posedness) of its mild solution. In other words, in this work, the following equation will be studied

$$\begin{aligned} &_qD_\psi^{\alpha_1(t), \alpha_2(t), \alpha_3(t), \dots, \alpha_n(t)} [x(t) - f(t, x(t))] = g(t, x(t)), \quad 0 < \alpha_1(t) < 1 \\ &_qI_\psi^{1-\Omega(t)} x(0) = x(0) = x_0, \quad _qI_\psi^{1-\Omega(t)} f(0, x(0)) = f(0, x(0)) = f_0, \quad t \in [0, T] \end{aligned} \tag{1.1}$$

where $\Omega(t) = 1 - \prod_{i=1}^{n}(1-\alpha_i(t))$ and $0 \leqslant \alpha_2(t), \alpha_3(t), \dots, \alpha_n(t) \leqslant 1$.

The existence of a continuous solution will be established using Krasnoselskii's fixed point theorem, the uniqueness will be demonstrated by Banach's fixed point theorem, and the Ulam–Hyers stability will also be illustrated.

2. Preliminary and Framework

This section summarizes the q-analog structures, spaces, definitions, and theorems used in the author's previous works, as they are necessary for understanding the present study. First, lets review the q-analog structures and spaces that form the mathematical foundation of this work.

2.1. (q, g)*-derivative and* q*-analog of special functions*

First, note that

$$ {}_qD_g f(t) = \frac{f(t) - f(qt)}{g(t) - g(qt)}, $$

where ${}_qD_g$ denotes the q-difference operator with respect to the function g (so-called (q, g)-derivative). The relevant q-analog structures are defined as follows.

Definition 2.1. For any $q \in (0, 1)$, the q-analog of power function is defined by

$$ \begin{aligned} (n-m)_q^{(k)} &= \prod_{i=0}^{k-1}(n - mq^i) \\ &= (n-m)(n-mq)(n-mq^2)\dots(n-mq^{k-1}). \end{aligned} $$

In more general sense, the notation is defined by

$$ (n-m)_q^{(k)} = n^k \prod_{i=0}^{\infty} \frac{n - mq^i}{n - mq^{i+k}}, \quad n \neq 0, \quad k \in \mathbb{R}. $$

Proof. Let $n \neq 0$ and $k, N \in \mathbb{R}$ such that $k < N$. Then,

$$ \begin{aligned} (n-m)_q^{(k)} &= \prod_{i=0}^{k-1}(n - mq^i) \\ &= (n-m)(n-mq)(n-mq^2)\dots(n-mq^{k-1}) \\ &= \frac{\prod_{i=0}^{N+k}(n-mq^i)}{\prod_{j=k}^{N+k}(n-mq^j)} \\ &= \frac{\prod_{i=0}^{k-1}(n-mq^i) \times \prod_{j=k}^{N}(n-mq^j) \times (n-mq^{N+1})(n-mq^{N+2})\dots(n-mq^{N+k})}{\prod_{j=k}^{N+k}(n-mq^j)} \end{aligned} $$

Expanding the product, it is obtained that

$$\begin{aligned}(n-m)_q^{(k)} &= \frac{\prod_{i=0}^{N}(n-mq^i)}{\prod_{i=0}^{N}(n-mq^{i+k})} \times (n-mq^{N+1})(n-mq^{N+2})\dots(n-mq^{N+k}) \\ &= \lim_{N\to\infty}\prod_{i=0}^{N}\frac{n-mq^i}{n-mq^{i+k}} \times \lim_{N\to\infty}(n-mq^{N+1})(n-mq^{N+2})\dots(n-mq^{N+k}) \\ &= n^k\prod_{i=0}^{\infty}\frac{n-mq^i}{n-mq^{i+k}}.\end{aligned}$$

This completes the proof. □

Definition 2.2. For any $q \in (0,1)$, the q-Gamma function is defined by

$$\Gamma_q(j) = \frac{(1-q)_q^{(j-1)}}{(1-q)^{j-1}}, j \in \mathbb{R}\setminus\{0,-1,-2,...\},$$

where $\Gamma_q(j+1) = [j]_q\Gamma_q(j)$ and

$$[h]_q = \frac{1-q^h}{1-q}, h \in \mathbb{R}.$$

Also, $\Gamma_q(j) = [j-1]!$ when $j \in \mathbb{N}\setminus\{0\}$.

Definition 2.3. Let $a, b \in \mathbb{R}^+$ and $q \in (0,1)$. The q-Beta function is defined by

$$B_q(a,b) = \int_0^1 s^{a-1}(1-qs)_q^{(b-1)}d_qs = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}.$$

2.2. q-*analog of spaces*

This subsection contains relevant spaces used in this paper.

Definition 2.4. [16] Let $q \in \mathbb{R}$, a subset $A \subset \mathbb{C}$ is called q-geometric if $qz \in A$ whenever $z \in A$.

Definition 2.5. [16] Let A be a q-geometric subset of $\mathbb{C}$ with $0 \in A$. A function f on A is said to be q-regular at 0 if, for every $z \in A$, the following limit exists:

$$\lim_{n\to\infty} f(zq^n) = f(0).$$

Definition 2.6. [16] For any $p \geqslant 1$, the space $L_q^p(a,b)$ is the space of the functions such that

$$\left(\int_a^b |f(t)|^p d_qt\right)^{\frac{1}{p}} < \infty.$$

For $p = 1$, this space is denoted by $L_q(a,b)$.

Definition 2.7. For any $p \geqslant 1$, the space $L^p_{q,\psi}(a,b)$ is the space of the functions such that

$$\left(\int_a^b |f(t)|^p D_q\psi(t) d_q t\right)^{\frac{1}{p}} < \infty.$$

For $p = 1$, this space is denoted by $L_{q,\psi}(a,b)$.

Definition 2.8. [16] For any $p \in \mathbb{R}^+$, the space $\mathbb{L}^p_q[a,b]$ is the space of the functions on interval $[a,b]$. The space $\mathbb{L}^p_q[a,b]$ is a Banach space with the supremum norm $\|\cdot\|_p$ defined by

$$\|f\|_p = \sup_{t\in[a,b]} \left(\int_a^b |f(t)|^p d_q t\right)^{\frac{1}{p}} < \infty.$$

For $p = 1$ it can be denoted the space as $\mathbb{L}_q[a,b]$.

Definition 2.9. For any $p \in \mathbb{R}^+$, the space $\mathbb{L}^p_{q,\psi}[a,b]$ is the space of the functions on interval $[a,b]$. The space $\mathbb{L}^p_{q,\psi}[a,b]$ is a Banach space with the supremum norm $\|\cdot\|_{p,\psi}$ defined by

$$\|f\|_{p,\psi} = \sup_{t\in[a,b]} \left(\int_a^b |f(t)|^p D_q\psi(t) d_q t\right)^{\frac{1}{p}} < \infty.$$

For $p = 1$ it can be denoted the space as $\mathbb{L}_{q,\psi}[a,b]$ and the supremum norm $\|\cdot\|_{p,\psi} = \|\cdot\|$.

Definition 2.10. [16] $C^n_q[a,b]$ denotes the space of continuous functions on $[a,b]$ such that $D^{n-1}_q f(t) \in C[a,b]$. Also, $C^n_q[a,b]$ is a Banach space with supremum norm $\|\cdot\|_{C^n_q}$ such that

$$\|f\|_{C^n_q} = \sup_{t\in[a,b]} \sum_{i=0}^{n-1} |D^i_q f(t)| < \infty.$$

For $n = 1$, this space reduces to $C[a,b]$, and for $q = 1$, to $C^n[a,b]$.

Definition 2.11. $C^n_{q,\psi}[a,b]$ denotes the space of continuous functions on $[a,b]$ such that ${}_qD^{n-1}_\psi f(t) \in C[a,b]$. Also, $C^n_{q,\psi}[a,b]$ is a Banach space with supremum norm $\|\cdot\|_{C^n_{q,\psi}}$ such that

$$\|f\|_{C^n_{q,\psi}} = \sup_{t\in[a,b]} \sum_{i=0}^{n-1} |{}_qD^i_\psi f(t)| < \infty.$$

Definition 2.12. [16] A function f on $[0,a]$ is called absolutely continuous if it is q-regular at 0 and there exists a constant $K > 0$ such that

$$\sum_{i=0}^{\infty} |f(tq^i) - f(tq^{i+1})| \leqslant K$$

for every $t \in (qa, a]$.

