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Novel Exact Solutions of a Higher-Dimensional Complex mKdV System with Conformable Derivative Using the Generalized Expansion Method

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Abstract

In this paper, we investigate the (2+1)-dimensional complex modified Korteweg-de Vries (CmKdV) system using the conformable derivative. The CmKdV system is a beneficial model in the field of nonlinear wave theory such as fluid flow, optical communication, and plasma physics. Explicit solutions are constructed, including periodic, solitary, and shock waves form using the Jacobi elliptic function expansion method. The solutions obtained are visually presented in various dimensions using Mathematica, providing a clear physical understanding of the effects of the conformable fractional derivative. This research enhances understanding of soliton behavior in complex nonlinear systems and demonstrates the effectiveness of combining conformable derivatives with analytical methods, while also providing new insights into the dynamics and diverse forms of propagating fluid waves.

Keywords: Complex mKdV system, Conformable derivative, Exact solutions, Nonlinear wave dynamics.

1. Introduction

Partial differential equations which are nonlinear in nature are one of the major points because they are very useful in modeling many phenomena especially in science and engineering. For example, they been used in fluid dynamics, plasma physics, etc. They have also been used in optic communication and nonlinear optics [1, 2]. The solutions of these equations, notably soliton solutions, provide invaluable information on the physical processes often beyond the reach of numerical methods [3, 4]. The exact solutions provide the analytical framework for a clear understanding of critical wave phenomena like solitons formation, stability and interaction. Over the year, a lot of research was done on

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some of the important nonlinear PDEs like Korteweg-de Vries, modified KdV (mKdV), the Improved modified KdV (ImKdV) equations etc. The models are important for comprehending multi-dimensional wave impact and complex nonlinear interactions [5, 6, 7]. In recent times, fractional calculus has caught the attention of researchers and used to extend models to better describe real-world systems exhibiting memory effects and anomalous diffusion. The enhanced nonlinear models with generalized derivatives, including the fractional and conformable derivatives, have resulted in better description of complex systems [8, 9, 10, 11]. The conformable derivative in particular has gained popularity due to its simplicity and being consistent with classical calculus. Its another benefit is to study nonlinear wave phenomena across a diverse range of applications. Research using analytical techniques such as the Tanh method [12], the modified Sardar sub-equation method [13, 14] as well as the generalized Riccati equation mapping method [15, 16] have been successful in finding exact solutions of these complex nonlinear PDEs and producing soliton, periodic and rational solutions, which can help comprehend the physical mechanism involved. The (2+1)-dimensional CmKdV system finds diverse applications across various scientific domains, including nonlinear optics, plasma physics, and a fluid dynamic. The mathematical formulation of this system in terms of the conformable derivative is expressed as:

$$D_t^{\alpha}h(x,y,t) + D_x^{\beta}\left(D_x^{\beta}\left(D_y^{\gamma}(h(x,y,t))\right)\right) + ih(x,y,t)g(x,y,t) + D_x^{\beta}\left(h(x,y,t)k(x,y,t)\right) = 0,$$

$$D_x^{\beta}g(x,y,t) + 2i\sigma\Big(h^*(x,y,t)\,D_x^{\beta}(D_y^{\gamma}(h(x,y,t))) - D_x^{\beta}\left(D_y^{\gamma}(h^*(x,y,t)\,h(x,y,t))\right)\Big) = 0,$$

$$D_x^{\beta}k(x,y,t) - 2\sigma D_y^{\gamma}\left(|h(x,y,t)|^2\right) = 0. \tag{1.1}$$

The function $h(x,y,t)$ is a complex function with its conjugate as $h^*(x,y,t)$, and $g(x,y,t), k(x,y,t)$ are real functions, $\sigma = \pm 1$, and the partial derivatives for x, y, and t are denoted by subscripts. The given model has been solved by various techniques to find its exact solutions, which has diverse applications in numerous fields, including fluid dynamics plasma physics, fluid optics, and optical communications [17, 18, 19, 20, 21, 22, 23, 24]. The CmKdV equation is known to support localized stable nonlinear waves that do not change shape or speed while traveling. These solitons are critical to the explanation of many phenomena. This paper aims to find new soliton solutions for CmKdV system in (2+1) dimensions based on the conformable derivative, using the Jacobi elliptic function expansion method. The Jacobi elliptic function expansion method is highly effective tool for obtaining wave solutions to nonlinear partial differential equations (PDEs) due to their remarkable efficiency. These functions not only provide solutions for solitary and shock waves, but also offer valuable insights into localized wave behaviors and periodic oscillations. Compared to more complex methods, such as the Hirota and Darboux transformations, the Jacobi Elliptic Function Expansion Method (JEFEM) stands out for its simplicity, flexibility, and efficiency. It allows for the derivation of multiple exact solutions without the need for intricate mathematical techniques, making it a powerful and accessible approach for solving nonlinear PDEs.

1.1. Definition and Properties of the Conformable Derivative

Definition 1.1. [25] Let $A : (0,\infty] \to \mathbb{R}$ be a function. The α-th order conformable derivative of A is defined by

$$D^{\alpha}(A)(t) = \lim_{\varepsilon \to 0} \frac{A\left(t + \varepsilon t^{1-\alpha}\right) - A(t)}{\varepsilon}, \quad \varepsilon > 0,\ \alpha \in (0,1).$$

The α-differentiability of A at t is characterized by the existence of the limit

$$A^{(\alpha)}(0) = \lim_{t \to 0^+} A^{(\alpha)}(t),$$

Theorem 1.2. *[25] Let $\alpha \in (0,1]$ and suppose h, g is α-differential at point $t > 0$. Then, the following properties are satisfied.*

1. The linear combination of the conformable derivative is given by

$$D^{\alpha}(\delta h + \gamma g) = \delta D^{\alpha}(h) + \gamma D^{\alpha}(g), \quad \forall \delta, \gamma \in \mathbb{R}.$$

2. The conformable derivative of the power function is given by

$$D^{\alpha}(t^{q}) = q t^{q-\alpha}, \quad \forall q \in \mathbb{R}.$$

3. The conformable derivative of a constant function equals zero:

$$D^{\alpha}(\chi) = 0, \quad \textit{for all constants } h(t) = \chi.$$

4. The product rule applies to the conformable derivative for the product of two functions:

$$D^{\alpha}(fg) = f D^{\alpha}(g) + g D^{\alpha}(f).$$

5. The conformable derivative of a fraction involving two functions is expressed by the following equation:

$$D^{\alpha}\left(\frac{h}{g}\right) = \frac{g D^{\alpha}(h) - h D^{\alpha}(g)}{g^2}.$$

6. Moreover, if h is differentiable, then the conformal derivative is equal to the classical derivative:

$$D^{\alpha}(h)(t) = t^{1-\alpha}\frac{dh}{dt}.$$

This paper is structured as follows. In Section 1, we introduce our research on the topic and the importance. Section 2 details the methodology employed in this work, that is the Jacobi elliptic function expansion method. Section 3 shows how to use this technique on the CmKdV system with conformable derivative. In Section 4, we address some of the graphical representations of the solutions. In Section 5, we address the conclusion of the study.

