


Fixed points of generalized integral type $\alpha - F$ contraction mappings in metric-like spaces

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Abstract

This article focuses on generalized integral type $\alpha - F$ contraction mappings in metric-like spaces and certain fixed point results in this setting. We also present some examples to support the validity of the results.

Keywords: Generalized integral type, $\alpha - F$ contraction mapping, F-contraction mapping, α -admissible mapping, metric-like spaces.

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1. Introduction

Contraction mappings play a crucial role in fixed point theory, solving existence problems across various disciplines of mathematics. Initially, Banach [3] established the classical contraction principle, which ensures the existence and uniqueness of fixed points. Because of its use, it has been generalized by employing other transformation types and modifying the structure of the space.

In 2002, Branciari [7], introduced the integral contraction as follows.

Theorem 1.1. Let (W_τ, d) be a complete metric space, $k \in (0, 1)$ and let $\Gamma : W_\tau \rightarrow W_\tau$ be a mapping such that for each $\gamma_\tau, \zeta_\tau \in W_\tau$

$$\int_0^{d(\Gamma\gamma_\tau, \Gamma\zeta_\tau)} \varphi(t) dt \leq k \int_0^{d(\gamma_\tau, \zeta_\tau)} \varphi(t) dt$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue-integrable map which is summable, (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$, then Γ has a unique fixed point.

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We begin by recalling a few definitions and lemmas.

Matthews [11, 12] introduced the concept of partial metric space (PMS) as follows:

Definition 1.2. Let W_r be a non-empty set. A function $\sigma_{ml} : W_r \times W_r \rightarrow [0, \infty)$ is said to be a partial metric on W_r if the following conditions hold:

- (PMS1) $\gamma_r = \zeta_r \Leftrightarrow \rho(\gamma_r, \gamma_r) = \rho(\zeta_r, \zeta_r) = \rho(\gamma_r, \zeta_r)$;
- (PMS2) $\rho(\gamma_r, \gamma_r) \leq \rho(\gamma_r, \zeta_r)$;
- (PMS3) $\rho(\gamma_r, \zeta_r) = \rho(\zeta_r, \gamma_r)$;
- (PMS4) $\rho(\gamma_r, \zeta_r) \leq \rho(\gamma_r, \eta_r) + \rho(\eta_r, \zeta_r) - \rho(\eta_r, \eta_r)$;

for all $\gamma_r, \zeta_r, \eta_r \in W_r$. The set W_r equipped with the metric σ_{ml} defined above is called a partial metric space and it is denoted by (W_r, ρ) (in short PMS). Each partial metric ρ on W_r generates a Γ_ρ topology τ_ρ on W_r , which has a base of the family of open σ_{ml} -balls

$$\{B_\rho(\gamma_r, \epsilon) : \gamma_r \in W_r, \epsilon > 0\},$$

where

$$B_\rho(\gamma_r, \epsilon) = \{\zeta_r \in W_r : |\rho(\gamma_r, \zeta_r) - \rho(\gamma_r, \gamma_r)| < \epsilon\},$$

for all $\gamma_r \in W_r$ and $\epsilon > 0$.

Harandi [2] introduced metric-like spaces as follows:

Definition 1.3. Let W_r be a non-empty set. A function $\sigma_{ml} : W_r \times W_r \rightarrow [0, \infty)$ is said to be a metric-like on W_r if the following conditions hold:

- (MLS1) $\sigma_{ml}(\gamma_r, \zeta_r) = 0 \Rightarrow \gamma_r = \zeta_r$;
- (MLS2) $\sigma_{ml}(\gamma_r, \zeta_r) = \sigma_{ml}(\zeta_r, \gamma_r)$;
- (MLS3) $\sigma_{ml}(\gamma_r, \zeta_r) \leq \sigma_{ml}(\gamma_r, \eta_r) + \sigma_{ml}(\eta_r, \zeta_r)$;

for all $\gamma_r, \zeta_r, \eta_r \in W_r$. Then (W_r, σ_{ml}) is called metric-like space. Each metric-like σ_{ml} on W_r generates a $\Gamma_{\sigma_{ml}}$ topology $\tau_{\sigma_{ml}}$ on W_r , which has a base of the family of open σ_{ml} -balls

$$\{B_{\sigma_{ml}}(\gamma_r, \epsilon) : \gamma_r \in W_r, \epsilon > 0\},$$

where

$$B_{\sigma_{ml}}(\gamma_r, \epsilon) = \{\zeta_r \in W_r : |\sigma_{ml}(\gamma_r, \zeta_r) - \sigma_{ml}(\gamma_r, \gamma_r)| < \epsilon\},$$

for all $\gamma_r \in W_r$ and $\epsilon > 0$.

Example 1.4. [2] Let $W_r = \{0, 1\}$ and define

$$\sigma_{ml}(\gamma_r, \zeta_r) = \begin{cases} 2 & \gamma_r = \zeta_r = 0 \\ 1 & \text{otherwise} \end{cases} \tag{1.1}$$

Then (W_r, σ_{ml}) is metric-like space but since $\sigma_{ml}(0, 0) \not\leq \sigma_{ml}(0, 1)$, (W_r, σ_{ml}) is not a partial metric space.

