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Fixed points of generalized integral type $\alpha - F$ contraction mappings in metric-like spaces

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Abstract

This article focuses on generalized integral type $\alpha - F$ contraction mappings in metric-like spaces and certain fixed point results in this setting. We also present some examples to support the validity of the results.

Keywords: Generalized integral type, $\alpha - F$ contraction mapping, F-contraction mapping, α-admissible mapping, metric-like spaces.
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1. Introduction

Contraction mappings play a crucial role in fixed point theory, solving existence problems across various disciplines of mathematics. Initially, Banach [3] established the classical contraction principle, which ensures the existence and uniqueness of fixed points. Because of its use, it has been generalized by employing other transformation types and modifying the structure of the space.

In 2002, Branciari [7], introduced the integral contraction as follows.

Theorem 1.1. *Let (W_r, d) be a complete metric space, $k \in (0,1)$ and let $\Gamma : W_r \to W_r$ be a mapping such that for each $\gamma_r, \zeta_r \in W_r$*

$$\int_0^{d(\Gamma\gamma_r,\Gamma\zeta_r)} \varphi(t)\, dt \leqslant k \int_0^{d(\gamma_r,\zeta_r)} \varphi(t)\, dt$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable map which is summable, (i.e., with finite integral) on each compact subset of $[0,\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\, dt > 0$, then Γ has a unique fixed point.

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We begin by recalling a few definitions and lemmas.

Matthews [11, 12] introduced the concept of partial metric space (PMS) as follows:

Definition 1.2. Let W_r be a non-empty set. A function $\sigma_{ml} : W_r \times W_r \to [0, \infty)$ is said to be a partial metric on W_r if the following conditions hold:

(PMS1) $\gamma_r = \zeta_r \Leftrightarrow \rho(\gamma_r, \gamma_r) = \rho(\zeta_r, \zeta_r) = \rho(\gamma_r, \zeta_r)$;
(PMS2) $\rho(\gamma_r, \gamma_r) \leqslant \rho(\gamma_r, \zeta_r)$;
(PMS3) $\rho(\gamma_r, \zeta_r) = \rho(\zeta_r, \gamma_r)$;
(PMS4) $\rho(\gamma_r, \zeta_r) \leqslant \rho(\gamma_r, \eta_r) + \rho(\eta_r, \zeta_r) - \rho(\eta_r, \eta_r)$;

for all $\gamma_r, \zeta_r, \eta_r \in W_r$. The set W_r equipped with the metric σ_{ml} defined above is called a partial metric space and it is denoted by (W_r, ρ) (in short PMS). Each partial metric ρ on W_r generates a Γ_ρ topology τ_ρ on W_r, which has a base of the family of open σ_{ml}-balls

$$\{B_\rho(\gamma_r, \epsilon) : \gamma_r \in W_r, \epsilon > 0\},$$

where

$$B_\rho(\gamma_r, \epsilon) = \{\zeta_r \in W_r : |\rho(\gamma_r, \zeta_r) - \rho(\gamma_r, \gamma_r)| < \epsilon\},$$

for all $\gamma_r \in W_r$ and $\epsilon > 0$.

Harandi [2] introduced metric-like spaces as follows:

Definition 1.3. Let W_r be a non-empty set. A function $\sigma_{ml} : W_r \times W_r \to [0, \infty)$ is said to be a metric-like on W_r if the following conditions hold:

(MLS1) $\sigma_{ml}(\gamma_r, \zeta_r) = 0 \Rightarrow \gamma_r = \zeta_r$;
(MLS2) $\sigma_{ml}(\gamma_r, \zeta_r) = \sigma_{ml}(\zeta_r, \gamma_r)$;
(MLS3) $\sigma_{ml}(\gamma_r, \zeta_r) \leqslant \sigma_{ml}(\gamma_r, \eta_r) + \sigma_{ml}(\eta_r, \zeta_r)$;

for all $\gamma_r, \zeta_r, \eta_r \in W_r$. Then (W_r, σ_{ml}) is called metric-like space. Each metric-like σ_{ml} on W_r generates a $\Gamma_{\sigma_{ml}}$ topology $\tau_{\sigma_{ml}}$ on W_r, which has a base of the family of open σ_{ml}-balls

$$\{B_{\sigma_{ml}}(\gamma_r, \epsilon) : \gamma_r \in W_r, \epsilon > 0\},$$

where

$$B_{\sigma_{ml}}(\gamma_r, \epsilon) = \{\zeta_r \in W_r : |\sigma_{ml}(\gamma_r, \zeta_r) - \sigma_{ml}(\gamma_r, \gamma_r)| < \epsilon\},$$

for all $\gamma_r \in W_r$ and $\epsilon > 0$.

Example 1.4. [2] Let $W_r = \{0, 1\}$ and define

$$\sigma_{ml}(\gamma_r, \zeta_r) = \begin{cases} 2 & \gamma_r = \zeta_r = 0 \\ 1 & \text{otherwise} \end{cases} \tag{1.1}$$

Then (W_r, σ_{ml}) is metric-like space but since $\sigma_{ml}(0, 0) \nleq \sigma_{ml}(0, 1)$, (W_r, σ_{ml}) is not a partial metric space.

Lemma 1.5. *[4] Let* (W_r, σ_{ml}) *be a metric-like space.*

(a) *A sequence* $\{\gamma_{r_n}\}$ *in* (W_r, σ_{ml}) *converges to a point* $\gamma_r \in W_r \Leftrightarrow$

$$\sigma_{ml}(\gamma_r, \gamma_r) = \lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_r),$$

(b) *A sequence* $\{\gamma_{r_n}\}$ *in* (W_r, σ_{ml}) *is a* σ_{ml}*-Cauchy sequence if* $\lim_{m,n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_m})$ *exists and finite,*

(c) (W_r, σ_{ml}) *is complete if every* σ_{ml}*-Cauchy sequence* $\{\gamma_{r_n}\}$ *in* W_r *converges to a point* $\gamma_r \in W_r$*, such that*

$$\sigma_{ml}(\gamma_r, \gamma_r) = \lim_{m,n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_m}) = \lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_r).$$

(d) *A mapping* $\Gamma : W_r \to W_r$ *is continuous, if following limit exists and*

$$\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_r) = \lim_{n\to\infty}\sigma_{ml}(\Gamma\gamma_r, \gamma_r).$$

Karapınar and Salimi [10] demonstrated the following key features in metric-like spaces:

Lemma 1.6. *Let* (W_r, σ_{ml}) *be a metric-like space.Then*

1. $\sigma_{ml}(\gamma_r, \zeta_r) = 0,\ \sigma_{ml}(\gamma_r, \gamma_r) = \sigma_{ml}(\zeta_r, \zeta_r) = 0,$
2. *If* $\{\gamma_{r_n}\}$ *is a sequence such that* $\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = 0,$

$$\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_n}) = \lim_{n\to\infty} \sigma_{ml}(\gamma_{r_{n+1}}, \gamma_{r_{n+1}}) = 0,$$

3. *if* $\gamma_r \neq \zeta_r$*, Then* $\sigma_{ml}(\gamma_r, \zeta_r) > 0,$
4. $\sigma_{ml}(\gamma_r, \gamma_r) \leqslant \frac{2}{n} \sum_{j=1}^{n} \sigma_{ml}(\gamma_r, \gamma_{r_j}),$

for all $\gamma_r, \gamma_{r_j} \in W_r$ *where* $1 \leqslant j \leqslant n$.

Lemma 1.7. *[19] Assume that* $\gamma_{r_n} \to \eta_r$ *as* $n \to \infty$ *in a metric-like space* (W_r, σ_{ml}) *such that* $\sigma_{ml}(\eta_r, \eta_r) = 0$*. Then* $\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \zeta_r) = \sigma_{ml}(\eta_r, \zeta_r)$ *for every* $\zeta_r \in W_r$.

