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## A Common Coincidence of Fixed Point for Generalized Caristi Fixed Point Theorem

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### Abstract

In this paper, the interpolative Caristi type weakly compatible contractive in a complete metric space is applied to show some common fixed points results related to such mappings. Our application shows that the function which is used to prove the obtained results is a bounded map. An example is provided to show the useability of the acquired results.

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### 1. Introduction and Preliminaries

In 1997, there shown at least five papers that were concerned with fixed point theorems (FPTs) for lower semi-continuous (LSC) multi maps with convex values; see [1, 2, 3, 4, 5]. Most of those theorems are based on one of the well-known selection theorems, for example, [6] and some have applications to the existence of equilibrium points of abstract economies. Our aim in this paper is to derive coincide and common FPTs for LSC multi maps by contractive weakly compatible mappings in Banach metric space. Several studies have extended the Banach contraction principle (PCB) since its inception, from those recent studies we refer to [7, 8, 9, 10]. This paper is inspired by some new work on the extension of the Banach Contraction Principle (BCP) to metric spaces with a partial order [11]. Caristi's fixed point theorem (CFPT) is possibly one of the nicest extensions of the

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BCP [12, 13]. This theorem states that a map  $F : X \rightarrow X$  has a fixed point, where  $X$  is a complete metric space and there exists an LSC map  $\psi : X \rightarrow [0, +\infty)$  such that

$$d(x, Fx) \leq \psi(x) - \psi(Fx), \quad \forall x \in X. \tag{1.1}$$

A fixed point of  $F$  if  $F(x) = x, \forall x \in X$ . In general, the fixed point theorems have established many applications in nonlinear analysis. For example, this result yields from basically all the known natural results of geometrical fixed point theorems in Banach spaces which using the proofing approaching Caristi’s theorem by various styles (see [14, 15, 16]). However, it is worth remembering that because of Caristi’s theorem’s close link to Ekeland’s [17] variational concept, many researchers indicate it as Caristi-Ekeland’s FPT. For information on Ekeland’s variational concept, and equality between Caristi-Ekeland’s FPT and the complete metric spaces, see [18]. The purpose of this paper is to present a new generalization of CFPT type on the BCP.

**Definition 1.1.** [19] The LSC map at  $x_*$  in a metric space  $X$  can be expressed as

$$\limsup_{x \rightarrow x_*} \psi(x) \leq \psi(x_*), \quad \forall x_*, x \in X. \tag{1.2}$$

The following result was introduced by Caristie in [12] as:

**Theorem 1.2.** *Let  $X$  be a complete metric space and  $\psi : X \rightarrow \mathbb{R}^+$  be an LSC and bonded below function, Suppose that  $F$  is a Caristie type mapping on  $X$  dominated by  $\psi$ . If  $F$  satisfies*

$$d(x, Fx) \leq \psi(x) - \psi(Fx), \quad \forall x \in X. \tag{1.3}$$

*Then,  $F$  has a fixed point in  $X$ .*

**Definition 1.3.** [20] Let  $F$  and  $T$  are selfing maps in  $X$ , then

- (i) A point  $x_0 \in X$  is said to be a common fixed point of  $F$  and  $T$  if  $x_0 = Fx_0 = Tx_0$ .
- (ii) A point  $x_0 \in X$  is called a coincidence point of  $F$  and  $T$  if  $x_0 = Fx = Tx$ .
- (iii) The mappings  $F, T : X \rightarrow X$  are said to be weakly compatible if  $FTx = TFx$  whenever  $Fx = Tx$ .

**Lemma 1.4.** [20] *Let  $X$  be a non-empty set and suppose that the mappings  $F, T : X \rightarrow X$  have a unique coincidence point  $x_0 \in X$ . If  $F$  and  $T$  are weakly compatible, then  $F$  and  $T$  have a unique common fixed point.*

Du in [19] has recalled another LSC function to get a direct proof of CFPT. Like this work depends on some development in BCP in metric space, see [21, 22, 23, 24]. Such these studies are still being replenished. A lot of recent papers study fixed point theorem by Caristi’s ways. Among these recent studies we will point out to [25, 26, 27, 28, 29, 30, 31]. Here, we use a LSC function to obtain some new FPTs of Caristi type for a weakly compatible contractive mapping in the Banach metric space.

## 2. Main results

In this section, we will prove the uniqueness of coincidence and a common fixed point in Banach space for generalization CFPT type.

**Theorem 2.1.** *Let  $F, T$  are weakly compatible contractive self-mappings on a Banach space  $X$ , and let  $\psi : X \rightarrow \mathbb{R}^+$  be a LSC function such that,*

$$\psi x \leq d(Fx, Tx), \tag{2.1}$$

for all  $x \in X$ , satisfies

$$d(Fx, Fy) \leq \psi(Tx) - \psi(Fx) + \psi(Ty) - \psi(Fy), \tag{2.2}$$

for all  $x, y \in X$  and  $x \neq y$ . Then  $F$  and  $T$  have a unique common fixed point.