Definition 2.13. [16] Let $AC_q[a, b]$ be a space of the absolutely continuous functions on $[a, b]$, then $f \in AC_q[a, b]$ if and only if there exists an arbitrary constant $\omega \in \mathbb{R}$ and the function $\phi(t) \in \mathbb{L}_q^p[a, b]$ such that

$$f(t) = \omega + \int_a^b \phi(s) d_q s.$$

For $q = 1$, it can be noted the space as $AC[a, b]$.

Definition 2.14. [16] The space $AC_q^{(n)}[a, b]$ consists of functions f on $[a, b]$ such that $D_q^{n-1} f(t) \in AC_q[a, b]$. For $q = 1$, it can be denoted as $AC^{(n)}[a, b]$

Definition 2.15. Let $AC_{q,\psi}[a, b]$ be a space of the absolutely continuous functions on $[a, b]$, then $f \in AC_{q,\psi}[a, b]$ if and only if there exists an arbitrary constant $\omega \in \mathbb{R}$ and the function $\phi(t) \in \mathbb{L}_q^p[a, b]$ such that

$$f(t) = \omega + \int_a^b \phi(s) D_q \psi(s) d_q s.$$

For $q = 1$, it can be noted the space as $AC[a, b]$.

Definition 2.16. The space $AC_{q,\psi}^{(n)}[a, b]$ is a space of function on $[a, b]$ such that ${}_qD_\psi^{n-1} f(t) \in AC_q[a, b]$.

The next section will revisit and address the issue in the previous paper.

3. Revisiting the Previous Work: Identification of Errors

Before we begin, as I am also a legal scholar, I would like to cite a quote from a U.S. case that best describes this section.

> A smart man knows when he is right; a wise man knows when he is wrong. Wisdom does not always find me, so I try to embrace it when it does — even if it comes late, as it did here.
>
> *Thomson Reuters Enter. Ctr. GmbH v. Ross Intel. Inc.*, 765 F. Supp. 3d 382, 390 (D. Del. 2025)

I thus revise my earlier formulation of the result and its proof in this paper.

3.1. What are the mistakes?

First, I would like to use ${}_q\mathrm{I}_\psi^n$ to refer to the incorrect illustration in the previous paper. In the previous paper, I stated that, since ψ is a strictly increasing function of s with $\psi(0) = 0$, writing $f(s) = g(s, \psi(s))$, where $s \mapsto g(s, \psi(s))$, and $f(s) = h(\psi(s))$, where $s \mapsto h(\psi(s))$, yields:

$$\begin{aligned}
{}_q\mathrm{I}_\psi^n f(t) &= \int_{\psi(0)}^{\psi(t)} \int_{\psi(0)}^{\psi_{n-1}} \dots \int_{\psi(0)}^{\psi_1} g(s, \psi(s)) d_q\psi_0 d_q\psi_1 \dots d_q\psi_{n-1} \\
&= \int_{\psi(0)}^{\psi(t)} \int_{\psi(0)}^{\psi_{n-1}} \dots \int_{\psi(0)}^{\psi_1} h(\psi(s)) d_q\psi_0 d_q\psi_1 \dots d_q\psi_{n-1} \\
&= \frac{1}{\Gamma_q(n)} \int_0^t (\psi(t) - q\psi(s))_q^{(n-1)} f(s) D_q\psi(s) d_q s.
\end{aligned}$$

It turns out that the aforementioned iterated integral — although valid only when ψ is homogeneous — is still not as trivial as it was presented in the previous paper. In this section, its origin will be explained, along with the corresponding mistakes. The mistake began here: after substitution, I carelessly applied ${}_qI_g$ when writing $f(t) = h(g(t))$, where $t \mapsto h(g(t))$, as follows:

$$ {}_qI_g f(t) = \int_0^t f(s) D_q g(s) d_q s = \int_{g(0)}^{g(t)} h(g(s)) d_q g(s) = (1-q) g(t) \sum_{n=0}^{\infty} q^n h(g(t) q^n). $$

After that, the incorrect iteration of ${}_qI_g$ was derived using induction as follows. First, where $g(0) = 0$, suppose that the following integral is true for $n = N$:

$$ \begin{aligned} {}_qI_g^n f(t) &= \int_{g(0)}^{g(t)} \int_{g(0)}^{g_{n-1}} \dots \int_{g(0)}^{g_1} h(g(s)) d_q g_0 d_q g_1 \dots d_q g_{n-1} \\ &= \frac{1}{[n-1]!} \int_{g(0)}^{g(t)} (g(t) - q g(s))_q^{(n-1)} h(g(s)) d_q g(s). \end{aligned} $$

We then obtain

$$ \begin{aligned} {}_qI_g^{N+1} f(t) &= \frac{1-q}{[n-1]!} \int_{g(0)}^{g(t)} \sum_{i=0}^{\infty} q^i \{ g(t) (g(t) - g(t) q^{i+1})_q^{(N-1)} h(g(t) q^i) \} d_q g(s) \\ &= \frac{(1-q)^2}{[n-1]!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^{i+j} (1 - q^{i+1})_q^{(N-1)} \{ [g(t)]^{N+1} q^{Nj} h(g(t) q^{i+j}) \}. \end{aligned} $$

This reduces to

$$ [N-1]! \, {}_qI_g^{N+1} f(t) = (1-q)^2 \sum_{s=0}^{\infty} q^s h(g(t) q^s) [g(t)]^{N+1} \sum_{j=0}^{\infty} q^j (q^j - q^{s+1})_q^{(N-1)}. $$

Applying the property stated [17] such that

$$ \sum_{i=0}^{m} q^j (q^j - q^{m+1})_q^{(n-1)} = \frac{(1 - q^{m+1})_q^{(n)}}{1 - q^n}, $$

it follows that

$$ \begin{aligned} {}_qI_g^{N+1} f(t) &= \frac{1-q}{[N]!} \sum_{s=0}^{\infty} q^s g(t) \{ (g(t) - g(t) q^{s+1})_q^{(N)} h(g(t) q^s) \} \\ &= \frac{1}{[N]!} \int_{g(0)}^{g(t)} (g(t) - q g(s))_q^{(N)} h(g(s)) d_q g(s) \\ &= \frac{1}{[N]!} \int_0^t (g(t) - q g(s))_q^{(N)} h(g(s)) D_q g(s) d_q s \\ &= \frac{1}{[N]!} \int_0^t (g(t) - q g(s))_q^{(N)} f(s) D_q g(s) d_q s. \end{aligned} $$

I also accidentally derived a theorem of the form

$$ {}_q I_\psi^\alpha {}_q D_\psi^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} {}_q I_\psi^{1+i-\alpha} f(0) \frac{(\psi(t)-\psi(0))^{\alpha-i-1}}{\Gamma_q(\alpha-i)} $$

where

$$ {}_q I_\psi^{1+i-\alpha} f(0) = \lim_{t\to 0^+} {}_q I_\psi^{1+i-\alpha} f(t). $$

Although, after correcting the earlier mistake, the final theorem remains the same, the version presented in the previous paper was derived through mathematically incorrect steps. While the method is not generally valid, for the reader's convenience, I will demonstrate how that earlier result was obtained. Beginning with

$$ \begin{aligned} {}_q I_\psi^\alpha {}_q D_\psi^\alpha f(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (\psi(t)-q\psi(s))_q^{(\alpha-1)} \\ &\quad \times \left\{ \frac{1}{\Gamma_q(1-\alpha)} \int_0^s (\psi(s)-q\psi(\tau))_q^{(\alpha-1)} f(\tau) D_q\psi(\tau) d_q\tau \right\} D_q\psi(s) d_q s. \end{aligned} $$

Substitute $\psi(\tau)=\theta$. And since $\psi(t)$ is a function of t, any function $f(t)$ can be expressed in the form $f(t)=g(t,\psi(t))$ for an appropriate two-variable function g, or in the form $f(t)=h(\psi(t))$ for an appropriate one-variable function h. The expressions $t\mapsto g(t,\psi(t))$ and $t\mapsto h(\psi(t))$ simply describe the same function f. For example, when $\psi(t)=t^2$ and $f(t)=t^3$, define $g(t,x)=t\cdot x$. Then $f(t)=g(t,\psi(t))=g(t,t^2)=t\cdot t^2=t^3$. Likewise, define $h(x)=x^{3/2}$. Then $f(t)=h(\psi(t))=h(t^2)=(t^2)^{3/2}=t^3$.
Here, $\psi(s)$ is treated as a constant. From now on, whenever $\psi(\cdot)$ is treated as a constant within an integral, denote $\psi(\cdot):=T$. Moreover, denote ${}_q D_{\psi(\cdot)}^\alpha f(\psi(\cdot)) = {}_q D_T^\alpha f(T) := G(T) = G(\psi(\cdot))$, interchangeably. It follows that

$$ \begin{aligned} {}_q I_\psi^\alpha {}_q D_\psi^\alpha f(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (\psi(t)-q\psi(s))_q^{(\alpha-1)} \\ &\quad \times \left\{ \frac{1}{\Gamma_q(1-\alpha)} \int_0^{\psi(s)} (\psi(s)-q\theta)_q^{(\alpha-1)} h(\theta) d_q\theta \right\} D_q\psi(s) d_q s \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (\psi(t)-q\psi(s))_q^{(\alpha-1)} \\ &\quad \times \left\{ \frac{1}{\Gamma_q(1-\alpha)} \int_0^{T} (T-q\theta)_q^{(\alpha-1)} h(\theta) d_q\theta \right\} D_q\psi(s) d_q s \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (\psi(t)-q\psi(s))_q^{(\alpha-1)} G(\psi(s)) D_q\psi(s) d_q s. \end{aligned} $$