Lemma 1.8. *[22] If* $\{\gamma_{r_n}\}$ *with* $\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = 0$ *is not a Cauchy sequence in metric-like space* (W_r, σ_{ml})*, and two sequences* $\{n(j)\}$ *and* $\{m(j)\}$ *of positive integers such that* $n(j) > m(j) > j$*, then following four sequences*

$$\sigma_{ml}(\gamma_{r_{m(j)}}, \gamma_{r_{n(j)+1}}), \sigma_{ml}(\gamma_{r_{m(j)}}, \gamma_{r_{n(j)}}),$$
$$\sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{n(j)+1}}), \sigma_{ml}(\gamma_{r_{m(j)-1}}, \gamma_{r_{n(j)}})$$

tend to $\mu_r^+ > 0$ *when* $j \to \infty$.

In recent years, numerous authors have established fixed point or common fixed point theorems in metric-like spaces, as seen in [19, 18, 21, 17, 14, 16, 5, 6].

In 2012, Samet et al. [20] introduced α-admissible mapping as follows:

Definition 1.9. Let $\Gamma : W_r \to W_r$ and $\alpha : W_r \times W_r \to [0, \infty)$. Γ is said to be α-admissible if

$$\alpha(\gamma_r, \zeta_r) \geqslant 1 \Rightarrow \alpha(\Gamma\gamma_r, \Gamma\zeta_r) \geqslant 1,$$

for all $\gamma_r, \zeta_r \in W_r$.

Further, Karapinar et al. [9], presented triangular α-admissible as follows:

Definition 1.10. Let $\Gamma : W_r \to W_r$ and $\alpha : W_r \times W_r \to [0, \infty)$ be functions. Then Γ is said to be triangular α-admissible if Γ is α-admissible and for $\gamma_r, \zeta_r, \eta_r \in W_r$, $\alpha(\gamma_r, \eta_r) \geqslant 1$ and $\alpha(\eta_r, \zeta_r) \geqslant 1 \Rightarrow \alpha(\gamma_r, \zeta_r) \geqslant 1$.

Lemma 1.11. *[9] Let $\Gamma : W_r \to W_r$ be triangular α-admissible mapping. Assume that there exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geqslant 1$. Define a sequence $\{\gamma_{r_n}\}$ by $\gamma_{r_{n+1}} = \Gamma\gamma_{r_n}$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(\gamma_{r_m}, \gamma_{r_n}) \geqslant 1$ for all $m, n \in \mathbb{N}$ with $m > n$.*

Wardowski [24, 25] introduced new class of contraction mappings as follows:

Definition 1.12. Let $\mathcal{F}$ be family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

(F1) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}^+$ if $a < b$ then $F(a) < F(b)$;
(F2) for each sequence $\{a_n\}$ of positive numbers, $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} F(a_n) = -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

Wardowski [24] defined F-contraction as follows:

Let (W_r, d) be a metric space, then the mapping $\Gamma : W_r \to W_r$ is said to be an F-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $\gamma_r, \zeta_r \in W_r$ with $d(\Gamma\gamma_r.\Gamma\zeta_r) > 0$ we have

$$\tau + F(d(\Gamma\gamma_r, \Gamma\zeta_r)) \leqslant F(d(\gamma_r, \zeta_r))$$

Piri and Kumam [15] extended Wardowski's [24] results by modifying the condition *(F3)* in Definition 1.12 as follows:

Definition 1.13. Let $\Delta_{\mathfrak{F}}$ be family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

(i) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}^+$ if $a < b$ then $F(a) < F(b)$;
(ii) for each sequence $\{a_n\}$ of positive numbers; $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} F(a_n) = -\infty$;
(iii) F is continuous on $(0, \infty)$.

Various authors have generalized Wardowski's result (refer to [13, 1, 8, 23]).