Lemma 1.5. [4] Let (W_r, σ_{ml}) be a metric-like space.

(a) A sequence $\{\gamma_{r_n}\}$ in (W_r, σ_{ml}) converges to a point $\gamma_r \in W_r \Leftrightarrow$

$$\sigma_{ml}(\gamma_r, \gamma_r) = \lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_r),$$

(b) A sequence $\{\gamma_{r_n}\}$ in (W_r, σ_{ml}) is a σ_{ml} -Cauchy sequence if $\lim_{m, n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_m})$ exists and finite,

(c) (W_r, σ_{ml}) is complete if every σ_{ml} -Cauchy sequence $\{\gamma_{r_n}\}$ in W_r converges to a point $\gamma_r \in W_r$, such that

$$\sigma_{ml}(\gamma_r, \gamma_r) = \lim_{m, n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_m}) = \lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_r).$$

(d) A mapping $\Gamma : W_r \rightarrow W_r$ is continuous, if following limit exists and

$$\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_r) = \lim_{n \rightarrow \infty} \sigma_{ml}(\Gamma \gamma_r, \gamma_r).$$

Karapinar and Salimi [10] demonstrated the following key features in metric-like spaces:

Lemma 1.6. Let (W_r, σ_{ml}) be a metric-like space. Then

1. $\sigma_{ml}(\gamma_r, \zeta_r) = 0, \sigma_{ml}(\gamma_r, \gamma_r) = \sigma_{ml}(\zeta_r, \zeta_r) = 0,$
2. If $\{\gamma_{r_n}\}$ is a sequence such that $\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = 0,$

$$\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_n}) = \lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_{n+1}}, \gamma_{r_{n+1}}) = 0,$$

3. if $\gamma_r \neq \zeta_r,$ Then $\sigma_{ml}(\gamma_r, \zeta_r) > 0,$
4. $\sigma_{ml}(\gamma_r, \gamma_r) \leq \frac{2}{n} \sum_{j=1}^n \sigma_{ml}(\gamma_r, \gamma_{r_j}),$

for all $\gamma_r, \gamma_{r_j} \in W_r$ where $1 \leq j \leq n.$

Lemma 1.7. [19] Assume that $\gamma_{r_n} \rightarrow \eta_r$ as $n \rightarrow \infty$ in a metric-like space (W_r, σ_{ml}) such that $\sigma_{ml}(\eta_r, \eta_r) = 0.$ Then $\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \zeta_r) = \sigma_{ml}(\eta_r, \zeta_r)$ for every $\zeta_r \in W_r.$

Lemma 1.8. [22] If $\{\gamma_{r_n}\}$ with $\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = 0$ is not a Cauchy sequence in metric-like space $(W_r, \sigma_{ml}),$ and two sequences $\{n(j)\}$ and $\{m(j)\}$ of positive integers such that $n(j) > m(j) > j,$ then following four sequences

$$\begin{aligned} &\sigma_{ml}(\gamma_{r_{m(j)}}, \gamma_{r_{n(j)+1}}), \sigma_{ml}(\gamma_{r_{m(j)}}, \gamma_{r_{n(j)}}), \\ &\sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{n(j)+1}}), \sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{n(j)}}) \end{aligned}$$

tend to $\mu_r^+ > 0$ when $j \rightarrow \infty.$

In recent years, numerous authors have established fixed point or common fixed point theorems in metric-like spaces, as seen in [19, 18, 21, 17, 14, 16, 5, 6].

In 2012, Samet et al. [20] introduced α -admissible mapping as follows:

Definition 1.9. Let $\Gamma : W_r \rightarrow W_r$ and $\alpha : W_r \times W_r \rightarrow [0, \infty).$ Γ is said to be α -admissible if

$$\alpha(\gamma_r, \zeta_r) \geq 1 \Rightarrow \alpha(\Gamma \gamma_r, \Gamma \zeta_r) \geq 1,$$

for all $\gamma_r, \zeta_r \in W_r$.

Further, Karapinar et al. [9], presented triangular α -admissible as follows:

Definition 1.10. Let $\Gamma : W_r \rightarrow W_r$ and $\alpha : W_r \times W_r \rightarrow [0, \infty)$ be functions. Then Γ is said to be triangular α -admissible if Γ is α -admissible and for $\gamma_r, \zeta_r, \eta_r \in W_r$, $\alpha(\gamma_r, \eta_r) \geq 1$ and $\alpha(\eta_r, \zeta_r) \geq 1 \Rightarrow \alpha(\gamma_r, \zeta_r) \geq 1$.

Lemma 1.11. [9] Let $\Gamma : W_r \rightarrow W_r$ be triangular α -admissible mapping. Assume that there exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geq 1$. Define a sequence $\{\gamma_{r_n}\}$ by $\gamma_{r_{n+1}} = \Gamma\gamma_{r_n}$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(\gamma_{r_m}, \gamma_{r_n}) \geq 1$ for all $m, n \in \mathbb{N}$ with $m > n$.