Substituting $\psi(s)=u$, it follows that

$$ \begin{aligned} {}_q I_\psi^\alpha {}_q D_\psi^\alpha f(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^{\psi(t)} (\psi(t)-qu)_q^{(\alpha-1)} G(u) d_q u \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^{T} (T-qu)_q^{(\alpha-1)} G(u) d_q u = {}_q I_T^\alpha G(T) = {}_q I_T^\alpha {}_q D_T^\alpha h(T). \end{aligned} $$

Therefore, it is obtained that

$$\begin{aligned}
{}_qI^{\alpha}_{\psi}\,{}_qD^{\alpha}_{\psi}f(t) &= {}_qI^{\alpha}_{T}\,{}_qD^{\alpha}_{T}h(T) \\
&= h(T) - \sum_{i=0}^{n-1} {}_qI^{1+i-\alpha}_{T}f(0)\frac{T^{\alpha-i-1}}{\Gamma_q(\alpha-i)} \\
&= f(t) - \sum_{i=0}^{n-1} {}_qI^{1+i-\alpha}_{\psi}f(0)\frac{(\psi(t)-\psi(0))^{\alpha-i-1}}{\Gamma_q(\alpha-i)}.
\end{aligned}$$

3.2. Why these are mistakes?

The main reason these expressions are incorrect is that

$$ {}_qI_g f(t) = (1-q)g(t)\sum_{n=0}^{\infty} q^n h(g(t)q^n) $$

where $f : t \mapsto h(g(t))$, is not the proper form of the operator ${}_qI_g$. Consider the following example. Let $f(t) = g(t) = t^n$. Then

$$\int_0^t f(s)D_q g(s)d_q s = \frac{1-q^n}{1-q}\int_0^t s^{2n-1}d_q s = \frac{1-q^n}{1-q}\cdot\frac{1-q}{1-q^{2n}}\cdot t^{2n} = \frac{1-q^n}{1-q^{2n}}t^{2n}.$$

However, applying ${}_qI_g$, the obtained result is

$${}_qI_g f(t) = \int_0^{g(t)} g(s)d_q g(s) = (1-q)g(t)\sum_{n=0}^{\infty} q^n g(t) q^n = \frac{1}{1+q}[g(t)]^2 = \frac{1}{1+q}t^{2n}.$$

These results are clearly not equal. This demonstrates that we cannot treat $g(t)$ as a free variable without rescaling q. For example, if we define, where $g(t) = t^n$, a rescaled q-integral:

$$\int_{g(0)}^{g(t)} h(g(s))d_q g(s) = (1-g(q))g(t)\sum_{i=0}^{\infty}[g(q)]^i h(g(t)[g(q)]^i),$$

then we obtain

$$\begin{aligned}
\int_0^t f(s)D_q g(s)d_q s = \int_{g(0)}^{g(t)} g(s)d_q g(s) &= (1-g(q))g(t)\sum_{i=0}^{\infty}[g(q)]^i g(t)[g(q)]^i \\
&= (1-g(q))[g(t)]^2\sum_{i=0}^{\infty}([g(q)]^2)^i \\
&= \frac{1-g(q)}{1-[g(q)]^2}[g(t)]^2 \\
&= \frac{1-q^n}{1-q^{2n}}t^{2n}.
\end{aligned}$$

To avoid the same pitfalls, I will not rescale anything in this paper.

4. q-fractional operators and (q,ψ)-fractional operators

4.1. q-fractional operators and theorems

This subsection begins with the q-fractional operators that are independent of the function ψ.

Definition 4.1. [8] Let $q \in (0,1)$ and $\alpha > 0$. Then the q-Riemann–Liouville fractional integral is defined by

$$ {}_qI_t^{\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{(\alpha-1)} f(s) d_q s. $$

Let $\alpha, \beta \geqslant 0$, and let $f(t)$ be a function defined on $[0,T]$. Then the following properties hold:
(1) ${}_qI_t^{\alpha}\, {}_qI_t^{\beta} f(t) = {}_qI_t^{\alpha+\beta} f(t)$
(2) ${}_qD_t^{\alpha}\, {}_qI_t^{\alpha} f(t) = f(t)$

Definition 4.2. [16] Let $n-1 < \alpha < n$. The q-Riemann–Liouville fractional derivative of a function $f(t)$ is defined by ${}_qD_t^{\alpha} f(t) = D_q^n\, {}_qI_t^{n-\alpha} f(t)$.

Definition 4.3. [16] Let $n-1 < \alpha < n$. The q-Caputo fractional derivative of a function $f(t)$ is defined by ${}_q^C D_t^{\alpha} f(t) = {}_qI_t^{n-\alpha} D_q^n f(t)$.

Theorem 4.4. [16] Suppose $n-1 < \alpha < n$ and $f \in \mathbb{L}_q[0,T]$ with ${}_qI_t^{n-\alpha} f(t) \in AC_q^{(n)}[0,T]$. Then

$$ {}_qI_t^{\alpha}\, {}_qD_t^{\alpha} f(t) = f(t) - \sum_{i=0}^{n-1} {}_qI_t^{1+i-\alpha} f(0) \frac{t^{\alpha-i-1}}{\Gamma_q(\alpha-i)} $$

where

$$ {}_qI_t^{1+i-\alpha} f(0) = \lim_{t \to 0^+} {}_qI_t^{1+i-\alpha} f(t). $$

Definition 4.5. [9] Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, and $0 < q < 1$. Then the q-Hilfer fractional derivative of a function $f(t)$ is defined by

$$ {}_qD_t^{\alpha,\beta} f(t) = {}_qI_t^{\beta(1-\alpha)} D_q\, {}_qI_t^{(1-\beta)(1-\alpha)} f(t) = {}_qI_t^{\gamma-\alpha}\, {}_qD_t^{\gamma} f(t), \quad \gamma = \alpha + \beta - \alpha\beta. $$

Theorem 4.6. [9] Suppose $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $f \in \mathbb{L}_q[0,T]$ with ${}_qI_t^{1-\gamma} f(t) \in AC_q^{(1)}[0,T]$, where $\gamma = \alpha + \beta - \alpha\beta$. Then,

$$ {}_qI_t^{\alpha}\, {}_qD_t^{\alpha,\beta} f(t) = f(t) - {}_qI_t^{1-\gamma} f(0) \frac{t^{\gamma-1}}{\Gamma_q(\gamma)}. $$

Definition 4.7. [10] Let $0 < \alpha_1 < 1$ and $\alpha_2, \alpha_3, \ldots, \alpha_n \in [0,1]$. The n-tuple order q-fractional derivative is defined as follows, where $\Omega = 1 - \prod_{i=1}^{n}(1-\alpha_i)$.

$$ {}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n} f(t) = {}_qI_t^{\left(1-\prod_{i=2}^{n}(1-\alpha_i)\right)(1-\alpha_1)} D_q\, {}_qI_t^{\prod_{i=1}^{n}(1-\alpha_i)} f(t) = {}_qI_t^{\Omega-\alpha_1}\, {}_qD_t^{\Omega} f(t). $$

Theorem 4.8. [10] Suppose $0 < \alpha_1 < 1$, $0 \leqslant \alpha_2, \alpha_3, \ldots, \alpha_n \leqslant 1$, $f \in \mathbb{L}_q[0,T]$ with ${}_qI_t^{1-\Omega} f(t) \in AC_q^{(1)}[0,T]$, where $\Omega = 1 - \prod_{i=1}^{n}(1-\alpha_i)$. Then,

$$ {}_qI_t^{\alpha_1}\, {}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n} f(t) = {}_qI_t^{\Omega}\, {}_qD_t^{\Omega} f(t) = f(t) - {}_qI_t^{1-\Omega} f(0) \frac{t^{\Omega-1}}{\Gamma_q(\Omega)}. $$

Next, we consider the q-fractional operators that depend on the function ψ.