2. Main Results

Let Φ be the family of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that φ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and for each $\epsilon > 0$,

$$\int_0^\epsilon \varphi(t)\, dt > 0.$$

Definition 2.1. Let (W_r, σ_{ml}) be a metric-like space and let $\Gamma : W_r \to W_r$ be a self map.Then Γ is said to be generalized integral type $\alpha - F$-contractive mapping if there exist two functions $\alpha : W_r \times W_r \to [0, \infty)$ and $F \in \Delta_{\mathfrak{F}}$ such that for $\tau > 0$ with $\sigma_{ml}(\Gamma\gamma_r.\Gamma\zeta_r) > 0$,

$$\tau + F\left(\alpha(\gamma_r, \zeta_r) \int_0^{\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r)} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{M(\gamma_r, \zeta_r)} \varphi(t)\, dt\right), \tag{2.1}$$

where $\varphi \in \Phi$ and

$$M(\gamma_r, \zeta_r) = \max\{\sigma_{ml}(\gamma_r, \zeta_r), \sigma_{ml}(\gamma_r, \Gamma\gamma_r), \sigma_{ml}(\zeta_r, \Gamma\zeta_r)\}.$$

Theorem 2.2. *Let (W_r, σ_{ml}) be a complete metric-like space and $\Gamma : W_r \to W_r$ be a self map. Suppose $\alpha : W_r \times W_r \to [0, \infty)$ be the mapping satisfying the conditions:*

(i) Γ *is triangular α-admissible mapping;*
(ii) Γ *is generalized integral type $\alpha - F$-contractive mapping;*
(iii) *There exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geqslant 1$;*
(iv) Γ *is continuous.*

Then Γ has a fixed point in W_r.

Proof. Let γ_{r_0} be an arbitrary point where $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geqslant 1$. Consider a sequence $\{\gamma_{r_n}\}$ in W_r such that $\gamma_{r_{n+1}} = \Gamma\gamma_{r_n}$ for all $n \in \mathbb{N}$. If $\gamma_{r_n} = \gamma_{r_{n+1}}$ for some $n \in \mathbb{N}$, then γ_{r_n} is a fixed point of Γ and the existence part of the proof is complete. Assume $\gamma_{r_n} \neq \gamma_{r_{n+1}}$ for all $n \in \mathbb{N}$, then $\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) = \sigma_{ml}(\Gamma\gamma_{r_{n-1}}, \Gamma\gamma_{r_n}) > 0$ by lemma 1.6. Now, since Γ is α-admissible, so

$$\alpha(\Gamma\gamma_{r_0}, \Gamma\gamma_{r_1}) = \alpha(\gamma_{r_1}, \gamma_{r_2}) \geqslant 1,$$
$$\alpha(\Gamma\gamma_{r_1}, \Gamma\gamma_{r_2}) = \alpha(\gamma_{r_2}, \gamma_{r_3}) \geqslant 1$$

and using induction we have $\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \geqslant 1$ for all $n \in \mathbb{N}$.

Now, by (2.1) we get

$$\begin{aligned} \tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t)\, dt\right) &\leqslant \tau + F\left(\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t)\, dt\right) \\ &= \tau + F\left(\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n-1}}, \Gamma\gamma_{r_n})} \varphi(t)\, dt\right) \\ &\leqslant F\left(\int_0^{M(\gamma_{r_{n-1}}, \gamma_{r_n})} \varphi(t)\, dt\right), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(\gamma_{r_{n-1}}, \gamma_{r_n}) &= \max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_{n-1}}, \Gamma\gamma_{r_{n-1}}), \sigma_{ml}(\gamma_{r_n}, \Gamma\gamma_{r_n})\} \\ &= \max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\} \\ &= \max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\}. \end{aligned} \tag{2.3}$$

Now, using (2.3) in (2.2) we get that

$$\tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{\max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\}} \varphi(t)\, dt\right). \tag{2.4}$$