Wardowski [24, 25] introduced new class of contraction mappings as follows:

Definition 1.12. Let \mathcal{F} be family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying:

- (F1) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}^+$ if $a < b$ then $F(a) < F(b)$;
- (F2) for each sequence $\{a_n\}$ of positive numbers, $\lim_{n \rightarrow \infty} a_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} F(a_n) = -\infty$;
- (F3) there exists $k \in (0, 1)$ such that $\lim_{a \rightarrow 0^+} a^k F(a) = 0$.

Wardowski [24] defined F -contraction as follows:

Let (W_r, d) be a metric space, then the mapping $\Gamma : W_r \rightarrow W_r$ is said to be an F -contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $\gamma_r, \zeta_r \in W_r$ with $d(\Gamma\gamma_r, \Gamma\zeta_r) > 0$ we have

$$\tau + F(d(\Gamma\gamma_r, \Gamma\zeta_r)) \leq F(d(\gamma_r, \zeta_r))$$

Piri and Kumam [15] extended Wardowski's [24] results by modifying the condition (F3) in Definition 1.12 as follows:

Definition 1.13. Let $\Delta_{\mathcal{F}}$ be family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying:

- (i) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}^+$ if $a < b$ then $F(a) < F(b)$;
- (ii) for each sequence $\{a_n\}$ of positive numbers;
 $\lim_{n \rightarrow \infty} a_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} F(a_n) = -\infty$;
- (iii) F is continuous on $(0, \infty)$.

Various authors have generalized Wardowski's result (refer to [13, 1, 8, 23]).

2. Main Results

Let Φ be the family of all functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that φ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and for each $\epsilon > 0$,

$$\int_0^\epsilon \varphi(t) dt > 0.$$

Definition 2.1. Let (W_r, σ_{ml}) be a metric-like space and let $\Gamma : W_r \rightarrow W_r$ be a self map. Then Γ is said to be generalized integral type $\alpha - F$ -contractive mapping if there exist two functions $\alpha : W_r \times W_r \rightarrow [0, \infty)$ and $F \in \Delta_{\mathcal{F}}$ such that for $\tau > 0$ with $\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r) > 0$,

$$\tau + F\left(\alpha(\gamma_r, \zeta_r) \int_0^{\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r)} \varphi(t) dt\right) \leq F\left(\int_0^{M(\gamma_r, \zeta_r)} \varphi(t) dt\right), \tag{2.1}$$

where $\varphi \in \Phi$ and

$$M(\gamma_r, \zeta_r) = \max\{\sigma_{ml}(\gamma_r, \zeta_r), \sigma_{ml}(\gamma_r, \Gamma\gamma_r), \sigma_{ml}(\zeta_r, \Gamma\zeta_r)\}.$$

Theorem 2.2. Let (W_r, σ_{ml}) be a complete metric-like space and $\Gamma : W_r \rightarrow W_r$ be a self map. Suppose $\alpha : W_r \times W_r \rightarrow [0, \infty)$ be the mapping satisfying the conditions:

- (i) Γ is triangular α -admissible mapping;
- (ii) Γ is generalized integral type $\alpha - F$ -contractive mapping;
- (iii) There exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geq 1$;
- (iv) Γ is continuous.

Then Γ has a fixed point in W_r .

Proof. Let γ_{r_0} be an arbitrary point where $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geq 1$. Consider a sequence $\{\gamma_{r_n}\}$ in W_r such that $\gamma_{r_{n+1}} = \Gamma\gamma_{r_n}$ for all $n \in \mathbb{N}$. If $\gamma_{r_n} = \gamma_{r_{n+1}}$ for some $n \in \mathbb{N}$, then γ_{r_n} is a fixed point of Γ and the existence part of the proof is complete. Assume $\gamma_{r_n} \neq \gamma_{r_{n+1}}$ for all $n \in \mathbb{N}$, then $\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = \sigma_{ml}(\Gamma\gamma_{r_{n-1}}, \Gamma\gamma_{r_n}) > 0$ by lemma 1.6. Now, since Γ is α -admissible, so

$$\begin{aligned} \alpha(\Gamma\gamma_{r_0}, \Gamma\gamma_{r_1}) &= \alpha(\gamma_{r_1}, \gamma_{r_2}) \geq 1, \\ \alpha(\Gamma\gamma_{r_1}, \Gamma\gamma_{r_2}) &= \alpha(\gamma_{r_2}, \gamma_{r_3}) \geq 1 \end{aligned}$$

and using induction we have $\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \geq 1$ for all $n \in \mathbb{N}$.