2. Methodology

In this section, we discuss the steps of the methodology. The following study will be carried out while carrying out this research.

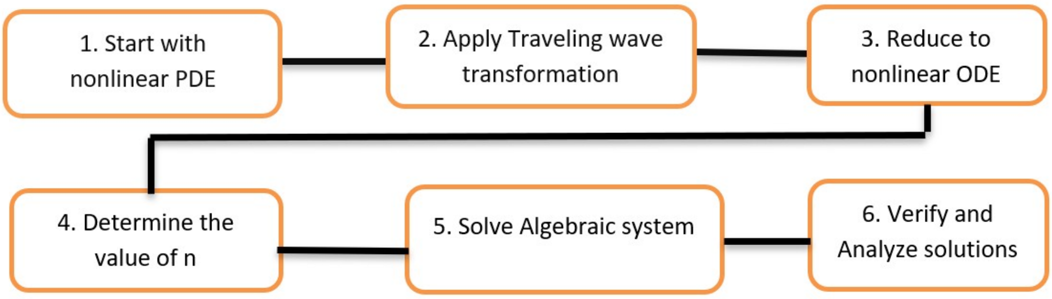


Figure 1: General Workflow of the Methodology.

Step 1: Let us take the given non-linear PDE say, in two variables.

$$F\Big(h, \frac{\partial^\alpha h}{\partial t^\alpha}, \frac{\partial^\alpha h}{\partial x^\alpha}, \frac{\partial^{2\alpha} h}{\partial t^{2\alpha}}, \frac{\partial^{2\gamma} h}{\partial x^{2\gamma}}, \dots\Big) = 0, \tag{2.1}$$

Step 2: Transforming Eq. by using the traveling wave transformation as

$$h = h(\xi), \quad \xi = \frac{wt^\alpha}{\alpha} + \frac{cx^\gamma}{\gamma}, \tag{2.2}$$

The Eq. (2.2) reduces the nonlinear partial differential equation to the ordinary differential equation of the following form:

$$F(h', h'', h''', \dots) = 0, \tag{2.3}$$

Apart from this prolonged and indirect approach, the main aim—besides increasing the probability of discovering solutions—is to employ an auxiliary ordinary differential equation (the first type of three-parameter Jacobi equation) in order to generate a substantial number of Jacobi elliptic solutions of the equation. The auxiliary equation is visualizable:

$$(F')^2(\xi) = m_2 F^4(\xi) + m_1 F^2(\xi) + m_0, \tag{2.4}$$

where $F' = \frac{\partial F}{\partial \xi}$, $\xi = (x, t)$, and m_2, m_1 and m_0 are real constants. The Eq. (2.4) has solutions in Table A, where $i^2 = -1$, the Jacobian elliptic functions $\text{sn}\xi = \text{sn}(\xi, \epsilon)$, $\text{cn}\xi = \text{cn}(\xi, \epsilon)$, and $\text{dn}\xi = \text{dn}(\xi, \epsilon)$, here ϵ $(0 < \epsilon < 1)$ is the modulus. Equation (2.4) can generate a variety of solutions for the function, depending on the specific choices of the parameters m_2, m_1, and m_0 as presented in Tables A and B [26, 27].

Table A: Possible solutions of $F(\xi)$ in Eq. (2.4) for the selected values m_2, m_1 and m_0.

No.	$\mathbf{m_2}$	$\mathbf{m_1}$	$\mathbf{m_0}$	**F**
1	ϵ^2	$-(1+\epsilon^2)$	1	sn, cd
2	$-\epsilon^2$	$2\epsilon^2 - 1$	$1 - \epsilon^2$	cn
3	-1	$2 - \epsilon^2$	$\epsilon^2 - 1$	dn

No.	m_2	m_1	m_0	F
4	1	$-(1-\epsilon^2)$	ϵ^2	ns, dc
5	$1-\epsilon^2$	$2\epsilon^2-1$	$-\epsilon^2$	nc
6	ϵ^2-1	$2-\epsilon^2$	-1	nd
7	$1-\epsilon^2$	$2-\epsilon^2$	1	sc
8	$-\epsilon^2(1-\epsilon^2)$	$2\epsilon^2-1$	1	sd
9	1	$2-\epsilon^2$	$1-\epsilon^2$	cs
10	1	$2\epsilon^2-1$	$\epsilon^2(1-\epsilon^2)$	-ds
11	$-1/4$	$(\epsilon^2+1)/2$	$-(1-\epsilon^2)^2/4$	ϵcn$\mp$dn
12	$1/4$	$(-2\epsilon^2+1)/2$	$1/4$	nc$\mp$cs
13	$(1-\epsilon^2)/4$	$(\epsilon+1)/2$	$(1-\epsilon^2)/4$	nc$\mp$sc
14	$1/4$	$(\epsilon-2)/2$	$\epsilon^4/4$	ns$\mp$ds
15	$\epsilon^2/4$	$(\epsilon^2-2)/2$	$\epsilon^4/4$	sn$\mp$icn, sn$/\sqrt{1-\epsilon \text{sn} \mp \text{cn}}$
16	$1/4$	$(1-2\epsilon^2)/2$	$1/4$	ϵcn$\mp$idn, sn/(1$\mp$cn)
17	$\epsilon^2/4$	$(\epsilon^2-2)/2$	$1/4$	sn/(1$\mp$dn)
18	$(\epsilon^2-1)/4$	$(\epsilon^2+1)/2$	$(\epsilon^2-1)/4$	dn/(1$\mp\epsilon$sn)
19	$(1-\epsilon^2)/4$	$(\epsilon^2+1)/2$	$(-\epsilon^2+1)/4$	cn/(1$\mp$sn)
20	$(1-\epsilon^2)^2/4$	$(\epsilon^2+1)/2$	$1/4$	sn/(dn$\mp$cn)
21	$\epsilon^4/4$	$(\epsilon^2-2)/2$	$1/4$	cn$/\sqrt{1-\epsilon^2 \mp \text{dn}}$

Table B: Limiting forms of Jacobi elliptic functions for $\epsilon \to 1$ and $\epsilon \to 0$.