4.2. q-fractional operators that respect ψ function

First, we observe that ${}_qI_g$ satisfies

$$\begin{aligned}{}_qI_gf(t) = \int_0^t f(s)D_qg(s)d_qs &= (1-q)\sum_{i=0}^{\infty} tq^i f(tq^i)D_qg(tq^i)\\ &= (1-q)\sum_{i=0}^{\infty} tq^i f(tq^i)\frac{g(tq^i)-g(tq^{i+1})}{(1-q)tq^i}\\ &= \sum_{i=0}^{\infty} f(tq^i)\left[g(tq^i)-g(tq^{i+1})\right].\end{aligned}$$

To ensure that the above representation corresponds to the proper integral form to counter ${}_qD_g$, we define the linear operator $\hat{M}_q$ by $\hat{M}_q[f(u)] = f(qu)$, with $\hat{M}_q^{n+1}[f(u)] = \hat{M}_q[\hat{M}_q^n[f(u)]]$. Setting ${}_qD_gF(x) = f(x)$, we have

$${}_qD_gF(x) = \frac{F(x)-F(qx)}{g(x)-g(qx)} = f(x).$$

Therefore,

$$f(x) = \frac{(1-\hat{M}_q)[F(x)]}{g(x)-g(qx)}.$$

Thus,

$$\begin{aligned}F(x) &= \frac{1}{(1-\hat{M}_q)}[(g(x)-g(qx))\cdot f(x)]\\ &= \sum_{i=0}^{\infty}\hat{M}_q^i[g(x)-g(qx))\cdot f(x)]\\ &= \sum_{i=0}^{\infty} f(xq^i)\left[g(xq^i)-g(xq^{i+1})\right].\end{aligned}$$

Consequently,

$${}_qI_gf(t) = \sum_{i=0}^{\infty} f(tq^i)\left[g(tq^i)-g(tq^{i+1})\right]$$

is the proper representation. We now demonstrate the iteration of ${}_qI_g$ as follows.

$$\begin{aligned}\int_0^t\int_0^s f(v)D_qg(v)d_qvD_qg(s)d_qs &= \sum_{i=0}^{\infty}[g(tq^i)-g(tq^{i+1})]\sum_{j=0}^{\infty} f(tq^{i+j})[g(tq^{i+j})-g(tq^{i+j+1})]\\ &= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}[g(tq^i)-g(tq^{i+1})]f(tq^k)[g(tq^k)-g(tq^{k+1})]\\ &= \sum_{n=0}^{\infty}[g(t)-g(tq^{n+1})]f(tq^n)[g(tq^n)-g(tq^{n+1})]\\ &= \int_0^t (g(t)-g(qs))f(s)D_qg(s)d_qs.\end{aligned}$$

This means that the following property yields:

$$\int_0^t \int_0^s f(v) D_q g(v) d_q v D_q g(s) d_q s = \int_0^t \int_{qv}^t f(v) D_q g(s) d_q s D_q g(v) d_q v$$

Repeating this process (see [19]) or applying integration by parts for the q-integral yields

$${}_qI_g^n f(t) = \frac{1}{[n-1]!} \int_0^t (g(t) - g(qs))_q^{(n-1)} f(s) D_q g(s) d_q s.$$

This obtained structure is used to define the (q, ψ)-Riemann–Liouville fractional integral as follows.

Definition 4.9. [19] Let $q \in (0, 1)$, $\alpha > 0$ and $D_q\psi(t) \neq 0$. Then the (q, ψ)-Riemann–Liouville fractional integral is defined by

$${}_qI_\psi^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (\psi(t) - \psi(qs))_q^{(\alpha-1)} f(s) D_q \psi(s) d_q s.$$

The operator ${}_qI_\psi^\alpha$ satisfies the following properties:
(1) ${}_qI_\psi^\alpha {}_qI_\psi^\beta f(t) = {}_qI_\psi^{\alpha+\beta} f(t)$
(2) ${}_qD_\psi^\alpha {}_qI_\psi^\alpha f(t) = f(t)$
It can be seen that the operator is reduced to the definition 4.1 when $\psi(t) = t$.

Definition 4.10. Let $n - 1 < \alpha < n$. The (q, ψ)-Riemann–Liouville fractional derivative is defined by ${}_qD_\psi^\alpha f(t) = {}_qD_\psi^n {}_qI_\psi^{n-\alpha} f(t)$
It can be observed that this operator reduces to the definition 4.2 when $\psi(t) = t$.

Definition 4.11. Let $n - 1 < \alpha < n$. The (q, ψ)-Caputo fractional derivative is defined by ${}_q^CD_\psi^\alpha f(t) = {}_qI_\psi^{n-\alpha} {}_qD_\psi^n f(t)$
It can be observed that this operator reduces to the definition 4.3 when $\psi(t) = t$.

Definition 4.12. Let $n - 1 < \alpha < n$. The (q, ψ)-Hilfer fractional derivative is defined by ${}_qD_\psi^{\alpha,\beta} f(t) = {}_qI_\psi^{\beta(n-\alpha)} {}_qD_\psi^n {}_qI_\psi^{(1-\beta)(n-\alpha)} f(t)$
It can be observed that this operator reduces to the definition 4.5 when $\psi(t) = t$ and $n = 1$.

Theorem 4.13. [19] Suppose $n - 1 < \alpha < n$ and $f \in \mathbb{L}_{q,\psi}[0, T]$ with ${}_qI_\psi^{n-\alpha} f(t) \in AC_{q,\psi}^{(n)}[0, T]$. Then

$${}_qI_\psi^\alpha {}_qD_\psi^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} {}_qI_\psi^{1+i-\alpha} f(0) \frac{(\psi(t) - \psi(0))^{\alpha-i-1}}{\Gamma_q(\alpha - i)}$$

where

$${}_qI_\psi^{1+i-\alpha} f(0) = \lim_{t \to 0^+} {}_qI_\psi^{1+i-\alpha} f(t).$$

Afterwards, I will introduce the n-tuple variable-order q-fractional derivative with respect to the function ψ, which extends the previously defined operators and unifies them within a single generalized framework.

5. n-tuple variable-order q-fractional derivative with respect to the function ψ

In this section, the author defines two new generalized operators and a relevant theorem.

Definition 5.1. Let $0<\alpha_1<1$, $\alpha_2,\alpha_3,\dots,\alpha_n\in[0,1]$, and $D_q\psi(t)\neq 0$. The n-tuple order (q,ψ)-fractional derivative is defined by

$$ {}_qD_{\psi}^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}f(t) = {}_qI_{\psi}^{(1-\prod_{i=2}^{n}(1-\alpha_i))(1-\alpha_1)}\,{}_qD_{\psi}\,{}_qI_{\psi}^{\prod_{i=1}^{n}(1-\alpha_i)}f(t) = {}_qI_{\psi}^{\Omega-\alpha_1}\,{}_qD_{\psi}^{\Omega}f(t), $$

where $\Omega = 1-\prod_{i=1}^{n}(1-\alpha_i)$. It can be observed that this operator reduces to the definition 4.7 when $\psi(t)=t$.
Moreover, the n-tuple order (q,ψ)-fractional derivative can be reduced to the derivatives of the classical variations. Some of the examples are as follows:
(1) Let $\alpha_2=\alpha_3=\alpha_4=\dots=\alpha_n=0$ and $\alpha_1=\alpha$. Then, the n-tuple order (q,ψ)-fractional derivative becomes ${}_qD_{\psi}\,{}_qI_{\psi}^{1-\alpha}f(t)={}_qD_{\psi}^{\alpha}f(t)$ which is the (q,ψ)-Riemann–Liouville fractional derivative (see definition 4.10).
(2) Let one of the $\alpha_i=1$ where $i=2,3,4,\dots,n$ and $\alpha_1=\alpha$. Then, the n-tuple order (q,ψ)-fractional derivative becomes ${}_qI_{\psi}^{1-\alpha}\,{}_qD_{\psi}f(t)={}_q^{C}D_{\psi}^{\alpha}f(t)$ which is the (q,ψ)-Caputo fractional derivative (see definition 4.11).
(3) Let $n=2$, $\alpha_1=\alpha$ and $\alpha_2=\beta$. Then, the n-tuple order (q,ψ)-fractional derivative becomes ${}_qI_{\psi}^{\beta(1-\alpha)}\,{}_qD_{\psi}\,{}_qI_{\psi}^{(1-\beta)(1-\alpha)}f(t)={}_qD_{\psi}^{\alpha,\beta}f(t)$ which is the (q,ψ)-Hilfer fractional derivative (see definition 4.12).