Now, by (2.1) we get

$$\begin{aligned} \tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) &\leq \tau + F\left(\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) \\ &= \tau + F\left(\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n-1}}, \Gamma\gamma_{r_n})} \varphi(t) dt\right) \\ &\leq F\left(\int_0^{M(\gamma_{r_{n-1}}, \gamma_{r_n})} \varphi(t) dt\right), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
M(\gamma_{r_{n-1}}, \gamma_{r_n}) &= \max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_{n-1}}, \Gamma\gamma_{r_{n-1}}), \sigma_{ml}(\gamma_{r_n}, \Gamma\gamma_{r_n})\} \\
&= \max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\} \\
&= \max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\}.
\end{aligned} \tag{2.3}$$

Now, using (2.3) in (2.2) we get that

$$\tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) \leq F\left(\int_0^{\max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\}} \varphi(t) dt\right). \tag{2.4}$$

Now, if $\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) > \sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n})$, then a contradiction follows from

$$\tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) \leq F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right).$$

Thus, we conclude that

$$\max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\} = \sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}). \tag{2.5}$$

Therefore, From (2.4) we get that

$$F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) \leq F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n})} \varphi(t) dt\right) - \tau. \tag{2.6}$$

Continuing in the same way, we obtain

$$F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n-1}})} \varphi(t) dt\right) \leq F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_{n-2}})} \varphi(t) dt\right) - \tau. \tag{2.7}$$

Using (2.7) in (2.6) we get that

$$\begin{aligned}
F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) &\leq F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n})} \varphi(t) dt\right) - \tau \\
&\leq F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_{n-2}})} \varphi(t) dt\right) - 2\tau.
\end{aligned}$$

On generalizing

$$F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) < F\left(\int_0^{\sigma_{ml}(\gamma_{r_0}, \gamma_{r_1})} \varphi(t) dt\right) - n\tau. \tag{2.8}$$

Letting the limit $n \rightarrow \infty$ in (2.8) and using the definition of F we get

$$\lim_{n \rightarrow \infty} F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t) dt\right) = -\infty \Leftrightarrow \lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = 0. \tag{2.9}$$

consequently, we get

$$\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = 0. \tag{2.10}$$

Now, we prove that the sequence $\{\gamma_{r_n}\}$ is a σ_{ml} -Cauchy sequence in W_r by supposing the contrary, i.e. $\lim_{n,m \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_m}) \neq 0$.

Then sequences in lemma 1.8 tends to $\mu_r^+ > 0$, when $j \rightarrow \infty$. So, we have

$$\lim_{j \rightarrow \infty} \sigma_{ml}(\gamma_{r_{n(j)}}, \gamma_{r_{m(j)}}) = \mu_r^+. \tag{2.11}$$

Further corresponding to $m(j)$, we can choose $n(j)$ in such a way that it is smallest integer with $n(j) > m(j) > j$. Then

$$\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_{n(j)-1}}, \gamma_{r_{m(j)}}) = \mu_r^+. \tag{2.12}$$

Again,

$$\sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{n(j)-1}}) \leq \sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{n(j)}}) + \sigma_{ml}(\gamma_{r_{n(j)}}, \gamma_{r_{n(j)-1}}).$$

Letting $j \rightarrow \infty$ and using lemma 1.8 we get

$$\lim_{j \rightarrow \infty} \sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{n(j)-1}}) = \mu_r^+. \tag{2.13}$$

Now as Γ is triangular α admissible we have $\alpha(\sigma_{ml}(\gamma_{r_{n(j)}}, \gamma_{r_{m(j)}})) \geq 1$, then replacing γ_r by $\gamma_{r_{n(j)}}$ and ζ_r by $\gamma_{r_{m(j)}}$ in (2.1) respectively, we get

$$\begin{aligned} \tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n(j)}}, \gamma_{r_{m(j)}})} \varphi(t) dt\right) &\leq \tau + F\left(\alpha(\gamma_{r_{n(j)}}, \gamma_{r_{m(j)}}) \int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n(j)-1}}, \Gamma\gamma_{r_{m(j)-1}})} \varphi(t) dt\right), \\ &\leq F\left(\int_0^{M(\gamma_{r_{n(j)-1}}, \gamma_{r_{m(j)-1}})} \varphi(t) dt\right), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} M(\gamma_{r_{n(j)-1}}, \gamma_{r_{m(j)-1}}) &= \max\{\sigma_{ml}(\gamma_{r_{n(j)-1}}, \gamma_{r_{m(j)-1}}), \sigma_{ml}(\gamma_{r_{n(j)-1}}, \Gamma\gamma_{r_{n(j)-1}}), \sigma_{ml}(\gamma_{r_{m(j)-1}}, \Gamma\gamma_{r_{m(j)-1}})\} \\ &= \max\{\sigma_{ml}(\gamma_{r_{n(j)-1}}, \gamma_{r_{m(j)-1}}), \sigma_{ml}(\gamma_{r_{n(j)-1}}, \gamma_{r_{n(j)}}), \sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{m(j)}})\}. \end{aligned} \tag{2.15}$$

Letting $j \rightarrow \infty$ in (2.15) and using (2.10), (2.11), (2.12), (2.13) and lemma 1.8 we get

$$\lim_{j \rightarrow \infty} M(\gamma_{r_{n(j)-1}}, \gamma_{r_{m(j)-1}}) = \mu_r^+. \tag{2.16}$$

Now Letting $j \rightarrow \infty$ in (2.14) and using (2.16) we get

$$\tau + F\left(\int_0^{\mu_r^+} \varphi(t) dt\right) \leq F\left(\int_0^{\mu_r^+} \varphi(t) dt\right).$$

Which is a contradiction.