No.	**F**	$\epsilon \to 1$	$\epsilon \to 0$	**No.**	**F**	$\epsilon \to 1$	$\epsilon \to 0$
1	snh	tanh	sinh	7	dch	1	sech
2	cnh	sech	cosh	8	nch	cosh	sech
3	dnh	sech	1	9	sch	sinh	tanh
4	cdh	cosh	1	10	nsh	coth	csch
5	sdh	sinh	sinh	11	dsh	csch	coth
6	ndh	cosh	1	12	csh	csch	coth

We build many solutions for the problem, such as periodic, hyperbolic, trigonometric solutions, and so on. The Jacobi elliptic function expansion method can be used to present $u(\xi)$ as a finite series of Jacobi elliptic functions.

$$u(\xi) = \sum_{i=0}^{n} r_i F^i(\xi), \tag{2.5}$$

where $F(\xi)$ is the solution of the nonlinear ordinary Eq. (2.4) and n, r_i $(i = 0, 1, 2, \ldots, n)$ are constants to be determined later. The integer n can be determined by balancing the highest-order linear term:

$$O\left(\frac{\partial^p u}{\partial \xi^p}\right) = n + p, \quad p = 1, 2, 3, \ldots \tag{2.6}$$

And the order of the highest nonlinear term is

$$O\left(u^q \frac{\partial^p u}{\partial \xi^p}\right) = (q+1)n + p, \quad q = 0,1,2,3,\dots, \; p = 1,2,3,\dots \tag{2.7}$$

In Eq. (2.3). Substituting Eq. (2.5) and setting all the coefficients of powers of F to zero, we obtain a system of nonlinear algebraic equations for r_i $(i = 0,1,2,\dots)$. By solving this system using all the values for m_2, m_1, m_0 in Eq. (2.4) from Table A, and then combining the results with Eq. (2.5) and the auxiliary equation we choose, we can derive exact solutions for Eq. (1.1).

3. Application of the Jacobi elliptic function expansion method

For employing the Jacobi elliptic function expansion method, we reduce Eq. (1.1) to an ODE by using the transformation

$$h(x,y,t) = N(x,y,t)\,\exp\left(i\left(\frac{a_1 x^\beta}{\beta} + \frac{a_2 y^\gamma}{\gamma} + \frac{a_3 t^\alpha}{\alpha}\right)\right). \tag{3.1}$$

Here a_1, a_2, a_3 are real constants. Then Eq. (1.1) reduces to

$$\begin{aligned} &D_t^\alpha N - 2a_1 a_2\, D_x^\beta N - a_1^2 D_y^\gamma N + D_x^\beta D_x^\beta D_y^\gamma N + D_x^\beta N\, k + N D_x^\beta k \\ &\quad + i\Big((a_3 - a_1^2 a_2)N + 2D_x^\beta D_y^\gamma N + a_2 D_x^\beta D_x^\beta N + a_1 N k + N g\Big) = 0, \end{aligned} \tag{3.2}$$

$$D_x^\beta g - 4\sigma\big(a_2 N D_x^\beta N + a_1 N D_y^\gamma N\big) = 0, \tag{3.3}$$

$$D_x^\beta k - 2\sigma D_y^\gamma (N^2) = 0. \tag{3.4}$$

Next, we use the wave transformation

$$\begin{aligned} &\xi = \frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c t^\alpha}{\alpha}, \\ &h(x,y,t) = N(\xi), \quad g(x,y,t) = g(\xi), \quad k(x,y,t) = k(\xi). \end{aligned}$$

Substituting this into Eqs. (3.2)–(3.4), we obtain

$$\begin{aligned} &(c - 2a_1 a_2 - a_1^2)N' + N''' + N'k + Nk' \\ &\quad + i\Big((a_3 - a_1^2 a_2)N + (2a_1 + a_2)N'' + a_1 N k + N g\Big) = 0, \end{aligned} \tag{3.5}$$

$$g' - 4\sigma(a_2 + a_1)NN' = 0, \tag{3.6}$$

$$k' - 2\sigma(N^2)' = 0. \tag{3.7}$$

Integrating Eqs. (3.6) and (3.7) once with respect to ξ and taking the integration constants to be zero, we obtain

$$g = 2\sigma(a_2 + a_1)N^2, \qquad k = 2\sigma N^2. \tag{3.8}$$

Substituting Eq. (3.8) into Eq. (3.5), we obtain the following ODE

$$\begin{aligned} &(c - 2a_1 a_2 - a_1^2)N' + N''' + 2\sigma(N^3)' \\ &\quad + i\Big((a_3 - a_1^2 a_2)N + (2a_1 + a_2)N'' + 2\sigma(2a_1 + a_2)N^3\Big) = 0. \end{aligned} \tag{3.9}$$

The prime denotes differentiation with respect to ξ. Separating the real and imaginary parts of Eq. (3.9), we have

$$(c - 2a_1a_2 - a_1^2)N' + N''' + 2\sigma(N^3)' = 0, \tag{3.10}$$

$$N'' + \frac{a_3 - a_1^2 a_2}{2a_1 + a_2} N + 2\sigma N^3 = 0. \tag{3.11}$$

Taking the antiderivative of Eq. (3.10) once with respect to ξ and setting the constant of integration to zero, we have

$$(c - 2a_1a_2 - a_1^2)N + N'' + 2\sigma N^3 = 0. \tag{3.12}$$

Eqs. (3.11) and (3.12) are the same if and only if the constraint condition holds:

$$c - 2a_1a_2 - a_1^2 = \frac{a_3 - a_1^2 a_2}{2a_1 + a_2}. \tag{3.13}$$

Solving for c, we obtain

$$c = 2a_1a_2 + a_1^2 + \frac{a_3 - a_1^2 a_2}{2a_1 + a_2}. \tag{3.14}$$

Thus Eq. (3.11) can be rewritten as

$$N'' + \frac{a_3 - a_1^2 a_2}{2a_1 + a_2} N + 2\sigma N^3 = 0. \tag{3.15}$$

The Jacobi elliptic function expansion method is then used to solve the transformed ODE (3.15).

3.1. Solutions by Jacobi elliptic function expansion method

Based on the technique, the solutions can be found using the series

$$N(\xi) = \sum_{i=0}^{n} F^i(\xi). \tag{3.16}$$

By applying the balancing procedure, balancing the highest-order nonlinear and the highest-order derivative term in Eq. (3.15), we obtain $n = 1$. Hence, Eq. (3.16) reduces to

$$N(\xi) = r_0 + r_1 F(\xi). \tag{3.17}$$

Here $F(\xi)$ is the solution defined in Eq. (5). Therefore,

$$N'(\xi) = r_1 F'(\xi). \tag{3.18}$$

Substituting Eq. (2.4) into Eq. (3.18), we get

$$N'(\xi) = r_1\sqrt{m_2F^4(\xi) + m_1F^2(\xi) + m_0}\,. \tag{3.19}$$

Also,

$$N''(\xi) = r_1F(\xi)\left(2m_2F^2(\xi) + m_1\right). \tag{3.20}$$

Now, substituting Eqs. (3.17) and (3.20) into Eq. (3.15), we obtain a set of algebraic equations. Solving the system with the help of Maple, we determine the coefficients involved in the series (3.17) as

$$r_0 = 0, \qquad r_1 = \pm\sqrt{-\tfrac{m_2}{\sigma}}, \qquad a_3 = a_1^2 a_2 - 2m_1 a_1 - m_1 a_2. \tag{3.21}$$

Substituting these values into Eq. (3.17), we construct the following exact solutions of the CmKdV system.