Definition 5.2. Let $0<\alpha_1(t)<1$, $0\leqslant\alpha_2(t),\alpha_3(t),\dots,\alpha_n(t)\leqslant 1$, and $D_q\psi(t)\neq 0$. The n-tuple variable-order (q,ψ)-fractional derivative is defined by

$$ {}_qD_{\psi}^{\alpha_1(t),\alpha_2(t),\alpha_3(t),\dots,\alpha_n(t)}f(t) = {}_qI_{\psi}^{(1-\prod_{i=2}^{n}(1-\alpha_i(t)))(1-\alpha_1(t))}\,{}_qD_{\psi}\,{}_qI_{\psi}^{\prod_{i=1}^{n}(1-\alpha_i(t))}f(t). $$

Theorem 5.3. Suppose $0<\alpha_1<1$, $0\leqslant\alpha_2,\alpha_3,\dots,\alpha_n\leqslant 1$, and $f\in\mathbb{L}_{q,\psi}[0,T]$ with ${}_qI_{\psi}^{1-\Omega}f(t)\in AC_{q,\psi}^{(1)}[0,T]$, where $\Omega=1-\prod_{i=1}^{n}(1-\alpha_i)$. Then

$$ {}_qI_{\psi}^{\alpha_1}\,{}_qD_{\psi}^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}f(t) = {}_qI_{\psi}^{\Omega}\,{}_qD_{\psi}^{\Omega}f(t) = f(t) - {}_qI_{\psi}^{1-\Omega}f(0)\,\frac{(\psi(t)-\psi(0))^{\Omega-1}}{\Gamma_q(\Omega)}. $$

Proof. The proof is trivial following theorem 4.13. □

6. Well-posedness of the solution

In this section, the fractional-order hybrid difference equation with the initial condition given by (1.1), where $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}$, and $x_0\in\mathbb{R}$ denote the initial value, is analyzed. Let $P=[0,T_1],(T_1,T_2],(T_2,T_3],\dots,(T_{N-1},T]$, where each $P_k\in P$ denotes the k-th subinterval of P. Let $\alpha_1:[0,T]\to(0,1)$ and $\alpha_i:[0,T]\to[0,1]$ for $i\in\mathbb{N}\setminus\{0,1\}$ be continuous functions. Secondly, we define the α_i-approximation function $\tilde{\alpha}_i(t):[0,T]\to J_i$, where $J_i=(0,1)$ for $i=1$ and $J_i=[0,1]$ for $i=2,3,\dots,n$, as a piecewise-continuous function with respect to the partition P. The function $\tilde{\alpha_i}$ is written by

$$\tilde{\alpha_i}(t)=\sum_{k=1}^{N}\alpha_i(t_k)I_k(t)=\sum_{k=1}^{N}\alpha_{(i,k)}I_k(t)=\begin{cases}\alpha_{(i,1)} & ,t\in[0,T_1]\\ \alpha_{(i,2)} & ,t\in(T_1,T_2]\\ \alpha_{(i,3)} & ,t\in(T_2,T_3]\\ \vdots & \\ \alpha_{(i,N)} & ,t\in(T_{N-1},T]\end{cases},\quad t_k\in P_k\subset[0,T] \tag{6.1}$$

where I_k denotes the indicator function of P_k, that is, $I_k(t)=1$ if $t\in P_k$ and $I_k(t)=0$ otherwise. Consequently, $\alpha_i(t)=\lim_{N\to\infty}\tilde{\alpha_i}(t)$. This limit holds as $|\alpha_{(i,k)}-\alpha_{(i,k-1)}|\to 0$ whenever $|t_k-t_{k-1}|\to 0$. Hence, (1.1) can be represented as

$$\begin{aligned}&\sum_{k=1}^{\infty}I_k(t)\,{}_qD_{\psi}^{\alpha_{(1,k)},\alpha_{(2,k)},\alpha_{(3,k)},\ldots,\alpha_{(n,k)}}[x(t)-f(t,x(t))]=g(t,x(t))\\ &{}_qI_{\psi}^{1-\Omega_k}x(0)=x(0)=x_0,\quad {}_qI_{\psi}^{1-\Omega_k}f(0,x(0))=f(0,x(0))=f_0,\end{aligned} \tag{6.2}$$

where $\Omega_k=1-\prod_{i=1}^{n}(1-\alpha_{(i,k)})$.

I now present the definition of the solution to problem (1.1), which is fundamental to this study. From theorem 5.3 and the equation (6.1), the integral representation of the solution $x_k(t)$ in the subinterval P_k is given by

$$\begin{aligned}x_k(t)=&\frac{C(\psi(t)-\psi(0))^{\Omega_k-1}}{\Gamma_q(\Omega_k)}+f(t,x_k(t))\\ &+\frac{1}{\Gamma_q(\alpha_{(1,k)})}\int_0^t(\psi(t)-\psi(qs))_q^{(\alpha_{(1,k)}-1)}g(s,x_k(s))D_q\psi(s)d_qs,\end{aligned} \tag{6.3}$$

where $C=x_0-f_0\in\mathbb{R}$ for $t\in P_k$.

Moreover, where $\Omega(t)=1-\prod_{i=1}^{n}(1-\alpha_i(t))$, the continuous mild solution $x(t)=\sum_{k=1}^{\infty}I_k(t)x_k(t)$ is written by

$$\begin{aligned}x(t)=&\frac{C(\psi(t)-\psi(0))^{\Omega(t)-1}}{\Gamma_q(\Omega(t))}+f(t,x(t))\\ &+\frac{1}{\Gamma_q(\alpha_1(t))}\int_0^t(\psi(t)-\psi(qs))_q^{(\alpha_1(t)-1)}g(s,x(s))D_q\psi(s)d_qs,\end{aligned} \tag{6.4}$$

The well-posedness of the solution will now be examined. Before proceeding, I will recall several theorems that will be useful later.

Theorem 6.1. [16] Suppose $\phi:[0,a]\to\mathbb{R}$ is a function, if $\phi\in\mathbb{L}_q[0,a]$, then ${}_qI_t^{\alpha}\phi\in\mathbb{L}_q[0,a]$, and $\|{}_qI_t^{\alpha}\phi\|_1\leqslant\frac{a^{\alpha}}{\Gamma_q(\alpha+1)}\|\phi\|_1$.

Theorem 6.2. Let $\max_{t\in[0,a]}(\psi(t)-\psi(0))=\Psi$; then, suppose $\phi:[0,a]\to\mathbb{R}$ and $\psi:[0,a]\to\mathbb{R}$ is a function. Then, ${}_qI_{\psi}^{\alpha}\phi\in\mathbb{L}_{q,\psi}[0,a]$, and $\|{}_qI_{\psi}^{\alpha}\phi\|\leqslant\frac{\Psi^{\alpha}}{\Gamma_q(\alpha+1)}\|\phi\|$.

Proof.

$$\begin{aligned}
\|{}_qI_\psi^\alpha\phi\| &= \left\|\frac{1}{\Gamma_q(\alpha)}\int_0^t(\psi(t)-\psi(qs))_q^{(\alpha-1)}\phi(s)D_q\psi(s)d_qs\right\| \\
&= \int_0^t |{}_qI_\psi^\alpha\phi| D_q\psi(s)d_qs \\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^t\left|\int_0^s(\psi(t)-\psi(qv))_q^{(\alpha-1)}\phi(v)D_q\psi(v)d_qv\right|D_q\psi(s)d_qs \\
&\leqslant \frac{1}{\Gamma_q(\alpha)}\int_0^t D_q\psi(s)\int_0^s D_q\psi(v)(\psi(t)-\psi(qv))_q^{(\alpha-1)}|\phi(v)|d_qvd_qs \\
&\leqslant \frac{1-q}{\Gamma_q(\alpha)}\int_0^t D_q\psi(s)(\psi(s)-\psi(0))^\alpha\times\sum_{i=0}^\infty q^i(1-q^{i+1})_q^{(\alpha-1)}\times|\phi(tq^i)|d_qs \\
&\leqslant \frac{(1-q)^2(\psi(t)-\psi(0))^{\alpha+1}}{\Gamma_q(\alpha)}\sum_{k=0}^\infty q^k|\phi(tq^k)|\sum_{n=0}^k q^{n\alpha}(1-q^{k-n+1})_q^{(\alpha-1)} \\
&\leqslant \frac{(\psi(t)-\psi(0))^\alpha}{\Gamma_q(\alpha)}B_q(1,\alpha)\int_0^t\phi(s)D_q\psi(s)d_qs \\
&= \frac{(\psi(t)-\psi(0))^\alpha}{\Gamma_q(\alpha+1)}\|\phi\| \\
&\leqslant \frac{\Psi^\alpha}{\Gamma_q(\alpha+1)}\|\phi\|
\end{aligned}$$

This completes the proof. □

Theorem 6.3. [9] For any $0<q<1$ and $0<\alpha(t)<1$, the inequality of q-gamma function holds,

$$(1-q)^{\alpha(t)}\leqslant\frac{1}{\Gamma_q(\alpha(t)+1)}<\left(\frac{1-q}{1-q^{\frac{\alpha(t)+1}{2}}}\right)^{\alpha(t)}.$$

6.1. *Existence of the solution*

First, we establish the existence of the solution on the subinterval. If solutions exist on all subintervals, then a continuous solution exists on the interval $[0,T]$.