This implies that $\{\gamma_{r_n}\}$ is a σ_{ml} -Cauchy sequence in (W_r, σ_{ml}) . So, there exists $\eta_r \in W_r$ such that

$$\sigma_{ml}(\eta_r, \eta_r) = \lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \eta_r) = \lim_{m, n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_m}) = 0. \tag{2.17}$$

Since, Γ is continuous, we get

$$\lim_{n \rightarrow \infty} \sigma_{ml}(\Gamma\gamma_{r_n}, \Gamma\eta_r) = \lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \eta_r) = \sigma_{ml}(\eta_r, \eta_r) = 0.$$

From Lemma 1.7, we also have

$$\lim_{n \rightarrow \infty} \sigma_{ml}(\gamma_{r_n}, \Gamma\eta_r) = \sigma_{ml}(\eta_r, \Gamma\eta_r). \tag{2.18}$$

Combining (2.17) and (2.18) and Lemma 1.6, we get that η_r is a fixed point of Γ i.e., $\Gamma\eta_r = \eta_r$. □

Theorem 2.3. Let (W_r, σ_{ml}) be a complete metric-like space and $\Gamma : W_r \rightarrow W_r$ be a self map. Suppose $\alpha : W_r \times W_r \rightarrow [0, \infty)$ be the mapping satisfying the conditions:

- (i) Γ is triangular α -admissible mapping;
- (ii) Γ is integral type generalized $\alpha - F$ -contractive mapping;
- (iii) There exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geq 1$;
- (iv) If $\{\gamma_{r_n}\}$ is a sequence in W_r such that $\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \geq 1$ for all n and $\gamma_{r_n} \rightarrow \eta_r \in W_r$ as $n \rightarrow \infty$, then there exists a subsequence $\gamma_{r_{n(i)}}$ of $\{\gamma_{r_n}\}$ such that $\alpha(\gamma_{r_{n(i)}}, \eta_r) \geq 1$ for all i .

Then Γ has a fixed point in W_r . Further if η_r, η_s are fixed points of Γ with $\alpha(\eta_r, \eta_s) \geq 1$, then Γ has a unique fixed point in W_r .

Proof. From the proof of the Theorem 2.2, the sequence $\{\gamma_{r_n}\}$ defined by $\gamma_{r_{n+1}} = \Gamma\gamma_{r_n}$ is a Cauchy sequence in (W_r, σ_{ml}) , as a result there exist $\eta_r \in W_r$ such that $\gamma_{r_n} \rightarrow \eta_r$. It is enough to show that $\eta_r \in W_r$ is the fixed point of Γ .

On contrary we suppose that $(\Gamma\eta_r, \eta_r) > 0$. Then from condition (iii) there exists a subsequence $\gamma_{r_{n(i)}}$ of $\{\gamma_{r_n}\}$ such that $\alpha(\gamma_{r_{n(i)}}, \eta_r) \geq 1$ for all i . By Using given contractive

condition (2.1) for $\gamma_r = \gamma_{r_{n(i)}}$ and $\zeta_r = \eta_r$ and property of F we have

$$\begin{aligned} \tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n(i)+1}, \Gamma\eta_r)} \varphi(t) dt}\right) &= \tau + F\left(\int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n(i)}, \Gamma\eta_r)} \varphi(t) dt}\right) \\ &\leq \tau + F\left(\alpha(\gamma_{r_{n(i)}, \eta_r} \int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n(i)}, \Gamma\eta_r)} \varphi(t) dt}\right) \\ &\leq F\left(\int_0^{M(\gamma_{r_{n(i)}, \eta_r)} \varphi(t) dt}\right), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} M(\gamma_{r_{n(i)}, \eta_r) &= \max\{\sigma_{ml}(\gamma_{r_{n(i)}, \eta_r), \sigma_{ml}(\gamma_{r_{n(i)}, \Gamma\gamma_{r_{n(i)}}), \sigma_{ml}(\eta_r, \Gamma\eta_r)\} \\ &= \max\{\sigma_{ml}(\gamma_{r_{n(i)}, \eta_r), \sigma_{ml}(\gamma_{r_{n(i)}, \gamma_{r_{n(i)+1}}), \sigma_{ml}(\eta_r, \Gamma\eta_r)\}. \end{aligned} \tag{2.20}$$

Letting $i \rightarrow \infty$ in (2.20) and taking (2.18) into account we get that

$$\lim_{i \rightarrow \infty} M(\gamma_{r_{n(i)}, \eta_r) = \sigma_{ml}(\eta_r, \Gamma\eta_r). \tag{2.21}$$

