Journal of Mathematical Analysis and Modeling
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2026)7(1) : 82-95
doi:10.48185/jmam.v7i1.1882

SOME NEW GRONWALL - BELLMAN AND BIHARI TYPE INTEGRAL INEQUALITIES AND ITS APPLICATIONS TO RIEMANN - LIOUVILLE FRACTIONAL DIFFERENTAL EQUATIONS

NIRMALA M. MORADE [a,*], ASHOK P. BHADANE [b]

[a] Department of Mathematics, MVP Samaj's SVKT College (affiliated to SPPU), Deolali Camp, Nashik (Maharashtra), India

[b] Department of Mathematics, MGV's LVH College (affiliated to SPPU), Panchavati, Nashik((Maharashtra), India

• *Received: 13 November 2025* • *Accepted: 22 March 2026* • *Published Online: 13 April 2026*

Abstract

In this paper, some new variants of Gronwall–Bellman and Bihari type integral inequalities are developed, which generalize some known weakly singular integral inequalities found in the literature. These can be used in the quantitative and qualitative analysis of the solution to certain fractional differential equations. Furthermore, these results are applied to a fractional differential equation to study uniqueness and the dependence of solution on initial conditions and nonlinear terms. The results contribute to the theoretical analysis of fractional differential equations and provide effective tools for studying their qualitative properties. We can also focus on extending these inequalities to other fractional operators and these results can be applied to study the existence, uniqueness, and stability of solutions for more general nonlinear fractional differential equations.

Keywords: Gronwall-Bellman inequalities, Riemann–Liouville operator, Fractional Differential Equations, Bihari type inequalities.
2010 MSC: 26D10, 26A33 .

1. Introduction

Integral inequalities and their various linear and nonlinear variants play a crucial role in the qualitative analysis of the solutions to differential and integral equations.The prominent Gronwall-Bellman and Bihari type inequalities provided explicit bounds on solving a kind of integral inequalities [20].However, to understand the singular integral equations dealt with fractional differential equations, we need different kinds of inequalities.Over the year, many researchers have extended these inequalities to more general forms in order to handle nonlinear and singular problems.

*Corresponding author: nmmorade@gmail.com

The weakly singular integral inequalities naturally arise in the theory of fractional differential equations, where integral repertory often contain kernel of the form $(t-s)^{\alpha-1}$, with $0<\alpha<1$.These types of kernels appear in problems governed by fractional derivatives as those defined in the sense of Riemann-Liouville and Caputo.Fractional Calculus (FC) is an important area of mathematics that generalizes classical differentiation and integration to non-integer orders [10] [14]. Due to the memory and hereditary properties of fractional operators, they provide powerful tools for modeling complex real–world phenomena that cannot be accurately described by classical calculus. Therefore, FC has wide applications in engineering, physics, control theory, biology, environment, and finance etc.

The study of weakly singular inequalities has gained attention of many researchers due to their fundamental role in qualitative analysis of fractional differential equations.These type inequality, characterized by kernels involving Gamma function, Mittag leffler function, generalize the classical results such as Gronwall, Bihari type inequalities and provide efficient tools for establishing bound, stability, existence , and uniqueness of solutions.Therefore the development and refinement of such inequalities play a essential role advancing in both theoretical and applied aspects of fractional order systems. The study of weakly singular inequalities provides an essential theoretical framework for obtaining explicit estimates and upper bounds for solutions of fractional differential equations.

Henry's work (1919) [9] establishes global existence and exponential decay results for a parabolic Cauchy problem by utilizing using the singular integral inequality:

$$\nu(\iota)\leqslant\chi(\iota)+\sigma\int_0^{\iota}(\iota-s)^{\delta-1}\nu(s)ds.$$

This inequality is generalized by Ye et al. (2007) by replacing constant σ by $\sigma(\iota)$, a bounded function and studied the dependence of the solutions on the parameters order and initial conditions of Cauchy type problem[20]. Subsequently,Ricardo Almeida et al.(2019) established the inequality:

$$\nu(\iota)\leqslant\chi(\iota)+\sigma(\iota)\int_0^{\iota}\varphi'(\iota)(\varphi(\iota)-\varphi(s))^{(\delta-1)}\nu(s)ds[1].$$

using the φ- fractional derivative (w.r.t. another another function).This inequality is the generalization of inequality studied by Ye Haiping et al..This inequality is also studed by Salah Boulares et al.(2020) by Tao Zhu's method [**?**].Ding and Jiang in 2012 generalized Henry's result with $\sigma=\sigma(\iota)$, kernel $(\iota-s)^{(\delta-1)}E_{(\delta,\delta)}(\lambda(\iota-\tau)^{\delta})$,and used it to analyze how the solution of semilinear fractional differential equations varies with initial circumstances [7].

Medved (1997) studied the linear Henry-type integral inequality:

$$\nu(\iota)\leqslant\chi(\iota)+\sigma(\iota)\int_0^{\iota}(\iota-s)^{(\delta-1)}\mu(s)\nu(s)ds[13].$$

Denton and Vatsala (2010) [6] discussed the above inequality when $\chi(\iota)=\nu_0\iota^{(\delta-1)}$. Medved (1997) [13], and Pecaric (2008) [12] presented the following integral inequality:

$$\nu(\iota)\leqslant\chi(\iota)+\int_0^{\iota}(\iota-s)^{(\delta-1)}\mu(s)w(\nu(s))ds.$$

Moreover,Tao Zhu (2018) presented the following linear integral inequalities using a new method and obtain an upper bound for the unknown function is simple form :

$$\nu(\iota) \leqslant \chi(\iota) + \frac{1}{\Gamma(\delta)} \int_0^{\iota} (\iota - s)^{(\delta-1)} \mu(s)\nu(s) ds[21].$$