Now, if $\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}}) > \sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n})$, then a contradiction follows from

$$\tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t)\, dt\right).$$

Thus, we conclude that

$$\max\{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}), \sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})\} = \sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n}). \tag{2.5}$$

Therefore, From (2.4) we get that

$$F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n})} \varphi(t)\, dt\right) - \tau. \tag{2.6}$$

Continuing in the same way, we obtain

$$F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n-1}})} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_{n-2}})} \varphi(t)\, dt\right) - \tau. \tag{2.7}$$

Using (2.7) in (2.6) we get that

$$\begin{aligned} F\left(\int_0^{\sigma_{ml}(\gamma_{r_n}, \gamma_{r_{n+1}})} \varphi(t)\, dt\right) &\leqslant F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_n})} \varphi(t)\, dt\right) - \tau \\ &\leqslant F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n-1}}, \gamma_{r_{n-2}})} \varphi(t)\, dt\right) - 2\tau. \end{aligned}$$

On generalizing

$$F\left(\int_0^{\sigma_{ml}(\gamma_{r_n},\gamma_{r_{n+1}})} \varphi(t)\,dt\right) < F\left(\int_0^{\sigma_{ml}(\gamma_{r_0},\gamma_{r_1})} \varphi(t)\,dt\right) - n\tau. \tag{2.8}$$

Letting the limit $n \to \infty$ in (2.8) and using the definition of F we get

$$\lim_{n\to\infty} F\left(\int_0^{\sigma_{ml}(\gamma_{r_n},\gamma_{r_{n+1}})} \varphi(t)\,dt\right) = -\infty \Leftrightarrow \lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n},\gamma_{r_{n+1}}) = 0. \tag{2.9}$$

consequently, we get

$$\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n},\gamma_{r_{n+1}}) = 0. \tag{2.10}$$

Now, we prove that the sequence $\{\gamma_{r_n}\}$ is a σ_{ml}-Cauchy sequence in W_r by supposing the contrary, i.e. $\lim_{n,m\to\infty} \sigma_{ml}(\gamma_{r_n},\gamma_{r_m}) \neq 0$.

Then sequences in lemma 1.8 tends to $\mu_r^+ > 0$, when $j \to \infty$.
So, we have

$$\lim_{j\to\infty} \sigma_{ml}(\gamma_{r_{n(j)}},\gamma_{r_{m(j)}}) = \mu_r^+. \tag{2.11}$$

Further corresponding to $m(j)$, we can choose $n(j)$ in such a way that it is smallest integer with $n(j) > m(j) > j$. Then

$$\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_{n(j)-1}},\gamma_{r_{m(j)}}) = \mu_r^+. \tag{2.12}$$

Again,

$$\sigma_{ml}(\gamma_{r_{m(j)-1}},\gamma_{r_{n(j)-1}}) \leqslant \sigma_{ml}(\gamma_{r_{m(j)-1}},\gamma_{r_{n(j)}}) + \sigma_{ml}(\gamma_{r_{n(j)}},\gamma_{r_{n(j)-1}}).$$

Letting $j \to \infty$ and using lemma 1.8 we get

$$\lim_{j\to\infty} \sigma_{ml}(\gamma_{r_{m(j)-1}},\gamma_{r_{n(j)-1}}) = \mu_r^+. \tag{2.13}$$

Now as Γ is triangular α admissible we have $\alpha(\sigma_{ml}(\gamma_{r_{n(j)}},\gamma_{r_{m(j)}})) \geqslant 1$, then replacing γ_r by $\gamma_{r_{n(j)}}$ and ζ_r by $\gamma_{r_{m(j)}}$ in (2.1) respectively, we get