3.1.1. The Jacobi elliptic function solutions

Using the data from Table A, combining with Eq. (3.17), we derived Jacobi elliptic function solutions as shown below:

$$h_1(x,y,t) = \pm\sqrt{-\frac{\epsilon^2}{\sigma}}\ \mathrm{sn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta} + a_2\frac{y^\gamma}{\gamma} + a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_1(x,y,t) = 2\sigma(a_2 + a_1)\left[\pm\sqrt{-\frac{\epsilon^2}{\sigma}}\ \mathrm{sn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_1(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{\epsilon^2}{\sigma}}\ \mathrm{sn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_2(x,y,t) = \pm\sqrt{\frac{\epsilon^2}{\sigma}}\ \mathrm{cn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta} + a_2\frac{y^\gamma}{\gamma} + a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_2(x,y,t) = 2\sigma(a_2 + a_1)\left[\pm\sqrt{\frac{\epsilon^2}{\sigma}}\ \mathrm{cn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_2(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{\epsilon^2}{\sigma}}\ \mathrm{cn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_3(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\ \mathrm{dn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta} + a_2\frac{y^\gamma}{\gamma} + a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_3(x,y,t) = 2\sigma(a_2 + a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{dn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_3(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{dn}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_4(x,y,t) = \pm\sqrt{-\frac{1}{\sigma}}\ \mathrm{ns}\left(\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta} + a_2\frac{y^\gamma}{\gamma} + a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_4(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{1}{\sigma}}\ \mathrm{ns}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_4(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{1}{\sigma}}\ \mathrm{ns}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_5(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\ \mathrm{dc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_5(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{dc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_5(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{dc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_6(x,y,t) = \pm\sqrt{\frac{1-\epsilon^2}{\sigma}}\ \mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_6(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1-\epsilon^2}{\sigma}}\ \mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_6(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1-\epsilon^2}{\sigma}}\ \mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_7(x,y,t) = \pm\sqrt{-\frac{\epsilon^2-1}{\sigma}}\ \mathrm{nd}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_7(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{\epsilon^2-1}{\sigma}}\ \mathrm{nd}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_7(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{\epsilon^2-1}{\sigma}}\ \mathrm{nd}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_8(x,y,t) = \pm\sqrt{\frac{1-\epsilon^2}{\sigma}}\ \mathrm{sc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_8(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1-\epsilon^2}{\sigma}}\ \mathrm{sc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_8(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1-\epsilon^2}{\sigma}}\ \mathrm{sc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_9(x,y,t) = \pm\sqrt{-\frac{\epsilon^2(1-\epsilon^2)}{\sigma}}\ \mathrm{sd}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_9(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{\epsilon^2(1-\epsilon^2)}{\sigma}}\ \mathrm{sd}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_9(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{\epsilon^2(1-\epsilon^2)}{\sigma}}\ \mathrm{sd}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_{10}(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\ \mathrm{cs}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{10}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{cs}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_{10}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{cs}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_{11}(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\ \mathrm{ds}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{11}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{ds}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$k_{11}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\ \mathrm{ds}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right]^2,$$

$$h_{12}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\left(\epsilon\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\mp\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{12}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\epsilon\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\mp\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{12}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\epsilon\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\mp\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{c\,t^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{13}(x,y,t) = \pm\sqrt{-\frac{1}{4\sigma}}\left(\mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{cs}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{13}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{1}{4\sigma}}\left(\mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{cs}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{13}(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{1}{4\sigma}}\left(\mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{cs}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{14}(x,y,t) = \pm\sqrt{\frac{1-\epsilon^2}{4\sigma}}\left(\mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{sc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{14}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1-\epsilon^2}{4\sigma}}\left(\mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{sc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{14}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1-\epsilon^2}{4\sigma}}\left(\mathrm{nc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{sc}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{15}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\left(\mathrm{ns}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{ds}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{15}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\mathrm{ns}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{ds}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{15}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\mathrm{ns}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{ds}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{16}(x,y,t) = \pm\sqrt{\frac{\epsilon^2}{4\sigma}}\left(\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\ \exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{16}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{\epsilon^2}{4\sigma}}\left(\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{16}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{\epsilon^2}{4\sigma}}\left(\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{17}(x,y,t) = \pm\sqrt{\frac{\epsilon^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{1-\epsilon\,\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\exp\left(i(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha})\right),$$

$$g_{17}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{\epsilon^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{1-\epsilon\,\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\right]^2,$$

$$k_{17}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{\epsilon^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{1-\epsilon\,\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\right]^2,$$

$$h_{18}(x,y,t) = \pm\sqrt{-\frac{1}{4\sigma}}\left(\epsilon\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\,\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\exp\left(i(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha})\right),$$

$$g_{18}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{1}{4\sigma}}\left(\epsilon\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\,\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{18}(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{1}{4\sigma}}\left(\epsilon\,\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\,\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{19}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\exp\left(i(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha})\right),$$

$$g_{19}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$k_{19}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$h_{20}(x,y,t) = \pm\sqrt{\frac{\epsilon^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp \mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{20}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{\epsilon^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp \mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$k_{20}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{\epsilon^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp \mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$h_{21}(x,y,t) = \pm\sqrt{\frac{1-\epsilon^2}{4\sigma}}\frac{\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp \mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{21}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1-\epsilon^2}{4\sigma}}\frac{\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp \mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$k_{21}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1-\epsilon^2}{4\sigma}}\frac{\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp \mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$h_{22}(x,y,t) = \pm\sqrt{-\frac{(1-\epsilon^2)^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp \mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{22}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{(1-\epsilon^2)^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp \mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$k_{22}(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{(1-\epsilon^2)^2}{4\sigma}}\frac{\mathrm{sn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp \mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$h_{23}(x,y,t) = \pm\sqrt{\frac{\epsilon^4}{4\sigma}}\frac{\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{1-\epsilon^2\mp \mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{23}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{\epsilon^4}{4\sigma}}\frac{\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{1-\epsilon^2\mp\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\right]^2,$$

$$k_{23}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{\epsilon^4}{4\sigma}}\frac{\mathrm{cn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{1-\epsilon^2\mp\mathrm{dn}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\right]^2.$$

3.1.2. The Solitary Wave Solutions

Using Table B, for $\epsilon \to 1$, the solutions represent the solitary wave solutions.

$$h_{24}(x,y,t) = \pm\sqrt{-\frac{1}{\sigma}}\,\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{24}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{1}{\sigma}}\,\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$k_{24}(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{1}{\sigma}}\,\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$h_{25}(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\,\cosh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{25}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\,\cosh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$k_{25}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\,\cosh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$h_{26}(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\,\mathrm{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{26}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\,\mathrm{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$k_{26}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\,\mathrm{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$h_{27}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\left(\mathrm{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\mathrm{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{27}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{27}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{28}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\left(\coth\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{csch}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{28}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\coth\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{csch}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{28}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\coth\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{csch}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{29}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\left(\coth\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{csch}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{29}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\coth\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{csch}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{29}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\coth\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp\operatorname{csch}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{30}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\left(\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{30}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$k_{30}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\mp i\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right)\right]^2,$$

$$h_{31}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\frac{\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{31}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\frac{\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$k_{31}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\frac{\tanh\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{1\mp\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}\right]^2,$$

$$h_{32}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\frac{\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\pm\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{32}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\frac{\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\pm\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\right]^2,$$

$$k_{32}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\frac{\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\pm\operatorname{sech}\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)}}\right]^2.$$