Ben Makhlouf et al. [3] developed a class of weakly singular integral inequalities of the form

$$u(t) \leqslant a(t) + \frac{1}{\Gamma(\beta)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\beta-1} w(s)u(s)\, ds, \tag{1.1}$$

and applied these inequalities to investigate the existence, uniqueness, and Ulam stability of solutions of tempered fractional differential equations. Their results provide important analytical tools for studying the qualitative behavior of solutions involving tempered fractional operators. More recently, Boulares et al. [4] studied weakly singular integral inequalities in the context of weighted fractional differential equations. They considered inequalities of the form

$$u(t) \leqslant a(t) + \frac{\phi^{-1}(t)}{\Gamma(\beta)} \int_0^t \phi(s)(t-s)^{\beta-1} w(s)u(s)\, ds, \tag{1.2}$$

and established new estimates that can be used to analyze the boundedness and qualitative properties of solutions of weighted fractional differential equations. Furthermore, Qiong Wu (2017) proposed Gronwall–Bellman type inequality, which arises from a class of integral equations with a combination of singular and nonsingular integrals [15]. Additionally, he investigated the existence and uniqueness of the solution to a stochastic differential equation.
Also,Yongsheng Li and Zizun Li (2026) developed weakly singular Wendroff-type integral inequalities with three variables and nonlinear terms, and apply to study qualitative properties of fractional partial differential equations [11].Such type of inequalities are also studied by researchers Cheung et al. (2008) [5], Wang and Zheng (2010) [16],Run xu et al.(2017)[18].Run Xu and Fanewl Meng (2016) Studied Henry - Gronwall retarded type inequalities and its applications to fractional differential equations [17].
A generalized version of the frequently referenced Gronwall-Bellman inequality , proposed by Ye Haipping et al. in 2007 for an integral equations with singular integrals , is presented below.

Lemma 1.1. *[20] Suppose* $\chi : [0, I) \to R_+$ *is a locally integrable function with* $I \leqslant \infty$ *and* $\xi : [0, I) \to R_+$ *is a nondecreasing continuous function,* $\xi(\iota) \leqslant M$*(constant), and suppose that* $\nu : [0, I) \to R_+$ *is a locally integrable with*
$\nu(\iota) \leqslant \chi(\iota) + \xi(\iota) \int_0^{\iota} (\iota - \sigma)^{(\delta-1)} \nu(\sigma) d\sigma$ *on this interval for* $\delta > 0$*. Then,*

$$\nu(\iota) \leqslant \chi(\iota) + \xi(\iota) \int_0^{\iota} \sum_{n=0}^{\infty} \frac{(\xi(\iota)\Gamma(\delta))^n}{\Gamma(n\delta)} (\iota - \sigma)^{n\delta-1} \chi(\sigma)\, d\sigma, \quad \iota \in [0, I).$$

Furthermore, $\chi(\iota)$*is a nondecreasing function on* $[0, I)$*;then*

$$\nu(\iota) \leqslant \chi(\iota)\, E_{\delta} \left(\xi(\iota)\Gamma(\delta)\iota^{\delta}\right),$$

where the Mittag-Leffler function $E_\delta(z)$ *is defined by*

$$E_\delta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\delta + 1)}.$$

The following Lemma concerns an integral inequality with singular and nonsingular kernels by Qiong Wu(2017).

Lemma 1.2. *[15] Suppose* $0 < \delta < 1, \chi : [0, I) \to R_+$ *is a locally integrable function with* $I \leqslant \infty$*,* $\sigma(\iota)$ *and* $\xi(\iota)$*is a nonnegative, nondecreasing continuous function, with both bounded by a positive constant,* M*. If* $\nu : [0, I) \to R_+$ *is a locally integrable function with*

$$\nu(\iota) \leqslant \chi(\iota) + \sigma(\iota) \int_0^\iota \nu(s)\, ds + \xi(\iota) \int_0^\iota (\iota - s)^{\delta - 1} \nu(s)\, ds,$$

for $\iota \in [0, I)$*. Then,*

$$\nu(\iota) \leqslant \chi(\iota) + \sum_{n=1}^{\infty} \sum_{i=0}^{n} \binom{n}{i} \sigma^{n-i}(\iota) \xi^i(\iota) \frac{[\Gamma(\delta)]^i}{\Gamma(i\delta + n - i)} \int_0^\iota (\iota - s)^{i\delta - (i+1-n)} \chi(s)\, ds.$$

Qiong Wu and Ye et al. employed an iterative technique, leading to estimates represented by integrals with singular kernels associated with functions expressed through complex power series. In the present work, we establish new integral inequalities that generalize earlier results. Moreover, the obtained explicit upper bounds for the unknown function are presented in a simplified form, analogous to the classical Gronwall–Bellman and Bihari inequalities. These results are then utilized to study the dependence of solutions of nonlinear fractional differential equations on the involved functions and initial conditions.

2. Methods

This study focuses on an extension and applying integral inequalities, particularly of Gronwall-Bellman and Bihari type in the context of fractional differential equations. New integral inequalities are derived using lemma (3.1), and comparison principle. These inequalities extend classical results like Gronwall and Bihari types. Additionally, this study investigates fractional Cauchy problems with Riemann–Liouville derivatives, which are reformulated as equivalent Volterra-type integral equations of second kind.

3. Preliminaries

The following result is useful to establish the main results.

Lemma 3.1. *[8] If* ρ *and* σ *are nonconstant functions such that, on any finite subinterval of the interval of integration* $[t_0, t_1]$*, one function is increasing while the other is decreasing (and vice versa), then the following inequality holds:*

$$\int_{t_0}^{t_1} \rho(\xi)\sigma(\xi)\, d\xi < \frac{1}{(t_0 - t_1)} \int_{t_0}^{t_1} \rho(\xi)\, d\xi \int_{t_0}^{t_1} \sigma(\xi)\, d\xi$$

Riemann-Liouville integral and derivative defined as follows.