Definition 6.4. A mapping $\Delta:[a,b]\times\mathbb{R}\to\mathbb{R}$ is said to be $\mathbf{L}^p$-Carathéodory in $\mathbb{L}^p_{q,\psi}[a,b]$ if the following conditions are satisfied:
(i) $t\to\Delta(t,x)$ is measurable for every $x\in\mathbb{R}$;
(ii) $x\to\Delta(t,x)$ is continuous for every $t\in[a,b]$;
(iii) for each $r>0$, there exists a function $\varphi_r\in\mathbb{L}^p_{q,\psi}[a,b]$ such that

$$|\Delta(t,x)|\leqslant\varphi_r(t),\quad t\in[a,b],\quad x\in\mathbb{R},\quad |x|<r.$$

Then, I will state the essential assumptions as follows:
(A0) There a positive constant M_f such that

$$\|f(t,u)-f(t,v)\|\leqslant M_f\|u-v\|,\quad |f(t,u)|\leqslant\tilde{M}_f(\|u\|+1)$$

for all $u, v \in \mathbb{L}_{q,\psi}[0,T]$.
(A1) There exists a positive constant M_g such that

$$\|g(t,u) - g(t,v)\| \leqslant M_g \|u - v\|$$

for all $u, v \in \mathbb{L}_{q,\psi}[0,T]$.
(A2) The function $g(t,x)$ is $\mathbf{L}^p$-Carathéodory in $\mathbb{L}^p_{q,\psi}[0,T]$.

Theorem 6.5. Suppose $f, g : [0,T] \to \mathbb{R}$ are continuous, $q \in (0,1)$, and $c, d > 1$ satisfy $\frac{1}{c} + \frac{1}{d} = 1$. Then the following (q,ψ)-analog of Hölder's inequality holds:

$$\int_0^t |f(s)||g(s)| D_q\psi(s) d_q s \leqslant \left(\int_0^t |f(s)|^c D_q\psi(s) d_q s\right)^{\frac{1}{c}} \left(\int_0^t |g(s)|^d D_q\psi(s) d_q s\right)^{\frac{1}{d}}.$$

Proof. Let $|x(t)|_{p,\psi} = |x|_{p,\psi} = \left(\int_0^t |x(s)|^p D_q\psi(s) d_q s\right)^{\frac{1}{p}}$. Then let $A = |f|_{c,\psi}$ and $B = |g|_{d,\psi}$ where $\frac{1}{c} + \frac{1}{d} = 1$. From Young's inequality, we get

$$\begin{aligned}
\frac{|f(t)||g(t)|}{AB} &\leqslant \frac{\left(\frac{|f(t)|}{A}\right)^c}{c} + \frac{\left(\frac{|g(t)|}{B}\right)^d}{d} \\
&\leqslant \frac{|f|^c}{cA^c} + \frac{|g|^d}{dB^d}.
\end{aligned}$$

Integrating both sides gives

$$\frac{|fg|_{1,\psi}}{AB} \leqslant \frac{|f|^c_{c,\psi}}{cA^c} + \frac{|g|^d_{d,\psi}}{dB^d}.$$

Then,

$$\begin{aligned}
|fg|_{1,\psi} &\leqslant AB\left(\frac{|f|^c_{c,\psi}}{cA^c} + \frac{|g|^d_{d,\psi}}{dB^d}\right) \\
&\leqslant |f|_{c,\psi}|g|_{d,\psi}\left(\frac{1}{c} + \frac{1}{d}\right) \\
&\leqslant |f|_{c,\psi}|g|_{d,\psi} = \left(\int_0^t |f(s)|^c D_q\psi(s) d_q s\right)^{\frac{1}{c}} \left(\int_0^t |g(s)|^d D_q\psi(s) d_q s\right)^{\frac{1}{d}}.
\end{aligned}$$

This completes the proof. □

Theorem 6.6. Let $\mathbb{K}$ be a closed, convex, and nonempty subset of $\mathbb{X}$. Let $\mathbb{M}, \mathbb{N} : \mathbb{K} \to \mathbb{X}$ be two operators satisfying:
(a) $\mathbb{M}x + \mathbb{N}y \in \mathbb{K}$ for all $x, y \in \mathbb{K}$;
(b) $\mathbb{M}$ is a contraction on $\mathbb{K}$;
(c) $\mathbb{N}$ is completely continuous on $\mathbb{K}$.
Then the operator equation $\mathbb{M}x + \mathbb{N}x = x$ has a solution.

Theorem 6.7. Suppose (A0)–(A2) hold. Then equation (1.1) has a solution if $\alpha_1(t) \in (0,1)$.

Proof. For each $k = 1,2,3,\dots,N$, we define $\mathbb{M},\mathbb{N} : \mathbb{L}_{q,\psi}[0,T] \to \mathbb{L}_{q,\psi}[0,T]$ such that

$$\mathbb{M}x_k(t) = \frac{C(\psi(t)-\psi(0))^{\Omega_k-1}}{\Gamma_q(\Omega_k)} + f(t,x_k(t))$$
$$\mathbb{N}x_k(t) = \frac{1}{\Gamma_q(\alpha_{(1,k)})}\int_0^t (\psi(t)-\psi(qs))_q^{(\alpha_{(1,k)}-1)} g(s,x_k(s))D_q\psi(s)d_qs.$$

From the proof of theorem 6.2, it is obtained that $|{}_qI_\psi^\alpha\phi| \leqslant \frac{(\psi(t)-\psi(0))^\alpha}{\Gamma_q(\alpha+1)}|\phi|$. Thus, it follows that

$${}_qI_\psi^\alpha\phi(0) = |{}_qI_\psi^\alpha\phi(0)| = \lim_{t\to0^+}|{}_qI_\psi^\alpha\phi(t)| \leqslant \lim_{t\to0^+}\frac{(\psi(t)-\psi(0))^\alpha}{\Gamma_q(\alpha+1)}|\phi| = 0$$

as $t \to 0$.
Consequently,

$${}_qI_\psi^{1-\Omega_k}x(0) = {}_qI_\psi^{1-\Omega_k}x(0)\frac{(\psi(t)-\psi(0))^{\Omega_k-1}}{\Gamma_q(\Omega_k)} = x(0) = 0$$

and

$${}_qI_\psi^{1-\Omega_k}f(0,x(0)) = {}_qI_\psi^{1-\Omega_k}f(0,x(0))\frac{(\psi(t)-\psi(0))^{\Omega_k-1}}{\Gamma_q(\Omega_k)} = f(0,x(0)) = 0.$$

Let $\mathbb{B}_R(x_0) := \{x \in \mathbb{L}_{q,\psi}[0,T] | |x-x_0| \leqslant \mathbb{R}\} \subseteq \mathbb{L}_{q,\psi}[0,T]$, which is closed and bounded in the Banach space $\mathbb{L}_q[0,T]$. Also, let $\max_{t\in[0,T]}\{\psi(t),(\psi(t)-\psi(0))\} = \tilde{\Psi}$. It can be shown that $\mathbb{M}x+\mathbb{N}y \in \mathbb{B}_R(x_0)$ when $x,y \in \mathbb{B}_R(x_0)$ for any $t \in (0,T_k]$ as follows.