3.1.3. The Shock Wave Solutions

Using Table B, for $\epsilon \to 0$, the solutions represent the shock wave solutions.

$$h_{33}(x,y,t) = \pm\sqrt{-\frac{1}{\sigma}}\csc\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{33}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{1}{\sigma}}\csc\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$k_{33}(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{1}{\sigma}}\csc\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$h_{34}(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\sec\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\exp\left(i\left(a_1\frac{x^\beta}{\beta}+a_2\frac{y^\gamma}{\gamma}+a_3\frac{t^\alpha}{\alpha}\right)\right),$$

$$g_{34}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\sec\left(\frac{x^\beta}{\beta}+\frac{y^\gamma}{\gamma}+\frac{ct^\alpha}{\alpha}\right)\right]^2,$$

$$k_{34}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\ \sec\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right]^{2},$$

$$h_{35}(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\ \sec\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\exp\Big(i\big(a_1\frac{x^{\beta}}{\beta}+a_2\frac{y^{\gamma}}{\gamma}+a_3\frac{t^{\alpha}}{\alpha}\big)\Big),$$

$$g_{35}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\ \sec\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right]^{2},$$

$$k_{35}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\ \sec\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big),\right]^{2},$$

$$h_{36}(x,y,t) = \pm\sqrt{\frac{1}{\sigma}}\ \tan\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\exp\Big(i\big(a_1\frac{x^{\beta}}{\beta}+a_2\frac{y^{\gamma}}{\gamma}+a_3\frac{t^{\alpha}}{\alpha}\big)\Big),$$

$$g_{36}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{\sigma}}\ \tan\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right]^{2},$$

$$k_{36}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{\sigma}}\ \tan\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right]^{2},$$

$$h_{37}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\ \left(\csc\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\mp\cot\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right)\exp\Big(i\big(a_1\frac{x^{\beta}}{\beta}+a_2\frac{y^{\gamma}}{\gamma}+a_3\frac{t^{\alpha}}{\alpha}\big)\Big),$$

$$g_{37}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\ \left(\csc\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\mp\cot\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right)\right]^{2},$$

$$k_{37}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\ \left(\csc\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\mp\cot\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right)\right]^{2},$$

$$h_{38}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\ \left(\sec\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\mp\tan\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right)\exp\Big(i\big(a_1\frac{x^{\beta}}{\beta}+a_2\frac{y^{\gamma}}{\gamma}+a_3\frac{t^{\alpha}}{\alpha}\big)\Big),$$

$$g_{38}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\ \left(\sec\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\mp\tan\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right)\right]^{2},$$

$$k_{38}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\left(\sec\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\mp\tan\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)\right)\right]^2,$$

$$h_{39}(x,y,t) = \pm\sqrt{\frac{1}{4\sigma}}\,\frac{\cos\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}{1\mp\sin\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}\exp\Big(i\big(a_1\frac{x^{\beta}}{\beta}+a_2\frac{y^{\gamma}}{\gamma}+a_3\frac{t^{\alpha}}{\alpha}\big)\Big),$$

$$g_{39}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{\frac{1}{4\sigma}}\,\frac{\cos\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}{1\mp\sin\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}\right]^2,$$

$$k_{39}(x,y,t) = 2\sigma\left[\pm\sqrt{\frac{1}{4\sigma}}\,\frac{\cos\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}{1\mp\sin\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}\right]^2,$$

$$h_{40}(x,y,t) = \pm\sqrt{-\frac{1}{4\sigma}}\,\frac{\sin\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}{1\mp\cos\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}\exp\Big(i\big(a_1\frac{x^{\beta}}{\beta}+a_2\frac{y^{\gamma}}{\gamma}+a_3\frac{t^{\alpha}}{\alpha}\big)\Big),$$

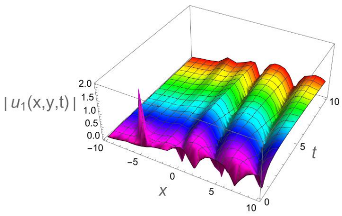
$$g_{40}(x,y,t) = 2\sigma(a_2+a_1)\left[\pm\sqrt{-\frac{1}{4\sigma}}\,\frac{\sin\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}{1\mp\cos\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}\right]^2,$$

$$k_{40}(x,y,t) = 2\sigma\left[\pm\sqrt{-\frac{1}{4\sigma}}\,\frac{\sin\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}{1\mp\cos\Big(\frac{x^{\beta}}{\beta}+\frac{y^{\gamma}}{\gamma}+\frac{ct^{\alpha}}{\alpha}\Big)}\right]^2.$$

4. Graphical Representation

This study presents new exact solutions of the (2+1)-dimensional Complex modified Korteweg–de Vries (CmKdV) system based on the conformable derivative using the Jacobi elliptic function expansion method. The obtained results include Jacobi elliptic functions, solitary wave solutions, and shock wave solutions. Numerical simulations in 3D, 2D, contour plots, and the influence of the parameter α on wave propagation are provided to highlight the physical implications for three different values of α via Mathematica. These simulations reveal localized soliton profiles, oscillatory patterns, and singular structures. The 2D plots provide detailed spatial slices, the 3D surfaces illustrate the evolution of amplitudes in space and time, and the contour maps highlight areas of high concentration or oscillation. Figure 1 shows the graph of the solution $|h_1(x,y,t)|$ in 3D, 2D, and contour form together with the influence of α, using the parameters $\sigma=1, c=1, m=0.8, \gamma=0.9, \beta=0.9, a_1=1, a_2=0.2, a_3=1$. Figure 2 shows the graph of the solution $|h_4(x,y,t)|$ with 3D, 2D, and contour plots under varying α, using the parameters $\sigma=1, y=0.1, c=0.1, m=0.5, \gamma=0.9, \beta=0.9, a_1=1, a_2=0.2, a_3=1$.

Figure 3 shows the graph of the solution $|h_{11}(x, y, t)|$ in 3D, 2D, and contour form for varying α, using the parameters $\sigma = 1, y = 0.1, c = 2.5, m = 0.8, \gamma = 0.9, \beta = 0.9, a_1 = 1, a_2 = 0.2, a_3 = 1$. Figure 4 shows the graph of the solution $|h_{11}(x, y, t)|$ for varying α in 3D, 2D, and contour plots, using the parameters $\sigma = 1, y = 0.1, c = 2.5, m = 0.8, \gamma = 0.9, \beta = 0.9, a_1 = 1, a_2 = 0.2, a_3 = 1$. Figure 5 shows the graph of the solution $|h_{24}(x, y, t)|$ for varying α, with 3D, 2D, and contour plots constructed using the parameters $\sigma = 1, y = 0, c = 0.1, m = 0.5, \gamma = 0.9, \beta = 0.9, a_1 = 1, a_2 = 0.2, a_3 = 1$. Finally, Figure 6 shows the graph of the solution $|h_{25}(x, y, t)|$ in 3D, 2D, and contour form for varying α, using the parameters $\sigma = 1, y = 0, c = 0.6, m = 0.8, \gamma = 0.9, \beta = 0.9, a_1 = 1, a_2 = 0.2, a_3 = 1$. This representation makes it straightforward to categorize the behavior of the waves and demonstrates that computational analysis is an effective tool for examining complex nonlinear dynamics.