Definition 3.2. [10] (page 69) Let $[\mathfrak{a},\mathfrak{b}](-\infty < \mathfrak{a} < \mathfrak{b} < \infty)$.The Riemann-Liouville fractional left-sided integral of function $\mathfrak{y}$ of order $\alpha \in C(R(\alpha) \geqslant 0)$ is defined by

$$(\mathfrak{I}^{\alpha}_{(\mathfrak{a}+)}\mathfrak{y})(\mathfrak{x}) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{a}}^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{t})^{(n-\alpha+1)}\mathfrak{y}(\iota)d\iota \quad (\mathfrak{x} < \mathfrak{a}; R(\alpha) < 0).$$

Definition 3.3. [10] (page 70) Let $[\mathfrak{a},\mathfrak{b}](-\infty < \mathfrak{a} < \mathfrak{b} < \infty)$. The Riemann-Liouville fractional left-sided derivative of $\mathfrak{y}(\mathfrak{x})$ of order $\alpha \in C(R(\alpha) \geqslant 0)$ is defined by

$$(\mathfrak{D}^{\alpha}_{(\mathfrak{a}+)}\mathfrak{y})(\mathfrak{x}) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{d\mathfrak{x}})^{n} \int_{\mathfrak{a}}^{\mathfrak{x}} (\mathfrak{x}-')^{(\alpha-1)}\mathfrak{y}(\iota)d\iota \quad (\mathfrak{x} > \mathfrak{a}; n = [R(\alpha)]+1)$$

where $[R(\alpha)]$ is integral part of $R(\alpha)$.

Cauchy type Problems [10](page 136): The nonlinear differential equation of fractional order $\alpha(0 < \alpha < 1$ on finite interval $[\mathfrak{a},\mathfrak{b}]$ is
$\mathfrak{D}^{\alpha}_{\mathfrak{a}+}\mathfrak{y}(\iota) = f(\iota,\mathfrak{y}(\iota)); (\iota > \mathfrak{a})$, with initial condition $(\mathfrak{D}^{(\alpha-1)}_{\mathfrak{a}+}\mathfrak{y}(\iota))(\mathfrak{a}+) = \eta$.
where $\mathfrak{D}^{\alpha}_{\mathfrak{a}+}$ is the Riemann-Liouville fractional derivative.
The above initial value problem is equivalent to the following nonlinear Volterra integral equation of the second kind:

$$\mathfrak{y}(\iota) = \frac{\eta}{\Gamma(\alpha)}(\iota-\mathfrak{a})^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\iota}(\iota-\tau)^{(\alpha-1)}f(\tau,\mathfrak{y}(\tau))d\tau, \iota > \mathfrak{a}. \tag{3.1}$$

4. Main Results

In this section, we are going to demonstrate integral inequalities that can be employed in a fractional differential equations.

Theorem 4.1. *Let ν, ϕ, φ, ψ , and ξ are continuous functions from $[a,b]$ to R_{+} and*

$$\nu(\iota) \leqslant \phi(\iota) + \varphi(\iota)\int_{a}^{\iota}(\iota-s)^{\delta-1}\left[\psi(s)\nu(s)+\xi(s))\right]\,ds, \tag{4.1}$$

$\iota \in [a,b]$ *and* $\delta > 0$. *Then,*

$$\nu(\iota) \leqslant \phi(\iota) + \vartheta(\iota)\int_{a}^{\iota}[\psi(s)\phi(s)+\xi(s)]\exp\left(\int_{s}^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma\right)ds, \quad 0 \leqslant \iota \leqslant b, \tag{4.2}$$

where $\vartheta(\iota) = \frac{\varphi(\iota)(\iota-a)^{(\delta-1)}}{\delta}$.

Proof. From inequality (4.1)for $\iota \in [a,b]$, we have

$$\nu(\iota) \leqslant \phi(\iota) + \varphi(\iota)\int_{a}^{\iota}(\iota-s)^{\delta-1}\left[\psi(s)\nu(s)+\xi(s))\right]\,ds.$$

Thus

$$\nu(\iota) \leqslant \phi(\iota) + \frac{\varphi(\iota)}{(\iota-a)}\int_{a}^{\iota}(\iota-s)^{\delta-1}\int_{a}^{\iota}\left[\psi(s)\nu(s)+\xi(s)\right]\,ds, \qquad \text{using Lemma (3.1)}$$

provided that $\psi(\iota)\nu(\iota)+\xi(\iota)$ is non-constant function, it is increasing if $0<\delta<1$ and decreasing if $\delta>1$ on any finite subinterval of $[a,b]$.

$$\nu(\iota)\leqslant \phi(\iota)+\frac{\varphi(\iota)(\iota-a)^{(\delta-1)}}{\delta}\int_a^{\iota}[\psi(s)\nu(s)+\xi(s)]\,ds.$$

Therefore,

$$\nu(\iota)\leqslant \phi(\iota)+\varphi(\iota)\int_a^{\iota}[\psi(s)\nu(s)+\xi(s)]\,ds \tag{4.3}$$

where $\vartheta(\iota)=\frac{\varphi(\iota)(\iota-a)^{(\delta-1)}}{\delta}$.
Let

$$z(\iota)=\int_a^{\iota}[\psi(s)\nu(s)+\xi(s)]\,ds$$

Therefore $z(a)=0$, and

$$z'(\iota)=\psi(\iota)\nu(\iota)+\xi(\iota)$$

Thus inequality (4.3) becomes

$$\nu(\iota)\leqslant \phi(\iota)+\varphi(\iota)z(\iota) \tag{4.4}$$

Using inequalities (4.3) and (4.4), we obtain

$$z'(\iota)\leqslant \psi(\iota)\phi(\iota)+\xi(\iota)+\psi(\iota)\varphi(\iota)z(\iota) \tag{4.5}$$

Multiply the above inequality by $\exp(-\int_a^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)$

$$\left[z'(\iota)-\psi(\iota)\varphi(\iota)z(\iota)\right]\exp(-\int_a^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)\leqslant[\psi(\iota)\phi(\iota)+\xi(\iota)]\exp(-\int_a^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)$$

$$\frac{d}{dt}\left[z(\iota)\exp(-\int_a^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)\right]\leqslant[\psi(\iota)\phi(\iota)+\xi(\iota)]\exp(-\int_a^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)$$

By setting $\iota=s$ and integrating it with respect to s from a to ι, we obtain

$$\left[z(\iota)\exp(-\int_a^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)\right]\leqslant\int_a^{\iota}[\psi(s)\phi(s)+\xi(s)]\exp(-\int_a^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)ds$$

$$z(\iota)\leqslant\int_a^{\iota}[\psi(s)\phi(s)+\xi(s)]\exp(\int_s^{\iota}\psi(\sigma)\varphi(\sigma)d\sigma)ds$$