Now, Letting $i \rightarrow \infty$ in (2.19) and using (2.21) and the continuity of F we get that

$$\tau + F\left(\int_0^{\sigma_{ml}(\eta_r, \Gamma\eta_r)} \varphi(t) dt}\right) \leq F\left(\int_0^{\sigma_{ml}(\eta_r, \Gamma\eta_r)} \varphi(t) dt}\right),$$

which is a contradiction since $\tau > 0$, Thus we have $\Gamma\eta_r = \eta_r$. This shows that η_r is a fixed point of Γ . Further, suppose η_r and η_s be two fixed points of Γ such that $\sigma_{ml}(\eta_r, \eta_s) > 0$. From (2.1) we have

$$\begin{aligned} \tau + F\left(\int_0^{\sigma_{ml}(\eta_r, \eta_s)} \varphi(t) dt}\right) &= \tau + F\left(\int_0^{\sigma_{ml}(\Gamma\eta_r, \Gamma\eta_s)} \varphi(t) dt}\right) \\ &\leq \tau + F\left(\alpha(\eta_r, \eta_s) \int_0^{\sigma_{ml}(\Gamma\eta_r, \Gamma\eta_s)} \varphi(t) dt}\right) \\ &\leq F\left(\int_0^{M(\eta_r, \eta_s)} \varphi(t) dt}\right), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} M(\eta_r, \eta_s) &= \max\{\sigma_{ml}(\eta_r, \eta_s), \sigma_{ml}(\eta_r, \Gamma\eta_r), \sigma_{ml}(\eta_s, \Gamma\eta_s)\} \\ &= \max\{\sigma_{ml}(\eta_r, \eta_s), \sigma_{ml}(\eta_r, \eta_r), \sigma_{ml}(\eta_s, \eta_s)\} \\ &= \sigma_{ml}(\eta_r, \eta_s). \end{aligned} \tag{2.23}$$

putting (2.23) in (2.22) we get

$$\tau + F\left(\int_0^{\sigma_{ml}(\eta_r, \eta_s)} \varphi(t) dt}\right) \leq F\left(\int_0^{\sigma_{ml}(\eta_r, \eta_s)} \varphi(t) dt}\right), \tag{2.24}$$

which is a contradiction. Hence Γ has a unique fixed point. This completes the proof. \square

Following are consequences of the theorems.

Corollary 2.4. Let (W_r, σ_{ml}) be a complete metric-like space and $\Gamma : W_r \rightarrow W_r$ be a self map. Suppose there exist two functions $\alpha : W_r \times W_r \rightarrow [0, \infty)$ and $F \in \Delta_{\mathfrak{F}}$ such that for $\tau > 0$ with $\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r) > 0$ and satisfying the conditions:

- (i) Γ is triangular α -admissible mapping;
- (ii) Γ is generalized $\alpha - F$ -contractive mapping i.e.

$$\tau + F\left(\alpha(\gamma_r, \zeta_r)(\Gamma\gamma_r, \Gamma\zeta_r)\right) \leq F\left(M(\gamma_r, \zeta_r)\right), \tag{2.25}$$

where

$$M(\gamma_r, \zeta_r) = \max\{\sigma_{ml}(\gamma_r, \zeta_r), \sigma_{ml}(\gamma_r, \Gamma\gamma_r), \sigma_{ml}(\zeta_r, \Gamma\zeta_r)\};$$

- (iii) There exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geq 1$;
- (iv) Γ is continuous or if $\{\gamma_{r_n}\}$ is a sequence in W_r such that $\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \geq 1$ for all n and $\gamma_{r_n} \rightarrow \eta_r \in W_r$ as $n \rightarrow \infty$, then there exists a subsequence $\gamma_{r_{n(i)}}$ of $\{\gamma_{r_n}\}$ such that $\alpha(\gamma_{r_{n(i)}}, \eta_r) \geq 1$ for all i .

Then Γ has a fixed point in W_r .

Corollary 2.5. Let (W_r, σ_{ml}) be a complete metric-like space and let $\Gamma : W_r \rightarrow W_r$ be a continuous self map. Suppose that there exists $k \in (0, 1)$ such that

$$\int_0^{\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r)} \varphi(t) dt \leq k \int_0^{\sigma_{ml}(\gamma_r, \zeta_r)} \varphi(t) dt \tag{2.26}$$

and $\varphi \in \Phi$. Then Γ has a unique fixed point in W_r .

Example 2.6. Let $W_r = [0, 1]$ and define $\sigma_{ml} : W_r \times W_r \rightarrow \mathbb{R}^+$ by $\sigma_{ml}(\gamma_r, \zeta_r) = \max\{\gamma_r, \zeta_r\}$. Then (W_r, σ_{ml}) is a complete metric-like space. Consider the mapping $\Gamma : W_r \rightarrow W_r$ defined by $\Gamma(\eta_r) = \frac{\eta_r}{4}$. Suppose that $\varphi(t) = 2t$. Define the function $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $F(a) = \ln a$ for all $a \in \mathbb{R}^+ > 0$ and $\alpha : W_r \times W_r \rightarrow [0, \infty)$ by $\alpha(\gamma_r, \zeta_r) = 4$ for all $\gamma_r, \zeta_r \in W_r$.

We show that contractive conditions of Theorem 2.2 are satisfied. Let $\gamma_r, \zeta_r \in W_r$, without loss of generality we assume that $\gamma_r \geq \zeta_r$. Suppose that $\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r) > 0$ and let $\tau = \ln(2)$, then

$$\begin{aligned} \tau + F\left(\alpha(\gamma_r, \zeta_r) \int_0^{\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r)} \varphi(t) dt\right) &= \tau + F\left(4 \int_0^{\sigma_{ml}(\frac{\gamma_r}{4}, \frac{\zeta_r}{4})} 2t dt\right) \\ &= \tau + F\left(\frac{\gamma_r^2}{4}\right) \\ &= \ln(2) + \ln\left(\frac{\gamma_r^2}{4}\right) = \ln\left(\frac{\gamma_r^2}{2}\right) \\ &\leq \ln(\gamma_r^2) = F(\gamma_r^2) = F\left(\int_0^{M(\gamma_r, \zeta_r)} \varphi(t) dt\right). \end{aligned} \tag{2.27}$$

Hence Γ has a fixed point, which in this case is 0.