$$\begin{aligned}
|\mathbb{M}x_k + \mathbb{N}y_k - x_0| &\leqslant |f(t,x_k(t)) - f_0| \\
&\quad + \frac{\left(\int_0^t |(\psi(t)-\psi(qs))_q^{(\alpha_{(1,k)}-1)}|^c D_q\psi(s)d_qs\right)^{\frac{1}{c}}}{\Gamma_q(\alpha_{(1,k)})}\left(\int_0^t |g(s,y_k(s))|^d D_q\psi(s)d_qs\right)^{\frac{1}{d}} \\
&\leqslant RM_f + 2\tilde{M}_f(\|x_0\|+1) + \frac{\left(\tilde{\Psi}^{c\alpha_{(1,k)}-c+1}\right)^{\frac{1}{c}}}{\Gamma_q(\alpha_{(1,k)})}\|\varphi_r\|_{d,\psi} \\
&< R
\end{aligned}$$

when R is large enough. Thus, $\mathbb{M}x+\mathbb{N}y \in \mathbb{B}_R(x_0)$. By assumption (A0), it is clear that $\mathbb{M}$ is a contraction on $\mathbb{B}_R(x_0)$. Since $\mathbb{N}x \leqslant \frac{\left(\tilde{\Psi}^{c\alpha_{(1,k)}-c+1}\right)^{\frac{1}{c}}}{\Gamma_q(\alpha_{(1,k)})}\|\varphi_r\|_{d,\psi}$, it follows that $\mathbb{N}$ is uniformly bounded in $\mathbb{L}_{q,\psi}[0,T]$. Moreover, $\mathbb{N}$ is also equicontinuous, since for any

$t_1, t_2 \in [0,T]$ such that $t_1 \leqslant t_2$:

$$\begin{aligned}
\left|\mathbb{N}x_k(t_1) - \mathbb{N}x_k(t_2)\right| &< \frac{1}{\Gamma_q(\alpha_{(1,k)})}\left|\int_0^{t_1} (\psi(t_1)-\psi(qs))_q^{(\alpha_{(1,k)}-1)} g(s,x_k(s))D_q\psi(s)\,d_qs\right.\\
&\quad \left. - \int_0^{t_2} (\psi(t_2)-\psi(qs))_q^{(\alpha_{(1,k)}-1)} g(s,x_k(s))D_q\psi(s)\,d_qs\right|\\
&< \frac{1}{\Gamma_q(\alpha_{(1,k)})}\left|\int_0^{t_1} \left[(\psi(t_1)-\psi(qs))_q^{(\alpha_{(1,k)}-1)} - (\psi(t_2)-\psi(qs))_q^{(\alpha_{(1,k)}-1)}\right]\right.\\
&\quad \left. \times g(s,x_k(s))\,D_q\psi(s)\,d_qs\right|\\
&\quad + \frac{1}{\Gamma_q(\alpha_{(1,k)})}\left|\int_{t_1}^{t_2} (\psi(t_2)-\psi(qs))_q^{(\alpha_{(1,k)}-1)} g(s,x_k(s))D_q\psi(s)\,d_qs\right|\\
&\to 0.
\end{aligned}$$

as $|t_1 - t_2| \to 0$. Hence, $\mathbb{N}$ is relatively compact in $\mathbb{B}_R(x_0)$. Therefore, the solution x_k exists in each subinterval P_k by theorem 6.7. Consequently, a continuous solution exists. □

6.2. Uniqueness of the solution

In this section, we first examine the uniqueness of the solution on each sub-interval and of the continuous solution.

Theorem 6.8. The equation (1.1) has a solution in $\mathbb{L}_{q,\psi}[0,T]$ if there exists a sequence of functions $x_j \in \mathbb{L}_{q,\psi}[0,T_j]$, $j = 1,2,\ldots,N$, satisfying (6.3) and the initial conditions ${}_qI_\psi^{1-\Omega_j}x(0) = x_0$ and ${}_qI_\psi^{1-\Omega_j}f(0,x(0)) = f_0$.

Theorem 6.9. Suppose that assumptions (A0)–(A1) hold. Then equation (6.3) has a unique solution in $\mathbb{L}_{q,\psi}[0,T_k]$ if there exists a contraction constant such that $M_f + \frac{M_g\Psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)} < 1$.

Proof. For each $k = 1,2,\ldots$, define the contraction mapping $\mathbb{Q} : \mathbb{L}_{q,\psi}[0,T_k] \to \mathbb{L}_{q,\psi}[0,T_k]$ by $\mathbb{Q}x_k = x_k$. Then,

$$\begin{aligned}
\mathbb{Q}x_k(t) =& \frac{C(\psi(t)-\psi(0))^{\Omega_k-1}}{\Gamma_q(\Omega_k)} + f(t,x_k(t))\\
&+ \frac{1}{\Gamma_q(\alpha_{(1,k)})}\int_0^t (\psi(t)-\psi(qs))_q^{(\alpha_{(1,k)}-1)} g(s,x_k(s))D_q\psi(s)d_qs.
\end{aligned}$$

Consequently,

$$\begin{aligned}\|Qx_k - Qy_k\| &\leqslant \|f(t,x_k(t)) - f(t,y_k(t))\| \\ &\quad + \|{}_qI_{\psi}^{\alpha_{(1,k)}}1\|\|g(t,x_k(t)) - g(t,y_k(t))\| \\ &\leqslant M_f\|x_k - y_k\| + \frac{\psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}(M_g\|x_k - y_k\|) \\ &= M_f + \frac{M_g\psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}\|x_k - y_k\|.\end{aligned}$$

By the Banach contraction theorem, since $M_f + \frac{M_g\psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)} < 1$ then x_k is a unique solution in $\mathbb{L}_{q,\psi}[0,T_k]$. This completes the proof. □

Theorem 6.10. Suppose that x_k is unique on $\mathbb{L}_{q,\psi}[0,T_k]$. Then $x(t)$ is a unique solution on $\mathbb{L}_{q,\psi}[0,T]$ if there exists a contraction function $\zeta : [0,T] \to (0,1)$ such that

$$\zeta(t) = M_f + M_g\psi^{\alpha_1(t)+1}\left(\frac{1-q}{1-q^{\frac{\alpha_1(t)+1}{2}}}\right)^{\alpha_1(t)} < 1.$$

Proof. Construct the approximate contraction function $\tilde{\zeta}$ on $[0,T]$ by aggregating the contraction constants in each subinterval P_k; we obtain

$$\tilde{\zeta}(t) = M_f + \sum_{k=1}^{N} I_k(t)\left(\frac{M_g\psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}\right), \quad t \in [0,T].$$

Thus, by taking the limit $N \to \infty$, the fundamental contraction function $\zeta^*(t)$ is given by

$$\zeta^*(t) = M_f + \sum_{k=1}^{\infty} I_k(t)\left(\frac{M_g\psi_k^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}\right) = M_f + \frac{M_g\psi^{\alpha_1(t)+1}}{\Gamma_q(\alpha_1(t)+1)}, \quad t \in [0,T].$$

According to theorem 6.3. Also, it is obvious that

$$\zeta^*(t) = M_f + \frac{M_g\psi^{\alpha_1(t)+1}}{\Gamma_q(\alpha_1(t)+1)} < M_f + M_g\psi^{\alpha_1(t)+1}\left(\frac{1-q}{1-q^{\frac{\alpha_1(t)+1}{2}}}\right)^{\alpha_1(t)} = \zeta(t).$$

Since there exists a contraction function $\zeta(t) < 1$, the continuous solution $x(t)$ is the unique solution on $\mathbb{L}_{q,\psi}[0,T]$. This completes the proof. □

6.3. *Stability of the solution*

In this section, I establish the Ulam–Hyers stability of the solutions.

Definition 6.11. The equation (1.1) is Ulam-Hyers stable in $\mathbb{L}_{q,\psi}[0,T]$ if, for every x_k in P_k and for every ϵ, whenever $x_k \in \mathbb{L}_{q,\psi}[0,T_k]$ satisfies the inequality

$$|{}_qD_{\psi}^{\alpha_{(1,k)},\alpha_{(2,k)},\alpha_{(3,k)},\ldots,\alpha_{(n,k)}}[x(t) - f(t,x(t))] - g(t,x(t))| \leqslant \epsilon,$$

there exist a constant $\vartheta > 0$ and a solution $u_k \in \mathbb{L}_{q,\psi}[0,T_k]$ of the equation (6.2) with

$$|x_k(t) - u_k(t)| \leqslant \vartheta_k\epsilon.$$

Theorem 6.12. Suppose (A0)–(A1) hold. Then the equation (1.1) is Ulam-Hyers stable

Proof. Let $x_k \in \mathbb{L}_{q,\psi}[0,T_k]$ satisfy the first inequality in the definition 6.11. It follows that

$$\begin{aligned}&\Bigg|x_k(t)-\frac{C(\psi(t)-\psi(0))^{\Omega_k-1}}{\Gamma_q(\Omega_k)}-f(t,x_k(t))\\&-\frac{1}{\Gamma_q(\alpha_{(1,k)})}\int_0^t(\psi(t)-\psi(qs))_q^{(\alpha_{(1,k)}-1)}g(s,x_k(s))D_q\psi(s)d_qs\Bigg|\\&\leqslant \epsilon\frac{\Psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}.\end{aligned}$$