Using above inequality and inequality(4.4), we obtain the result as inequality (4.1). □

Theorem 4.2. *Suppose* $\nu,\phi,\ \varphi,$*and* ξ *are continuous functions from* $[a,b]$ *to* R_+ *and*

$$\nu(\iota)\leqslant\phi(\iota)+\varphi(\iota)\int_a^{\iota}\psi(s)\nu(s)ds+\xi(\iota)\int_a^{\iota}(\iota-s)^{(\delta-1)}\psi(s)\nu(s)ds,\quad \delta>0. \tag{4.6}$$

Then,

$$\nu(\iota)\leqslant\phi(\iota)+\left[\varphi(\iota)+\frac{\xi(\iota)(\iota-a)^{(\delta-1)}}{\delta}\right]\int_a^{\iota}\psi(s)\phi(s)\exp\left[\int_s^{\iota}\psi(\sigma)\left(\varphi(\sigma)+\frac{\xi(\sigma)(\sigma-a)^{(\delta-1)}}{\delta}\right)d\sigma\right]ds \tag{4.7}$$

on $[a,b]$.

Proof. From inequality (4.6)for $\iota \in [a,b]$, we have

$$\nu(\iota) \leqslant \phi(\iota) + \varphi(\iota)\int_a^\iota \psi(s)\nu(s)ds + \xi(\iota)\int_a^\iota (\iota - s)^{(\delta-1)}\psi(s)\nu(s)ds$$

$$\nu(\iota) \leqslant \phi(\iota) + \varphi(\iota)\int_a^\iota \psi(s)\nu(s)ds + \frac{\xi(\iota)(\iota-a)^{(\delta-1)}}{\delta}\int_a^\iota \psi(s)\nu(s)ds \text{ using Lemma 3.1}$$

provided that $\psi(\iota)\nu(\iota)$ is nonconstant function, it is increasing if $0 < \delta < 1$ and decreasing if $\delta > 1$ on any finite subinterval of $[a,b]$.

$$\nu(\iota) \leqslant \phi(\iota) + Q_1(\iota)\int_a^\iota \psi(s)\nu(s)ds \quad \text{where} \quad Q_1(\iota) = \left(\varphi(\iota) + \frac{\xi(\iota)(\iota-a)^{(\delta-1)}}{\delta}\right)$$

The proof follows by the same argument as in the proof of Theorem 4.1,we obtain

$$\nu(\iota) \leqslant \phi(\iota) + Q_1(\iota)\int_a^\iota \psi(s)\phi(s)\exp\left(\int_s^\iota \psi(\sigma)Q_1(\sigma)d\sigma\right)ds, \quad 0 \leqslant \iota \leqslant b.$$

□

The following inequalities are extension of GronwalI-Bellman-Bihari Inequalities given by Pachpatte(1975a).

Theorem 4.3. *Let ν, φ, ξ and ρ be continuous functions from R_+ to R_+. Let $w(\nu)$ be a continuous nondecreasing and submultiplicative function defined on R_+ and $w(\nu) > 0$ on $(0,\infty)$. If*

$$\nu(\iota) \leqslant \nu_0 + \xi(\iota)\int_0^\iota (\iota-s)^{(\delta-1)}\psi(s)\nu(s)ds + \int_0^\iota \rho(s)w(\nu(s))ds, \tag{4.8}$$

for all $\iota \in R_+, \delta > 0$, where ν_0 is a positive constant, then for $0 < \iota < \iota_1$,

$$\nu(\iota) \leqslant \chi(\iota)G^{-1}\left(G(\nu_0) + \int_0^\iota \psi(s)w(\chi(s))ds\right), \tag{4.9}$$

where $\chi(\iota) = 1 + \xi(\iota)\int_0^\iota \psi(s)\exp\left(\int_s^\iota f(\sigma)G(\sigma)d\sigma\right)ds$ *for* $\iota \in R_+$ *and* $G(r) = \int_{r_0}^r \frac{1}{w(s)}ds, 0 < r_0, 0 < r$ *and* G^{-1} *is inverse function of* G, *and* $\iota_1 \in R_+$ *is chosen so that* $G(\nu_0) + \int_0^\iota \psi(s)w(\chi(s))ds \in \mathrm{Dom}\left(G^{-1}\right)$, $\iota \in [0,\iota_1]$.

Proof. From inequality (4.7)for $\iota \in R_+$, we have

$$\nu(\iota) \leqslant \nu_0 + \xi(\iota)\int_0^\iota (\iota-s)^{(\delta-1)}\psi(s)\nu(s)ds + \int_0^\iota \rho(s)w(\nu(s))ds.$$

$$\nu(\iota) \leqslant \nu_0 + \frac{\xi(\iota)t^\delta}{\delta}\int_0^\iota (\iota-s)^{(\delta-1)}\psi(s)\nu(s)ds + \int_0^\iota \rho(s)w(\nu(s))ds. \tag{4.10}$$

using Lemma 3.1 provided that $\psi(\iota)\nu(\iota)$ is nonconstant function, it is increasing if $0 < \delta < 1$ and decreasing if $\delta > 1$ on any finite subinterval of $[0,\iota]$.
Let

$$n(\iota) = \nu_0 + \int_0^\iota \rho(s)w(\nu(s))ds. \tag{4.11}$$

Then inequality (4.10) becomes

$$\nu(\iota) \leqslant n(\iota) + g_1(\iota)\int_0^{\iota} \psi(s)\nu(s)ds \quad \text{with} \quad g_1(\iota) = \frac{\xi(\iota)t^{(\delta-1)}}{\delta}.$$

Since $n(\iota)$ is positive monotonic nondecreasing on R_+, by applying Theorem 1.3.3(page 15) [2],we have

$$\nu(\iota) \leqslant \chi(\iota)n(\iota), \tag{4.12}$$

where $\chi(\iota) = 1 + g_1(\iota)\int_0^{\iota} \psi(s)\exp\left(\int_s^{\iota} \psi(\sigma)g_1(\sigma)d\sigma\right)ds$.
From equation (4.11) and inequality (4.12), we have

$$n^{'}(\iota) = \rho(\iota)w(\nu(\iota)) \leqslant \rho(\iota)w(\chi(\iota)n(\iota)) \leqslant \rho(\iota)w(\chi(\iota))w(n(\iota)).$$