Example 2.7. Let $W_r = \{0, 1, 2\}$ and let $\sigma_{ml} : W_r \times W_r \rightarrow \mathbb{R}^+$ be a metric-like function defined by

$$\begin{aligned} \sigma_{ml}(0, 0) &= \sigma_{ml}(2, 2) = 0, \sigma_{ml}(1, 1) = 1 \\ \sigma_{ml}(1, 2) &= \sigma_{ml}(2, 1) = 2 \\ \sigma_{ml}(2, 0) &= \sigma_{ml}(0, 2) = 3 \\ \sigma_{ml}(0, 1) &= \sigma_{ml}(1, 0) = \frac{3}{2}. \end{aligned}$$

Then (W_r, σ_{ml}) is a complete metric-like space. Let $\Gamma : W_r \rightarrow W_r$ be defined by $\Gamma 0 = \Gamma 1 = 0$ and $\Gamma 2 = 1$. Define $\alpha : W_r \times W_r \rightarrow [0, \infty)$ by

$$\alpha(\gamma_r, \zeta_r) = \begin{cases} 1 & \gamma_r, \zeta_r \in \{0, 1, 2\} \\ 0 & \text{otherwise} \end{cases} \tag{2.28}$$

Suppose that $F(t) = e^t$, $\varphi(t) = t$ and $\tau = \frac{1}{16}$. We show that conditions of Corollary 2.4 are satisfied. We have the following cases:

Case 1 $\gamma_r = 0, \zeta_r = 2$

Then $\sigma_{ml}(\Gamma 0, \Gamma 2) = \sigma_{ml}(0, 1) = \frac{3}{2} > 0$, and $M(0, 2) = \max\{\sigma_{ml}(0, 2), \sigma_{ml}(0, \Gamma 0), \sigma_{ml}(2, \Gamma 2)\} = \max\{\sigma_{ml}(0, 2), \sigma_{ml}(0, 0), \sigma_{ml}(2, 1)\} = 3$

$$\begin{aligned} \tau + F(\alpha(0, 2)\sigma_{ml}(\Gamma 0, \Gamma 2)) &= \frac{1}{16} + F\left(\frac{3}{2}\right) \\ &= \frac{1}{16} + e^{\frac{3}{2}} \\ &\leq e^3 = F(M(0, 2)) \end{aligned}$$

Case 2 $\gamma_r = 1, \zeta_r = 2$

Then $\sigma_{ml}(\Gamma 1, \Gamma 2) = \sigma_{ml}(0, 1) = \frac{3}{2} > 0$, and $M(1, 2) = \max\{\sigma_{ml}(1, 2), \sigma_{ml}(1, \Gamma 1), \sigma_{ml}(2, \Gamma 2)\} = \max\{\sigma_{ml}(1, 2), \sigma_{ml}(1, 0), \sigma_{ml}(2, 1)\} = 2$

$$\begin{aligned} \tau + F(\alpha(1, 2)\sigma_{ml}(\Gamma 1, \Gamma 2)) &= \frac{1}{16} + F\left(\frac{3}{2}\right) \\ &= \frac{1}{16} + e^{\frac{3}{2}} \\ &\leq e^2 = F(M(1, 2)) \end{aligned}$$

Case 3 $\gamma_r = 2, \zeta_r = 2$

Then $\sigma_{ml}(\Gamma 2, \Gamma 2) = \sigma_{ml}(1, 1) = 1 > 0$, and

$$M(2, 2) = \max\{\sigma_{ml}(2, 2), \sigma_{ml}(2, \Gamma 2), \sigma_{ml}(2, \Gamma 2)\} = \max\{\sigma_{ml}(2, 2), \sigma_{ml}(2, 1), \sigma_{ml}(2, 1)\} = 2$$

$$\begin{aligned} \tau + F(\alpha(2, 2)\sigma_{ml}(\Gamma 2, \Gamma 2)) &= \frac{1}{16} + F(1) \\ &= \frac{1}{16} + e \\ &\leq e^2 = F(M(2, 2)). \end{aligned}$$

Therefore, it satisfies the condition of Corollary 2.4. Hence Γ has a fixed point, which in this case is 0.

3. Conclusion

In this article, we presented the generalized integral type $\alpha - F$ contraction mappings in complete metric-like spaces and established some fixed point results for such mappings. We also provided some consequences of established results and examples.