*Proof.* First: Existence.

Consider  $Fx_{n-1} = Tx_n = x_n$  for all  $n \in \mathbb{N}$ . Now,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Fx_{n-1}, Fx_n) \leq \psi(Tx_{n-1}) - \psi(Fx_{n-1}) + \psi(Tx_n) - \psi(Fx_n) \\ &= \psi(x_{n-1}) - \psi(x_n) + \psi(x_n) - \psi(x_{n+1}) \\ &= \psi(x_{n-1}) - \psi(x_{n+1}) \\ &\leq d(Fx_{n-1}, Tx_{n-1}) - d(Fx_{n+1}, Tx_{n+1}). \end{aligned}$$

We obtain,

$$d(x_n, x_{n+2}) \leq d(x_{n-1}, x_n) \tag{2.3}$$

Since  $F, T$  are contractive mappings then they have continuous property, by Gopal in [32], the continuity and down bounded limit for  $\psi$  lead us to,  $x_n$  is decreasing to some points  $x_0 \in X$ , for all  $x \in X$

$$\psi(x_0) \leq \psi(x).$$

Since  $X$  is a Banach space thus,  $X \supset \{x_n\} \rightarrow x_0$ . Also,  $\{x_n\} \supset \{x_{n_k}\} \rightarrow x_0$  for all  $n, k \in \mathbb{N}$ , we have

$$\lim_{n,k \rightarrow \infty} x_{n_k} = F(\lim_{n,k \rightarrow \infty} x_{n_k}) = F(x_0), \tag{2.4}$$

$$\lim_{n \rightarrow \infty} x_n = T(\lim_{n \rightarrow \infty} x_n) = T(x_0). \tag{2.5}$$

Hence,  $x_0$  is a coincidence fixed point of  $F$  and  $T$  and by Lemma 1.4, we obtain that  $x_0$  is a common fixed point of  $F$  and  $T$ . Existence has been proven.

Second: Uniqueness.

Suppose that  $F(x_*) = T(x_*) = x_*$  for all  $x_* \in X$  such that  $x_0 \neq x_*$ , so by (2.2) we get

$$\begin{aligned} d(x_*, x_0) &= d(Fx_*, Fx_0) \\ &\leq \psi(Tx_*) - \psi(Fx_*) + \psi(Tx_0) - \psi(Fx_0) \\ &\leq \psi x_* - \psi x_* + \psi x_0 - \psi x_0 \\ &\leq d(Fx_*, Tx_*) - d(Fx_*, Tx_*) + d(Fx_0, Tx_0) - d(Fx_0, Tx_0) \\ &\leq 0. \end{aligned}$$

Hence,  $x_* = x_0$ . Uniqueness has been proven. □

We will generalize Theorem 2.1 for all repetitions  $m \in \mathbb{N}$ .

**Theorem 2.2.** *Let  $F^m, T^m$ ,  $m \in \mathbb{N}$  are weakly compatible contractive self-maps on Banach space  $X$ , and let  $\psi : X \rightarrow \mathbb{R}^+$  be a SLC function such that,*

$$\psi x \leq d(F^m x, T^m x), \tag{2.6}$$

for all  $x \in X$ , satisfies

$$d(F^m x, F^m y) \leq \psi(T^m x) - \psi(F^m x) + \psi(T^m y) - \psi(F^m y), \tag{2.7}$$

for all  $x, y \in X$  and  $x \neq y$ . Then  $F$  and  $T$  have a unique common fixed point.

*Proof.* First: Existence.

$F^m$  and  $T^m$  are continuous maps, since  $F^m$  and  $T^m$  are contractive. In the same way as proof Theorem 2.1, we conclude that  $F^m$  and  $T^m$  have a unique common fixed point in  $X$ . Suppose  $F^m x_0 = T^m x_0 = x_0$ . Let  $F^r x_0 \neq T^r x_0 \neq x_0$  for  $r = 1, 2, \dots, m-1$ . Also, let  $F^r x_0 = T^r x_0 = x_*$  for all  $x_0 \neq x_*$  then

$$\begin{aligned} d(F^r x_0, F^m x_0) &= d(FF^{r-1} x_0, FF^{m-1} x_0) \\ &\leq \psi(TF^{r-1} x_0) - \psi(FF^{r-1} x_0) + \psi(TF^{m-1} x_0) - \psi(FF^{m-1} x_0) \\ &\leq \psi(TF^{r-1} x_0) - \psi(F^r x_0) + \psi(TF^{m-1} x_0) - \psi(F^m x_0) \\ &= 0, \quad m > 1. \end{aligned}$$

Therefore,  $F^r x_0 = T^r x_0 = F^m x_0 = x_0$ , for all  $r = 1, 2, \dots, m-1$ . Thus,  $F x_0 = T x_0 = x_0$ , and since  $F$  and  $T$  are weakly compatible maps. Hence,  $x_0$  is a common fixed point for  $F$  and  $T$ .