The above inequality can be written as

$$\frac{d}{dt}G(n(\iota)) = \frac{n^{'}(\iota)}{w(n(\iota)} \leqslant \rho(\iota)w(\chi(\iota))$$

. By taking $\iota = s$ and integrating it from 0 to ι, we obtain

$$G(n(\iota)) \leqslant G(\nu_0) + \int_0^{\iota} \psi(s)w(\chi(s))ds.$$

Therefore,

$$n(\iota) \leqslant G^{-1}(G(\nu_0) + \int_0^{\iota} \psi(s)w(\chi(s))ds.$$

From inequality (4.12), we obtain

$$\nu(\iota) \leqslant \chi(\iota)G^{-1}(G(\nu_0) + \int_0^{\iota} \psi(s)w(\chi(s))ds).$$

□

Theorem 4.4. *Let ν,ψ, ξ and ρ be continuous functions R_+ to R_+. Let $w(\nu)$ be a continuous nondecreasing subadditive and submultiplicative function defined on R_+ and $w(\nu) > 0$ on $[0,\infty)$.Let $\phi(\iota) > 0$, $\varphi(\iota) \geqslant 0$ be continuous nondecresing functions defined on R_+ and $\varphi(0) = 0$.If*

$$\nu(\iota) \leqslant \phi(\iota) + \xi(\iota)\int_0^{\iota} (\iota - s)^{(\delta-1)}\psi(s)\nu(s)ds + \varphi(\int_0^{\iota} \rho(s)w(\nu(s))ds) \tag{4.13}$$

for all $\iota \in R_+$ and $\delta > 0$ where ν_0 is a positive constant, then for $0 < \iota < \iota_2$,

$$\nu(\iota) \leqslant \chi(\iota)\left[\phi(\iota) + \varphi\left(F^{-1}\left(F(A(\iota) + \int_0^{\iota} \rho(s)w(\chi(s))ds\right)\right)\right] \tag{4.14}$$

where $\chi(\iota) = 1 + g_1(\iota)\int_0^{\iota} \psi(s)\exp(\int_s^{\iota} \psi(\sigma)g_1(\sigma)d\sigma)ds$ for $\iota \in R_+$ and $F(r) = \int_{r_0}^{r} \frac{1}{w(\varphi(s))}ds, 0 < r_0, 0 < r$ and F^{-1} is inverse function of F, $\iota_2 \in R_+$ is chosen so that $F(A(\iota)) + \int_0^{\iota} \rho(s)w(\chi(s))ds \in \mathrm{Dom}(F^{-1}), \iota \in [0, \iota_2]$.

Proof. From inequality (4.13),

$$\nu(\iota) \leqslant \phi(\iota) + \xi(\iota)\int_0^{\iota}(\iota - s)^{(\delta-1)}\psi(s)\nu(s)ds + \varphi(\int_0^{\iota}\rho(s)w(\nu(s))ds)$$

Then using Lemma 3.1

$$\nu(\iota) \leqslant \phi(\iota) + g_1(\iota)\int_0^{\iota}(\iota - s)^{(\delta-1)}\psi(s)\nu(s)ds + \varphi(\int_0^{\iota}\rho(s)w(\nu(s))ds).$$

where $g_1(\iota) = \frac{\xi(\iota)t^{(\delta-1)}}{\delta}$.
Let

$$n(\iota) = \phi(\iota) + \varphi(\int_0^{\iota}\rho(s)w(\nu(s))ds). \tag{4.15}$$

Therefore the above inequality becomes

$$\nu(\iota) \leqslant n(\iota) + g_1(\iota)\int_0^{\iota}(\iota - s)^{(\delta-1)}\psi(s)\nu(s)ds.$$

Since $n(\iota)$ is positive monotonic nondecreasing on R_+, by applying Theorem 1.3.3(page 15) in [2], we have,

$$\nu(\iota) \leqslant \chi(\iota)n(\iota). \tag{4.16}$$

where $\chi(\iota) = 1 + g_1(\iota)\int_0^{\iota}\psi(s)\exp(\int_s^{\iota}\psi(\sigma)P(\sigma)d\sigma)ds$ for $t \in R_+$.
From inequalities (4.15) and (4.16),we have

$$n(\iota) - \phi(\iota) = \varphi(\int_0^{\iota}\rho(s)w(\nu(s))ds) \leqslant \varphi(\int_0^{\iota}\rho(s)w(\chi(s)n(s))ds).$$

The above inequality can be written as,

$$n(\iota) - \phi(\iota) \leqslant \varphi(\int_0^{\iota}\rho(s)w(\chi(s)\phi(s) + \chi(s)(n(s) - \phi(s))ds).$$

Therefore,

$$n(\iota) - \phi(\iota) \leqslant \varphi(A(\iota) + \int_0^{\iota}\rho(s)w(\chi(s))w((n(s) - \phi(s))ds). \tag{4.17}$$

where $A(\iota) = \int_0^{\iota}\rho(s)w(\chi(s)\phi(s)ds$.

Let $T \in R_+$ be any arbitrary number. Define a function $v(\iota)$ by

$$v(\iota) = A(\iota) + \int_0^{\iota}\rho(s)w(\chi(s))w((n(s) - \phi(s))ds). \tag{4.18}$$

Therefore suing inequalities (4.17) and (4.18) for $0 \leqslant \iota \leqslant T$,

$$v(\iota) \leqslant \chi(\iota) + \int_0^{\iota}\rho(s)w(\chi(s))w(\varphi(v(s))ds). \tag{4.19}$$

Define a function $y(\iota)$ by the right-hand side of 4.19.Then $y(0) = \chi(\iota)$, $\nu(\iota) \leqslant y(\iota)$, and

$$y^{'}(\iota) = \rho(\iota)w(\chi(\iota))w(\varphi(y(\iota)).$$

Thus,

$$\frac{d}{dt}F(y(\iota)) = \frac{y^{'}(\iota)}{w(\varphi(y(\iota))} \leqslant \rho(\iota)w(\chi(\iota)).$$