References

- [1] Alam K. H., Rohen Y. and Saleem N., *Fixed points of (α, β, F^*) and (α, β, F^{**}) weak Geraghty contractions with an application*, Symmetry, **15** (2023). <https://doi.org/10.3390/sym15010243>
- [2] Amini-Harandi A., *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl., (2012), 2012:204. <https://doi.org/10.1186/1687-1812-2012-204>
- [3] Banach S., *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fund. Math., **3** (1922), 133-181.
- [4] Beloul S., *A common fixed point theorem for generalized almost contractions in metric-like spaces*, Applied Mathematics E-Notes, **18** (2018), 127-139.
- [5] Bouhadjera H., *Fixed point results in incomplete metric and dislocated metric spaces*, Annals of West University of Timisoara Mathematics and Computer Science **59** (2023), 130-150. <https://doi.org/10.2478/awutm-2023-0011>
- [6] Bouhadjera H., Saifia O. and Ouldji S., *Unique common fixed points in metric-like space*, General Mathematics **30** (2), (2022), 17-28. <https://doi.org/10.2478/gm-2022-0011>
- [7] Branciari A., *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Inter. J. Math. Math. Sci., **29** (9) (2002), 531-536. <https://doi.org/10.1155/S01611712020075241>
- [8] Hammad H. A., Bota M. F. and Guran L., *Wardowski's contraction and fixed point technique for solving systems of functional and integral equations*, Journal of Function Spaces, Vol. 2021, Article ID 7017046. <https://doi.org/10.1155/2021/7017046>
- [9] Karapinar E., Kumam P. and Salimi P., *On $\alpha - \psi$ -Meir-Keeler contractive mappings*, Fixed Point Theory Appl., 2013 (2013). <https://doi.org/10.1186/1687-1812-2013-94>
- [10] Karapinar E., and Salimi P., *Dislocated metric space to metric spaces with some fixed point theorems*, Fixed Point Theory and Applications, vol. 2013, (2013), article 222. <https://doi.org/10.1186/1687-1812-2013-222>
- [11] Matthews S. G., *Partial metric topology*, Research report 212, Department of Computer Science, University of Warwick, (1992).
- [12] Matthews S. G., *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, 728 (1994), 183-197.
- [13] Mebawondu A. A. and Mewomo O. T., *Application of fixed point results for modified generalized F-contraction mappings to solve boundary value problems*, PanAmerican Mathematical Journal, **4** (29), (2019), 45 - 68.

- [14] Mohammed S. S., Al-Ansari M., Azam A. and Kanwal S., *Fixed points of (φ, F) -weak contractions on metric-like spaces with applications to integral equations on time scales*, Bol. Soc. Mat. Mexicana, **27** (2) (2021), 1-21.
- [15] Piri H. and Kumam P., *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory Appl., 2014 (2014). <https://doi.org/10.1186/1687-1812-2014-210>
- [16] Qawaqneh H., Noorani M., Shatanawi W. and Alsamir H., *Common fixed point theorems for generalized Geraghty (σ, ψ, φ) -quasi contraction type mapping in partially ordered metric-like spaces*, Axioms **7** (4) (2018). <https://doi.org/10.3390/axioms7040074>
- [17] Radenovic S., Mirkov N. and Paunovic L. R., *Some new results on F-contractions in 0-complete partial metric spaces and 0-complete metric-like spaces*, Frac. Fract., **5** (2) (2021). <https://doi.org/10.3390/fractalfract5020034>
- [18] Rhoda C., Shagari M. S. and Balarabe M., *Advancements in metric-like spaces with related fixed point results*, Mathematical Analysis and its Contemporary Applications **5** (1), (2023), 1–32. <https://doi.org/10.30495/maca.2022.1967959.1064>
- [19] Saleh A., Al-Mezel, Chen C. M. , Karapinar E. and Rakocevic V., *Fixed point results for various α -admissible contractive mappings on metric-like spaces*, Abstract and Applied Analysis, Vol. 2014, (2013), Article ID 379358. <https://doi.org/10.1155/2014/379358>
- [20] Samet B., Vetro C. and Vetro P., *Fixed point theorem for $\alpha - \psi$ contractive type mappings*, Nonlinear Anal., **75**(4) (2012), 2154-2165. <https://doi.org/10.1016/j.na.2011.10.014>
- [21] Shatanawi W. , Mlaiki N., Rizk D. and Onunwor E, *Fredholm-type integral equation in controlled metric-like spaces*, Adv. Diff. Eq., **2021** (1), 1-13. <https://doi.org/10.1186/s13662-021-03516-4>
- [22] Vujakovic J., Mitrovic S., Mitrovic Z. D. and Radenovic S., *On F-contractions for Weak α -admissible Mappings in metric-like spaces*, Mathematics, (2020), 1-12. <https://doi.org/10.3390/math8091629>
- [23] Wang M., Saleem N., Liu X. Ansari A. H. and Zhou M., *Fixed point of (α, β) -admissible generalized Geraghty F-contraction with application*, Symmetry, **15** (2022). <https://doi.org/10.3390/sym14051016>
- [24] Wardowski D., *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., 2012 (2012). <https://doi.org/10.1186/1687-1812-2012-94>
- [25] Wardowski D. and Dung N. V., *Fixed points of F-weak contractions on complete metric spaces*, Demonstr. Math., **47** (1) (2014), 146–155. <https://doi.org/10.2478/dema-2014-0012>