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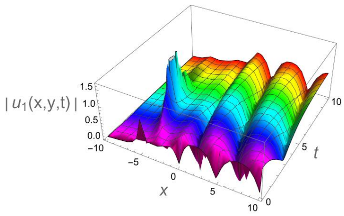

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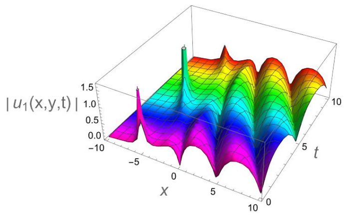

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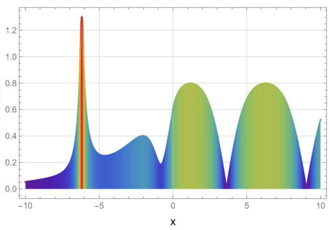

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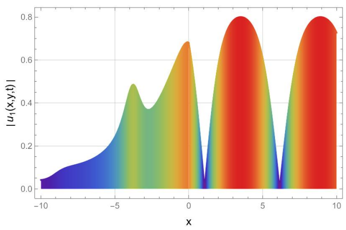

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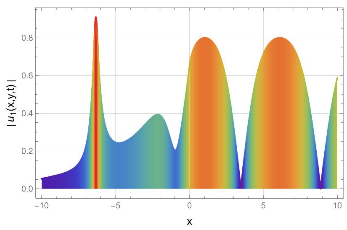

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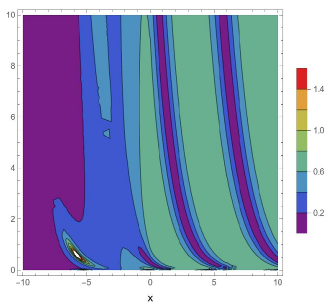

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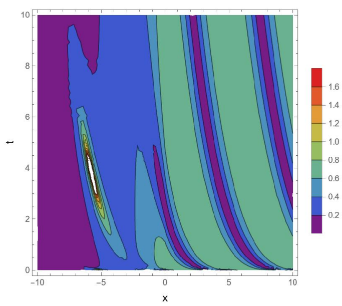

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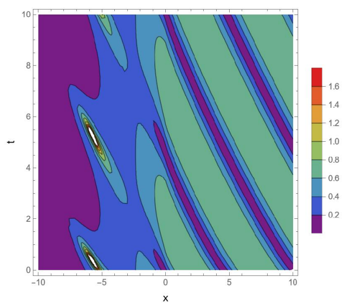

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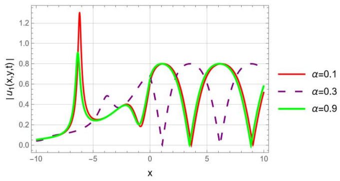
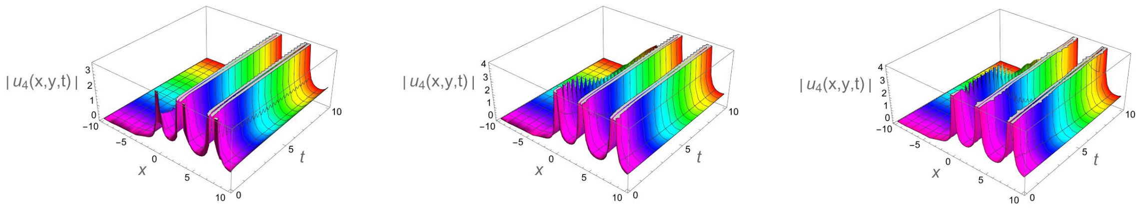
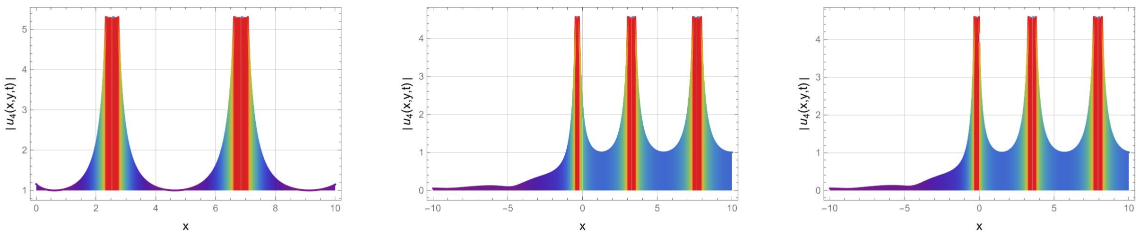


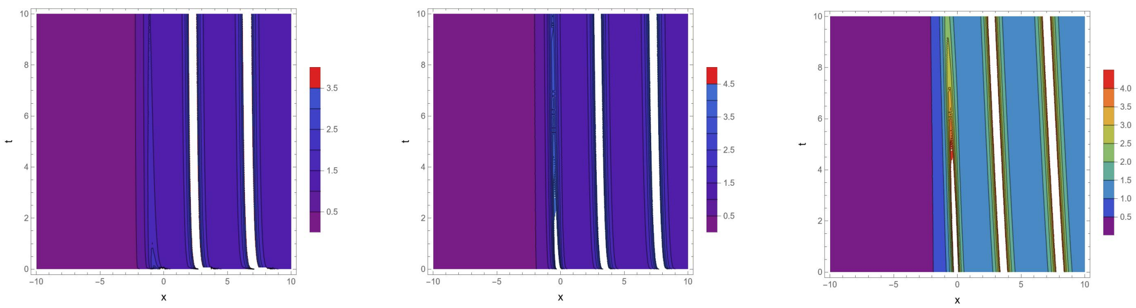
Figure 2: The graphs showing 3D, 2D, and contour representations, as well as the influence of α on wave propagation, for the solution $|h_1(x, y, t)|$.