By taking $\iota = s$ and integrating it from 0 to T , we obtain

$$F(y(\iota)) \leqslant F(A(\iota) + \int_0^\iota \rho(s)w(\chi(s))ds, 0 \leqslant \iota \leqslant T < \iota_2.$$

From inequality (4.17),

$$n(\iota) - \phi(\iota) \leqslant \varphi(y(\iota)) \leqslant \varphi(F^{-1}(F(A(\iota) + \int_0^\iota \rho(s)w(\chi(s))ds)), 0 < \iota < T < \iota_2.$$

Since T is arbitrary and $\nu(\iota) \leqslant \chi(\iota)n(\iota)$, we have

$$\nu(\iota) \leqslant \chi(\iota)\left[\phi(\iota) + \varphi(F^{-1}(F(A(\iota) + \int_0^\iota \rho(s)w(\chi(s))ds))\right].$$

□

Theorem 4.5. *Let ν, ψ, ξ, and ρ be continuous functions from R_+ to R_+. Let $w(\iota, \nu)$ be a nonnegative continuous monotonic nondecreasing in $\nu \geqslant 0$, for each fixed ι in R_+ and $w(\nu) > 0$ on $[0, \infty)$.Let $\phi(\iota) > 0$, $\varphi(\iota) \geqslant 0$ be continuous nondecresing functions defined on R_+ and $\varphi(0) = 0$.If*

$$\nu(\iota) \leqslant \phi(\iota) + \xi(\iota)\int_0^\iota (\iota - s)^{(\delta-1)}\psi(s)\nu(s)ds + \varphi(\int_0^\iota \rho(s)w(s, \nu(s))ds), \tag{4.20}$$

then for all ι in R_+,

$$\nu(\iota) \leqslant \chi(\iota)[\phi(\iota) + \varphi(r(\iota))] \tag{4.21}$$

where $\chi(\iota) = 1 + g_1(\iota)\int_0^\iota \psi(s)\exp(\int_s^\iota \psi(\sigma)g_1(\sigma)d\sigma)ds$ for $\iota \in R_+$ and $r(\iota)$ is the maximal solution of

$$r^{'}(\iota) = \rho(\iota)w(\iota, \chi(\iota)[\phi(\iota) + \varphi(r(\iota)]), \tag{4.22}$$

$r(0) = 0$ *existing on* R_+.

Proof. Using inequality (4.20) and Lemma(3.1), we obtain

$$\nu(\iota) \leqslant \phi(\iota) + g_1(\iota)\int_0^\iota (\iota - s)^{(\delta-1)}\psi(s)\nu(s)ds + \varphi(\int_0^\iota \rho(s)w(s, \nu(s))ds),$$

where $g_1(\iota) = \frac{\xi(\iota)t^{\delta-1}}{\delta}$ Let

$$n(\iota) = \phi(\iota) + \varphi(\int_0^\iota \rho(s)w(s, \nu(s))ds).$$

Therefore the above inequality becomes

$$\nu(\iota) \leqslant n(\iota) + g_1(\iota)\int_0^{\iota} \psi(s)\nu(s)ds$$

Since $n(\iota)$ is positive monotonic nondecreasing on R_+, by applying Theorem 1.3.3(page 15) [2] We have,

$$\nu(\iota) \leqslant \chi(\iota)n(\iota),$$

where $\chi(\iota) = 1 + g_1(\iota)\int_0^{\iota} \psi(s)\exp\left(\int_s^{\iota} \psi(\sigma)g_1(\sigma)d\sigma\right)ds, \quad t \in R_+.$

$$\nu(\iota) \leqslant \chi(\iota)[\phi(\iota) + \varphi(\int_0^{\iota} \rho(s)w(s, \nu(s))ds)].$$

$$\nu(\iota) \leqslant \chi(\iota)[\phi(\iota) + \varphi(v(\iota)] \quad \text{where} \quad v(\iota) = \int_0^{\iota} \rho(s)w(\nu(s))ds.$$

$$v'(\iota) = \rho(\iota)w(\iota, \nu(\iota)) \leqslant \rho(\iota)w(\iota, \chi(\iota)[\phi(\iota) + \varphi(v(\iota))]).$$

Using comparison principle (theorem 2.2.2, page 101 in [2]) we obtain, $v(\iota) \leqslant r(\iota)$ where $r(\iota)$ is the maximal solution of equation(4.22) such that $r(0) = v(0) = 0$. We have,

$$\nu(\iota) \leqslant \chi(\iota)[\phi(\iota) + \varphi(r(\iota))].$$

□

5. Application : Dependence of solutions on the functions

In this section, we investigate the solution's dependency on the functions of a fractional equation with Riemann-Liouville fractional derivatives.The existence and uniqueness of the Cauchy problem as well as the dependence of a solution on initial conditions studied in] [14] if $0 < \alpha < 1$ on $[0, I)$. The dependence of a solution on the order and the initial conditions studied in [20] if $0 < \alpha < 1$ on $[0, I)$. Also, we determine the relationship between two initial value problems involving two different functions, which will explain how the solutions vary when the functions are slightly modified.

Theorem 5.1. *Let* $0 < \alpha < 1$.*Let* $\psi, \xi : [a, b]XR \to R$ *be continuous functions satisfying the conditions i)* $|\psi(\iota, y) - \psi(\iota, z)| \leqslant L|y - z|$, *ii) Given* $\epsilon > 0, |\psi(\iota, y) - \xi(\iota, y)| < \epsilon$ *for all* $\iota, y \in [a, b]$. *Let* $y_1(\iota)$ *and* $y_2(\iota)$ *be the solutions of the following initial value problems (5.1) and (5.2) respectively with* $[\iota, y_1(\iota)]$ *and* $\iota, y_2(\iota)] \in [a, b]XR$.

$$\mathfrak{D}^{\alpha}_{a+}y(\iota) = \psi(\iota, y(\iota), (\mathfrak{D}^{(\alpha-1)}_{a+}y(\iota))(a+) = \eta, \tag{5.1}$$

$$\mathfrak{D}^{\alpha}_{a+}y(\iota) = \xi(\iota, y(\iota), (\mathfrak{D}^{(\alpha-1)}_{a+}y(\iota))(a+) = \eta_1. \tag{5.2}$$

Then,

$$|y_1(\iota) - y_2(\iota)| \leqslant \frac{|\eta - \eta_1|}{\Gamma(\alpha)h^{(\alpha-1)}} + \frac{\epsilon}{\alpha\Gamma(\alpha)h^{\alpha}} + \frac{\epsilon L h^{(\alpha-1)}}{\alpha^2\Gamma^2(\alpha)}\int_a^{\iota}(s-a)^{\alpha}\exp(K(h^{\alpha} - (s-a)^{\alpha}))ds, \tag{5.3}$$

where $h = \iota - a, K = \frac{L}{\alpha^2\Gamma^2(\alpha)}$.