Now, suppose $u_k \in \mathbb{L}_{q,\psi}[0,T_k]$ is a solution of the equation (6.2). Hence, it satisfies

$$\begin{aligned}u_k(t)=&\frac{C(\psi(t)-\psi(0))^{\Omega_k-1}}{\Gamma_q(\Omega_k)}+f(t,u_k(t))\\&+\frac{1}{\Gamma_q(\alpha_{(1,k)})}\int_0^t(\psi(t)-\psi(qs))_q^{(\alpha_{(1,k)}-1)}g(s,u_k(s))D_q\psi(s)d_qs.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|x_k(t)-u_k(t)|\leqslant{}&\epsilon\frac{\Psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}+|f(t,x_k(t))-f(t,u_k(t))|\\&+\left|\frac{1}{\Gamma_q(\alpha_{(1,k)})}\int_0^t(\psi(t)-\psi(qs))_q^{(\alpha_{(1,k)}-1)}[g(s,x_k(s))-g(s,u_k(s))]D_q\psi(s)d_qs\right|\\&\leqslant \epsilon\frac{\Psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}+M_f|x_k(t)-u_k(t)|+\frac{M_g\Psi^{\alpha_{(1,k)}+1}}{\Gamma_q(\alpha_{(1,k)}+1)}|x_k(t)-u_k(t)|\\&\leqslant \epsilon\Psi^{\alpha_{(1,k)}+1}\left(\frac{1-q}{1-q^{\frac{\alpha_{(1,k)}+1}{2}}}\right)^{\alpha_{(1,k)}}\\&+\left(M_f+M_g\Psi^{\alpha_{(1,k)}+1}\left(\frac{1-q}{1-q^{\frac{\alpha_{(1,k)}+1}{2}}}\right)^{\alpha_{(1,k)}}\right)|x_k(t)-u_k(t)|.\end{aligned}$$

Let $L_k := \Psi^{\alpha_{(1,k)}+1}\left(\frac{1-q}{1-q^{\frac{\alpha_{(1,k)}+1}{2}}}\right)^{\alpha_{(1,k)}}$. Then

$$|x_k(t)-u_k(t)|\leqslant \epsilon\left(\frac{L_k}{1-M_f-M_gL_k}\right)=\vartheta_k\epsilon.$$

Therefore, the equation (1.1) is Ulam-Hyers stable. □

7. Example

In this section, I present an example to illustrate the main result. Consider the following equation.

$$\begin{aligned}
&{}_{\frac{1}{2}}D_{\sqrt{t}}^{\frac{1}{2}\sin^2(\cos(\frac{t}{2})),\frac{1}{2}\cos^2(\cos(\frac{t}{2})),\frac{1}{2}}\left[x(t)-\frac{x(t)}{100x^2(t)+100}\right]=\frac{\tan^{-1}x(t)}{100}, \quad t\in[0,1]\\
&{}_{\frac{1}{2}}I_{\sqrt{t}}^{1-\Omega(t)}x(0)=x(0)=0, \quad {}_{\frac{1}{2}}I_{\sqrt{t}}^{1-\Omega(t)}f(0,x(0))=f(0,x(0))=0,
\end{aligned} \tag{7.1}$$

where $\Omega(t)=\frac{3}{4}-\frac{1}{32}\sin^2(2\cos(\frac{t}{2}))$.

7.1. Existence of the solution

As $|g(t,x)|\leqslant\frac{\pi}{200}$, it follows that $g(t,x)$ is $\mathbf{L}^2$-Carathéodory, and assumption (A2) is satisfied. Moreover, since

$$\left|\frac{\tan^{-1}x(t)}{100}-\frac{\tan^{-1}y(t)}{100}\right|\leqslant\frac{1}{100}\|x-y\|=M_g\|x-y\|,$$

assumption (A1) is satisfied. Similarly, assumption (A0) is satisfied, since

$$\left|\frac{x(t)}{100x^2(t)+100}-\frac{y(t)}{100y^2(t)+100}\right|\leqslant\frac{1}{100}\|x-y\|=M_f\|x-y\|.$$

Since $f_0=0$, it follows that $|f(t,x)|=|f(t,x)-f_0|\leqslant\frac{1}{100}|x|\leqslant\frac{1}{100}(\|x\|+1)$. This guarantees the existence of the solution x_k on each subinterval and the continuous solution to equation (7.1) by theorem 6.7.

7.2. Uniqueness of the solution

Now, since $\alpha_1(t)=\frac{1}{2}\sin^2(\cos(\frac{t}{2}))$, the maximum and minimum values of $\alpha_1(t)$ for $t\in[0,1]$ satisfy $0.59<\min_{t\in[0,1]}\alpha_1(t)<\max_{t\in[0,1]}\alpha_1(t)<0.71$. Moreover, since $\psi(t)=\sqrt{t}$, the constant Ψ equals 1. Thus, the contraction function $\zeta(t)$ is given by

$$0<0.0208<\zeta(t)=\frac{1}{100}+\frac{1}{100}\left(\frac{0.5}{1-0.5^{\frac{\frac{1}{2}\sin^2(\cos(\frac{t}{2}))+1}{2}}}\right)^{\frac{1}{2}\sin^2(\cos(\frac{t}{2}))}<0.0211<1.$$

Therefore, the solution x_k on each subinterval and the continuous solution to (7.1) are unique by theorem 6.10.

7.3. Stability of the solution

Now for any L_k, it follows that we have $1.0825<L_k<1.1027$. Consequently, for any ϑ_k, the following inequality holds:

$$1.10552<\vartheta_k<1.12638.$$

Thus, the solution x_k in each subinterval and the continuous solution to (7.1) are Ulam-Hyers stable by theorem 6.12.

8. Discussion

In this work, new generalized fractional operators were introduced, namely the n-tuple order q-fractional derivative and its variable-order extension with respect to a function ψ. These operators provide a broader analytical framework for the study of q-fractional calculus, as the n-tuple derivative can be reduced to classical structures such as the Riemann-Liouville, Caputo, and Hilfer derivatives. Furthermore, when $\psi(t) = t$, the q-fractional operators reduce to the traditional time-scale operator. The paper also examined the well-posedness of the solution to the associated hybrid difference equation. In the special case where the variable order becomes a constant function, the derivative simplifies to a standard operator of constant non-integer order. In future research, if any mathematicians would conduct using this operator, this definition may be further generalized by shifting the initial point from $t_0 = 0$ to an arbitrary $t_0 = a$, by setting $\alpha_i(t)$ at higher degree, for example, $1 < \alpha_1(t) < 2$, or may be further studied by analyzing the stability of the solution to (q, ψ)-difference system involving this operator.

Closing statement

As this is no longer the primary focus of the paper, I would like to be a bit more informal. The first question for readers, except those who personally know me, perhaps, is not about the math but my affiliation — you may wonder if I hold a graduate degree in mathematics, but I do not hold one. I resigned from my mathematics school due to a reason relevant to the future of my mathematical career in my country, and I have never regretted it. The only sorrow I have is that I have much less time to practice mathematics, the subject I love the most. So, at every moment I am free, I cannot resist finding myself reading mathematical publications or re-reading my own works, which is my happiness. Simultaneously, I also realize that the older I get, the more responsibilities in the legal field will arise, and after this academic year, I will no longer have time to conduct mathematical research. I also realize that I never made a proper goodbye to this area of study, which is coincidental to my encounter in early 2025 with the paper from Kamsrisuk et al. (see [14]), which made me realize that I had made an error, as the previous paper is not generally true. So, after almost four years during which I have not seriously touched mathematics, I came back to make a heavy revision to correct the mistakes I made in the previous results, which takes much longer than it used to in my prime time. And after I reached the current result, which is coherent with what was obtained by Thabet et al. (see [19]), I decided to generalize the n-tuple operator I had previously defined and use this opportunity to introduce a generalized version of such an operator, while also correcting my mistakes from four years ago. Lastly, I would like to turn this last rite into a final piece, a fitting goodbye, so that this cherished memory will never be forgotten and will never grow old.

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In memory of Assoc. Prof. Kamsing Nonlaopon

I would also like to express my deepest condolences on the passing of Assoc. Prof. Kamsing Nonlaopon — an unfortunate victim of a dysfunctional academic system — which occurred while this paper was under review. He made profound contributions to the fields of fractional calculus and q-calculus, and his work continues to shape the discipline. It is truly a tragedy that he was never able to return to the role he fulfilled with such devotion in teaching and researching.

Conflicts of Interest

The authors declare no conflicts of interest.

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