$\alpha = 0.1$ $\alpha = 0.3$ $\alpha = 0.9$



$\alpha = 0.1$ $\alpha = 0.3$ $\alpha = 0.9$



$\alpha = 0.1$ $\alpha = 0.3$ $\alpha = 0.9$

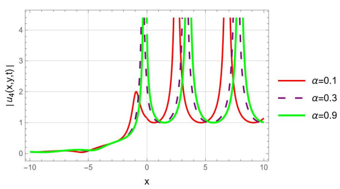
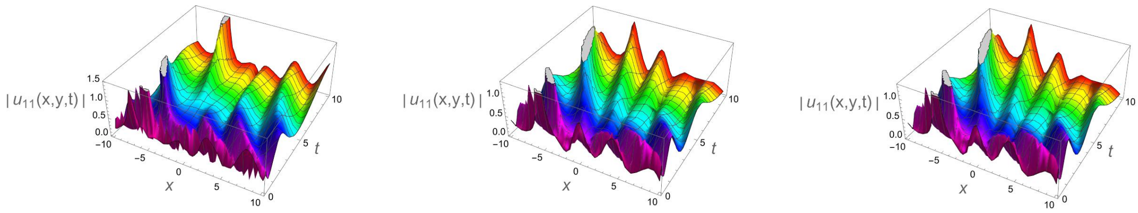
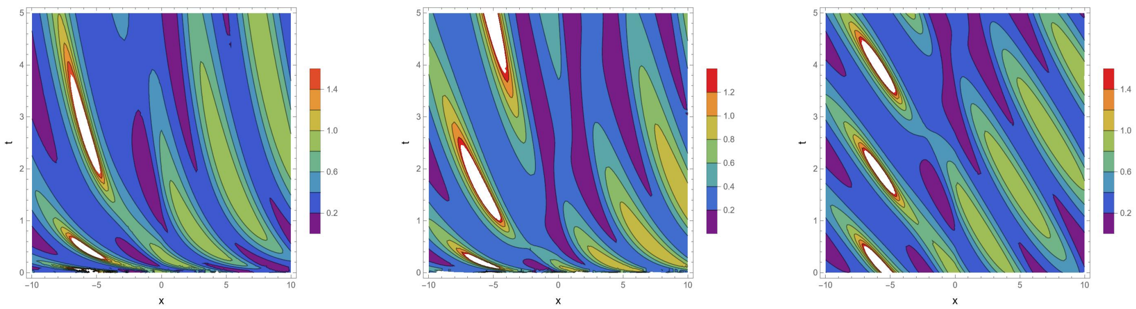


Figure 3: The graphs showing 3D, 2D, and contour representations, as well as the influence of α on wave propagation, for the solution $|h_4(x, y, t)|$.



$\alpha = 0.1$ $\alpha = 0.3$ $\alpha = 0.9$



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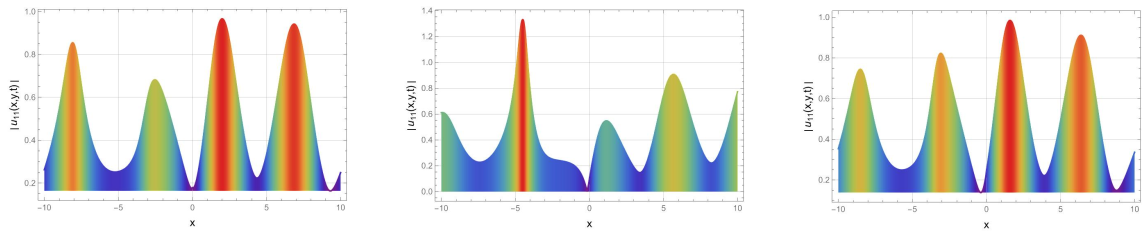

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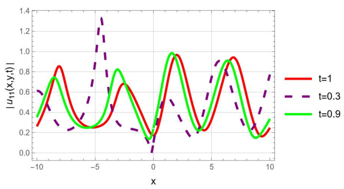
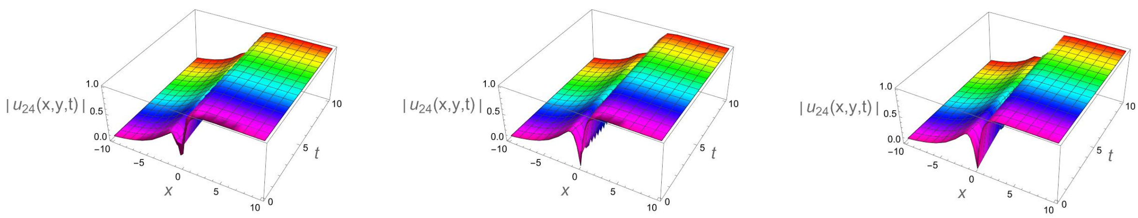
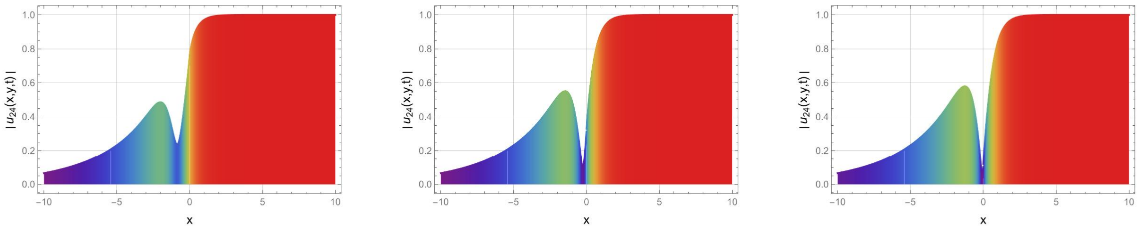


Figure 4: The graphs showing 3D, 2D, and contour representations, as well as the influence of α on wave propagation, for the solution $|h_{11}(x, y, t)|$.



$\alpha = 0.1$ $\alpha = 0.3$ $\alpha = 0.9$



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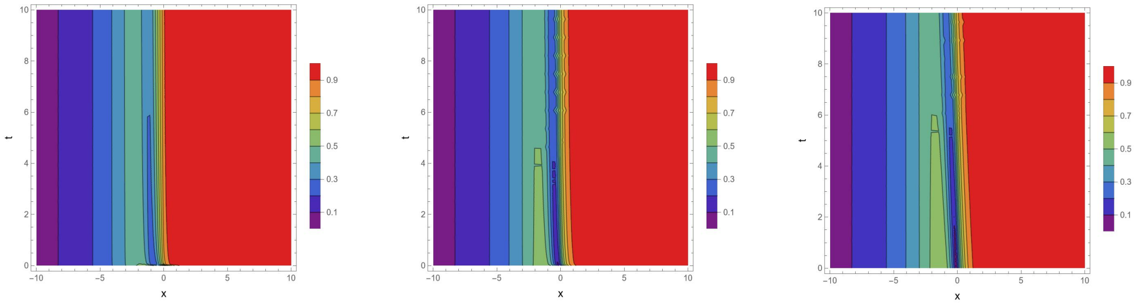

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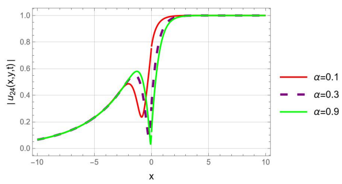
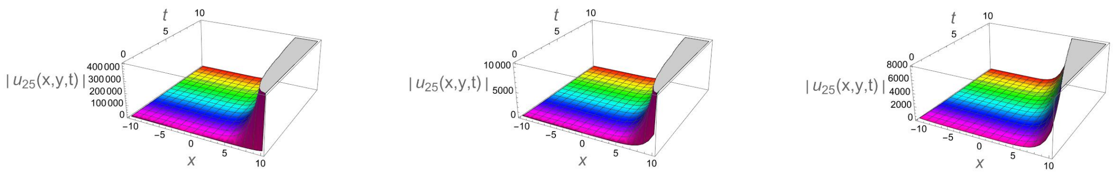


Figure 5: The graphs showing 3D, 2D, and contour representations, as well as the influence of α on wave propagation, for the solution $|h_{24}(x, y, t)|$.