Proof. The solutions of the given initial value problems (5.1) and (5.2) are given by

$$y_1(\iota) = \frac{\eta}{\Gamma(\alpha}(\iota - a)^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}\psi(\tau, y(\tau))d\tau, \iota > a$$

and

$$y_2(\iota) = \frac{\eta_1}{\Gamma(\alpha}(\iota - a)^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}\xi(\tau, y(\tau))d\tau, \iota > a$$

respectively. It follows that

$$\begin{aligned}
&|y_1(\iota) - y_2(\iota)| \\
&\leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)}(\iota-a)^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}|\psi(\tau,y_1(\tau)) - \xi(\tau, y_2(\tau))|d\tau \\
&\leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)}(\iota-a)^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}|\psi(\tau,y_1(\tau) - \psi(\tau, y_2(\tau))|d\tau \\
&+ \frac{1}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}|\psi(\tau,y_2(\tau) - \xi(\tau, y_2(\tau))|d\tau \\
&\leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)}(\iota-a)^{(\alpha-1)} + \frac{L}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}|y_1(\tau) - y_2(\tau)|d\tau + \frac{\epsilon}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}d\tau \\
&\leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)}(\iota-a)^{(\alpha-1)} + \frac{\epsilon}{\alpha\Gamma(\alpha)}(\iota-a)^{\alpha} + \frac{L}{\Gamma(\alpha)}\int_a^{\iota}(\iota-\tau)^{(\alpha-1)}|y_1(\tau) - y_2(\tau)|d\tau.
\end{aligned}$$

An application of theorem (1) yields

$$\begin{aligned}
|y_1(\iota) - y_2(\iota)| &\leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)}(\iota-a)^{(\alpha-1)} + \frac{\epsilon}{\alpha\gamma(\alpha)}(\iota-a)^{\alpha} \\
&+ \frac{L(\iota-a)^{(\alpha-1)}}{\alpha\Gamma(\alpha)}\int_a^{\iota}\frac{\epsilon}{\alpha\Gamma(\alpha)}(s-a)^{\alpha}\exp(\int_s^{\iota}\frac{L(\sigma-a)^{(\alpha-1)}}{\alpha\Gamma(\alpha)}d\sigma)ds \\
&\leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)}(\iota-a)^{(\alpha-1)} + \frac{\epsilon}{\alpha\Gamma(\alpha)}(\iota-a)^{\alpha} \\
&+ \frac{L(\iota-a)^{(\alpha-1)}}{\alpha\Gamma(\alpha)}\int_a^{\iota}\frac{\epsilon}{\alpha\Gamma(\alpha)}(s-a)^{\alpha}\exp(\int_s^{\iota}\frac{L(\sigma-a)^{(\alpha-1)}}{\alpha\Gamma(\alpha)})d\sigma)ds \text{ Thus,} \\
&\leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)}(\iota-a)^{(\alpha-1)} + \frac{\epsilon}{\alpha\Gamma(\alpha)}(\iota-a)^{\alpha} \\
&+ \frac{L\epsilon(\iota-a)^{(\alpha-1)}}{\alpha^2\Gamma^2(\alpha)}\int_a^{\iota}(s-a)^{\alpha}\exp(K(\iota-a)^{\alpha} - (s-a)^{\alpha})ds
\end{aligned}$$

where $K = \dfrac{L}{\alpha^2\Gamma^2(\alpha)}$.

$$|y_1(\iota) - y_2(\iota)| \leqslant \frac{|\eta-\eta_1|}{\Gamma(\alpha)h^{(\alpha-1)}} + \frac{\epsilon}{\alpha\Gamma(\alpha)h^{\alpha}} + \frac{\epsilon L h^{(\alpha-1)}}{\alpha^2\Gamma^2(\alpha)}\int_a^{\iota}(s-a)^{\alpha}\exp(K(h^{\alpha} - (s-a)^{\alpha}))ds$$

for $h = \iota - a$.

□

Remark 5.2. The solutions $y_1(\iota)$ and $y_2(\iota)$, corresponding to functions f and g respectively, stay close to each other if initial conditions are same or close to each other. The difference $|y_1(\iota) - y_2(\iota)|$ is controlled and bounded by an expression involving ε, α, $\Gamma(\alpha)$, the Lipschitz constant L of ψ, and the time variable ι if initial conditions are same.

Corollary 5.3. *Let ψ be continuous and satisfies the Lipschitz condition on $\mathbb{R}$. If $\mathfrak{x}$ and $\mathfrak{y}$ are two solutions of*

$$(\mathfrak{D}^{\alpha}_{a+}\mathfrak{y}(\iota) = \psi(\iota, \mathfrak{y}(\iota), (\mathfrak{D}^{(\alpha-1)}_{a+}\mathfrak{y}(\iota))(a+) = \eta, 0 < \alpha < 1 \tag{5.4}$$

on the interval $\mathfrak{I}$ containing a, then $\mathfrak{x}(\iota) = \mathfrak{y}(\iota)$ for all ι in the interval $\mathfrak{I}$.