Second: Uniqueness

Proving the uniqueness is the same steps as proving the uniqueness of the theorem 2.1.  $\square$

In the last section, we shall introduce an application and example to justify the viability and usability of our results.

### 3. application

**Theorem 3.1.** *A function  $\psi$  which is defined in (2.6) can be considered as a Lipschitz bounded function  $\forall x \in X$  and  $\forall m \in \mathbb{N}$ .*

*Proof.* Since,  $\psi x \leq d(F^n x, T^n x)$ ,  $\forall n \in \mathbb{N}$ . We have for all  $x_1, x_2 \in X$  such that  $x_1 \neq x_2$ .

$$\begin{aligned} \psi x_1 - \psi x_2 &\leq d(F^n x_1, T^n x_1) - d(F^n x_2, T^n x_2) \\ &= d(F^n x_1, T^n x_1) - d(F^n x_2, T^n x_2) + d(F^n x_1, T^n x_2) - d(F^n x_1, T^n x_2) \\ &= d(F^n x_1, T^n x_1) - d(F^n x_1, T^n x_2) + d(F^n x_1, T^n x_2) - d(F^n x_2, T^n x_2) \\ &= d(T^n x_1, T^n x_2) + d(F^n x_1, F^n x_2) \\ &= d(x_1, x_2) + d(F^n x_1, F^n x_2) \\ &\leq M d(x_1, x_2), \quad M \in \mathbb{N}. \end{aligned} \tag{3.1}$$

Then,  $M \geq 0$  where,  $\psi : X \rightarrow \mathbb{R}^+$ . Now we must prove that  $M > 1$ , so we will study all possible possibilities of  $M$  as following

- i When  $M = 0$ , then  $x_1 = x_2$  which is contradiction.
- ii When  $M = 1$ , then  $d(F^n x_1, F^n x_2) = 0$  which is contradiction since  $F$  has a unique fixed point.
- iii When  $0 < M < 1$  then, there exist  $\mathcal{A}$  from (3.1) satisfies

$$d(F^n x_1, F^n x_2) = \mathcal{A}d(x_1, x_2). \tag{3.2}$$

Inequalities (3.1) and (3.2) show that  $\mathcal{A}$  is negative real number, which is a contradiction.

Therefore,  $\mathcal{A} > 1$ . Hence,  $\psi$  is bounded Lipschitz function. □

**Example 3.2.** Consider,  $X = [0, 1)$ . Take  $\psi : [0, 1) \rightarrow \mathbb{R}^+$  defined by

$$\psi u = \begin{cases} 0, & \text{if } u \geq 0, \\ 1, & \text{if } u < 1. \end{cases}$$

Suppose,  $F(u) = u^2$  and  $T(u) = u$ . Then, for all  $u, v \in X$  we get

$$\begin{aligned} d(F(u), F(v)) &= d(u^2, v^2) \\ &< d(u, v). \end{aligned} \tag{3.3}$$

Clear that  $F$  is a contractive mapping. By inequalities (2.1) and (2.2) we get

$$\begin{aligned} d(Fu, Fv) &\leq d(F(Tu), T^2u) - d(F^2u, T(Fu)) + d(F(Tv), T^2v) - d(F^2v, T(Fv)) \\ &= d(u^2, u) - d(u^4, u^2) + d(v^2, v) - d(v^4, v^2) \end{aligned} \tag{3.4}$$

$$\leq d(u^4, u) + d(v^4, v). \tag{3.5}$$

We have two possibilities,

First possibility: If  $u = v$ , then  $F$  and  $T$  have a coincidence fixed point, and since  $F$  and  $T$  are commutable mappings then  $F$  and  $T$  have a unique common fixed point which is  $u = 0 \in X$ .

Second possibility: If  $u \neq v$ , then (3.5) is satisfies for every  $u, v \in X$ .

#### 4. conclusion

In this work, we have obtained the uniqueness of some common fixed points by utilized the interpolative Caristi type weakly compatible contractive results concerning such mappings through the Banach metric space. On other hand, Bakhtin in [33] introduced the concept of a quasi-metric space and showed some fixed point theorems on such space. Recently, Kamran in [34] generalized the quasi metric space and showed new primary notions. The problem considered in this paper can be studied on quasi metric space, which is a direction we are working on.

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