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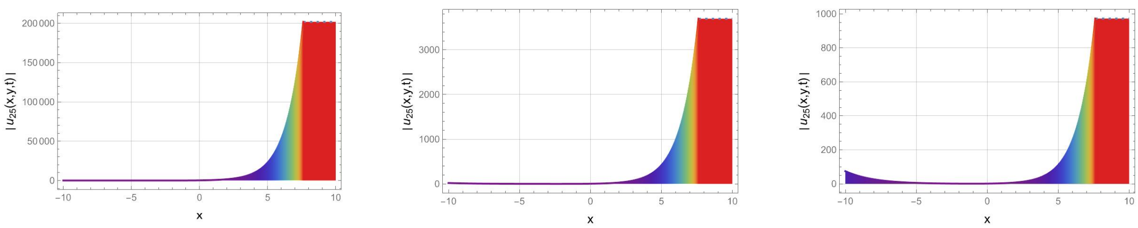

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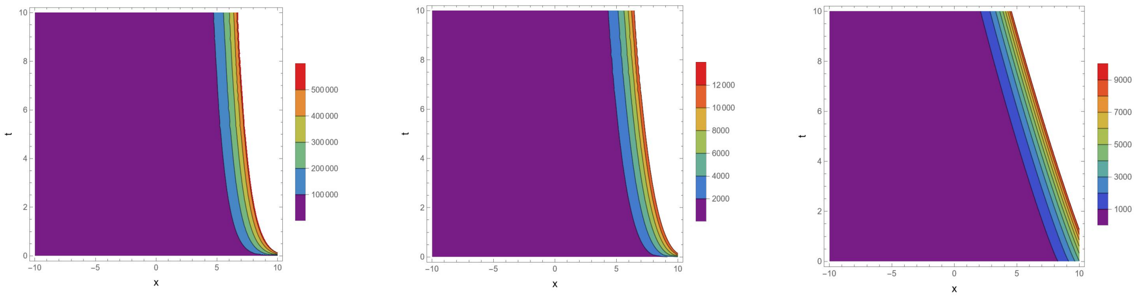

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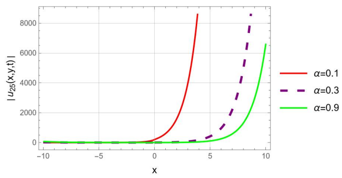
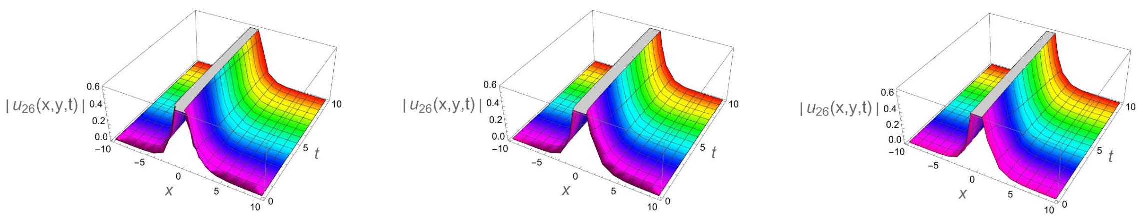


Figure 6: The graphs showing 3D, 2D, and contour representations, as well as the influence of α on wave propagation, for the solution $|h_{25}(x, y, t)|$.



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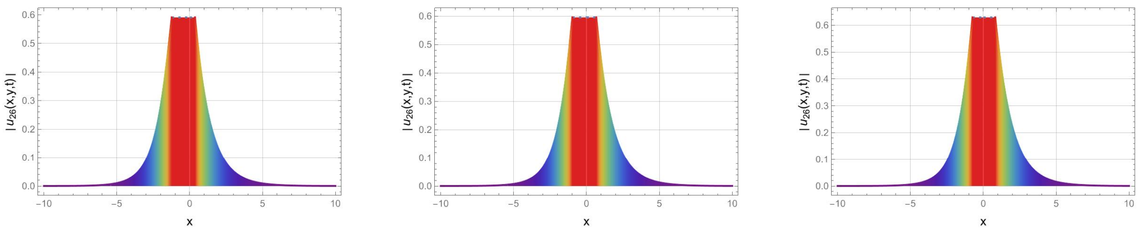

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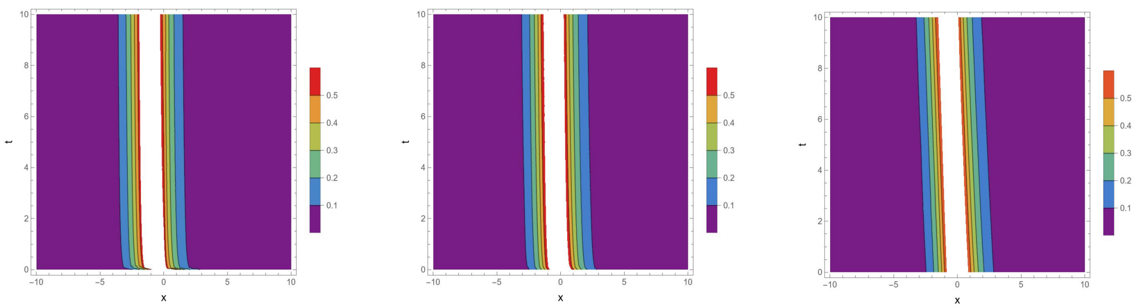

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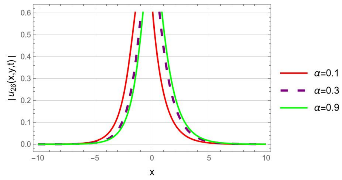

Figure 7: The graphs showing 3D, 2D, and contour representations, as well as the influence of α on wave propagation, for the solution $|h_{26}(x, y, t)|$.

5. Conclusion

This research successfully conducted a comprehensive investigation of the (2 + 1)-dimensional Complex modified KdV system using the conformable derivative and Jacobi elliptic function expansion method. Various types of solutions were obtained, including Jacobi Elliptic functions, solitary waves, and shock wave solutions. These solutions give a detailed insight into the dynamics of Nonlinear waves and solitons in higher dimensions. Using Mathematica, we generated 2D, 3D, and contour plots to illustrate the effects of the conformable derivative and various parameters on wave propagation. This analytical method effectively studies complex wave phenomena and provides a solid basis for further research and applications in nonlinear waves. Future work could extend it to other NLPDEs and higher-dimensional systems, broadening its usefulness in mathematical physics.

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