Proof. From theorem 2, we have

$$|\mathfrak{x}(\iota) - \mathfrak{y}(\iota)| \leqslant \frac{\epsilon}{\alpha\Gamma(\alpha)h^{\alpha}} + \frac{\epsilon L h^{(\alpha-1)}}{\alpha^2\Gamma^2(\alpha)} \int_a^{\iota} (s-a)^{\alpha} \exp(K(h^{\alpha} - (s-a)^{\alpha}))ds, \tag{5.5}$$

where $h = \iota - a, K = \frac{L}{\alpha^2\Gamma^2(\alpha)}$. for all $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, taking $\varepsilon \to 0$ in the above inequality, we obtain $\mathfrak{x}(\iota) = \mathfrak{y}(\iota)$ which establishes the uniqueness of the solution of the initial value problem (5.1). □

6. Discussion

The integral inequalities in theorems (4.1) to (4.5) established in this study provide a unified framework that encompasses both classical and weakly singular cases. Notably, when $\delta = 1$, the proposed results reduce to the well-established Gronwall-Bellman and Bihari-type inequalities, which are fundamental tools in the analysis of differential and integral equations. This demonstrates the consistency of present approach with existing literature. However, for $0 < \delta < 1$, the derived inequalities involve weakly singular kernels, thereby introducing a more generalized form of the classical results. These weakly singular inequalities extend several known results and are particularly significant in the context of fractional differential equations, where such singularities naturally occur due to the non-local nature of fractional operators. The generalization presented here contributes to the broader applicability of integral inequalities in analyzing complex systems with memory effects and hereditary properties.

The theorem (5.1) tells us that small errors in model dynamics and initial values lead to controlled errors in the solution, and the bound is explicitly given in terms of all relevant parameters: ϵ, L, α, h, and the Gamma function. So, the fractional-order systems are stable under small perturbations, which is important in applications like control theory, signal processing, and biological modeling where exact models are often approximated.

7. Conclusion

In this paper, we have established generalizations of Gronwall–Bellman-type and Bihari-type integral inequalities. These inequalities extend existing results and provide a valuable framework for analyzing the qualitative behavior of solutions to fractional differential equations. The study focused on fractional Cauchy problems governed by Riemann–Liouville derivatives, which were reformulated into equivalent Volterra-type integral equations of second kind. The derived estimates offer explicit and simplified upper bounds for the unknown solutions. Furthermore, the results have been applied to examine the sensitivity of solutions with respect to initial conditions and nonlinear terms.

References

[1] Almeida R, Malinowska AB and Odzijewicz T (2018). *An extension of the fractional Gronwall inequality*. In Conf. on Non-Integer Order Calculus and Its Applications, Springer, Cham, pp. 20–28.

[2] Ames WF and Pachpatte BG (1997). *Inequalities for Differential and Integral Equations*. Academic Press, Vol. **197**.

[3] Ben Makhlouf A, Boucenna D, Nagy AM and Mchiri L (2022). *Some weakly singular integral inequalities and their applications to tempered fractional differential equations*. J. Math. **2022**: 1682942. https://doi.org/10.1155/2022/1682942

[4] Boulares S, Boucenna D, Rhaima M, Mchiri L and Ben Makhlouf A (2025). *Weighted weakly singular integral inequalities: New explicit bounds and applications in fractional differential equations*. J. Math. **2025**: 8311840. https://doi.org/10.1155/2025/8311840

[5] Cheung WS, Ma QH and Tseng S (2008). *Some new nonlinear weakly singular integral inequalities of Wendroff type with applications*. J. Inequal. Appl. **2008**: 909156. https://doi.org/10.1155/2008/909156

[6] Denton Z and Vatsala AS (2010). *Fractional integral inequalities and applications*. Comput. Math. Appl. **59**: 1087–1094. https://www.sciencedirect.com/science/article/pii/S0898122109003538

[7] Ding XL and Jiang YL (2012). *Semilinear fractional differential equations based on a new integral operator approach*. Commun. Nonlinear Sci. Numer. Simul. **17**: 5143–5150.

[8] Dunkel O (1924). *Integral inequalities with applications to the calculus of variations*. Amer. Math. Monthly **31**: 326–337. https://www.jstor.org/stable/2299386

[9] Henry D (1981). *Geometric Theory of Semilinear Parabolic Equations*. Springer, Berlin.

[10] Kilbas AA, Srivastava HM and Trujillo JJ (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier.

[11] Li Y and Li Z (2026). *Weakly singular Wendroff-type integral inequalities of multiple variables with multiple nonlinear terms and their applications*. Mathematics **14**: 944. https://doi.org/10.3390/math14060944

[12] Ma QH and Pečarić J (2008). *Estimates on solutions of some new nonlinear retarded Volterra–Fredholm type integral inequalities*. Nonlinear Anal. **69**: 393–407.

[13] Medved M (1997). *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*. J. Math. Anal. Appl. **214**: 349–366.

[14] Podlubny I (1999). *Fractional Differential Equations*. Academic Press, New York.

[15] Qiong W (2017). *A new type of the Gronwall–Bellman inequality and its application to fractional stochastic differential equations*. Cogent Math. **4**: 1279781. https://doi.org/10.1080/23311835.2017.1279781

[16] Wang H and Zheng KL (2010). *Some nonlinear weakly singular integral inequalities with two variables and applications*. J. Inequal. Appl. **2010**: 345701. https://doi.org/10.1155/2010/345701

[17] Xu R and Meng F (2016). *Some new weakly singular integral inequalities and their applications to fractional differential equations*. J. Inequal. Appl. **2016**: 78. https://doi.org/10.1186/s13660-016-1015-2

[18] Xu R (2017). *Some new nonlinear weakly singular integral inequalities and their applications*. J. Math. Inequal. **11**: 1007–1018. https://files.ele-math.com/articles/jmi-11-76.pdf

[19] Yan Y (2015). *On some new weakly singular Volterra integral inequalities with maxima and their applications*. J. Inequal. Appl. **2015**: 369. https://doi.org/10.1186/s13660-015-0892-0

[20] Ye H, Gao J and Ding Y (2007). *A generalized Gronwall inequality and its application to a fractional differential equation*. J. Math. Anal. Appl. **328**: 1075–1081.

[21] Zhu T (2018). *New Henry–Gronwall integral inequalities and their applications to fractional differential equations*. Bull. Braz. Math. Soc. **49**: 647–657.