$$\begin{aligned}\tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n(j)}},\gamma_{r_{m(j)}})} \varphi(t)\,dt\right) &\leqslant \tau + F\left(\alpha(\gamma_{r_{n(j)}},\gamma_{r_{m(j)}}) \int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n(j)-1}},\Gamma\gamma_{r_{m(j)-1}})} \varphi(t)\,dt\right),\\ &\leqslant F\left(\int_0^{M(\gamma_{r_{n(j)-1}},\gamma_{r_{m(j)-1}})} \varphi(t)\,dt\right),\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}M(\gamma_{r_{n(j)-1}},\gamma_{r_{m(j)-1}}) =& \max\{\sigma_{ml}(\gamma_{r_{n(j)-1}},\gamma_{r_{m(j)-1}}), \sigma_{ml}(\gamma_{r_{n(j)-1}},\Gamma\gamma_{r_{n(j)-1}}), \sigma_{ml}(\gamma_{r_{m(j)-1}},\Gamma\gamma_{r_{m(j)-1}})\}\\ =& \max\{\sigma_{ml}(\gamma_{r_{n(j)-1}},\gamma_{r_{m(j)-1}}), \sigma_{ml}(\gamma_{r_{n(j)-1}},\gamma_{r_{n(j)}}), \sigma_{ml}(\gamma_{r_{m(j)-1}},\gamma_{r_{m(j)}})\}.\end{aligned} \tag{2.15}$$

Letting $j \to \infty$ in (2.15) and using (2.10), (2.11), (2.12), (2.13) and lemma 1.8 we get

$$\lim_{j\to\infty} M(\gamma_{r_{n(j)-1}}, \gamma_{r_{m(j)-1}}) = \mu_r^+. \tag{2.16}$$

Now Letting $j \to \infty$ in (2.14) and using (2.16) we get

$$\tau + F\left(\int_0^{\mu_r^+} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{\mu_r^+} \varphi(t)\, dt\right).$$

Which is a contradiction.

This implies that $\{\gamma_{r_n}\}$ is a σ_{ml}-Cauchy sequence in (W_r, σ_{ml}). So, there exists $\eta_r \in W_r$ such that

$$\sigma_{ml}(\eta_r, \eta_r) = \lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \eta_r) = \lim_{m,n\to\infty} \sigma_{ml}(\gamma_{r_n}, \gamma_{r_m}) = 0. \tag{2.17}$$

Since, Γ is continuous, we get

$$\lim_{n\to\infty} \sigma_{ml}(\Gamma\gamma_{r_n}, \Gamma\eta_r) = \lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \eta_r) = \sigma_{ml}(\eta_r, \eta_r) = 0.$$

From Lemma 1.7, we also have

$$\lim_{n\to\infty} \sigma_{ml}(\gamma_{r_n}, \Gamma\eta_r) = \sigma_{ml}(\eta_r, \Gamma\eta_r). \tag{2.18}$$

Combining (2.17) and (2.18) and Lemma 1.6, we get that η_r is a fixed point of Γ i.e., $\Gamma\eta_r = \eta_r$. □

Theorem 2.3. *Let (W_r, σ_{ml}) be a complete metric-like space and $\Gamma : W_r \to W_r$ be a self map. Suppose $\alpha : W_r \times W_r \to [0, \infty)$ be the mapping satisfying the conditions:*

(i) *Γ is triangular α-admissible mapping;*
(ii) *Γ is integral type generalized $\alpha - F$-contractive mapping;*
(iii) *There exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geqslant 1$;*
(iv) *If $\{\gamma_{r_n}\}$ is a sequence in W_r such that $\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \geqslant 1$ for all n and $\gamma_{r_n} \to \eta_r \in W_r$ as $n \to \infty$, then there exists a subsequence $\gamma_{r_{n(i)}}$ of $\{\gamma_{r_n}\}$ such that $\alpha(\gamma_{r_{n(i)}}, \eta_r) \geqslant 1$ for all i.*

Then Γ has a fixed point in W_r. Further if η_r, η_s are fixed points of Γ with $\alpha(\eta_r, \eta_s) \geqslant 1$, then Γ has a unique fixed point in W_r.

Proof. From the proof of the Theorem 2.2, the sequence $\{\gamma_{r_n}\}$ defined by $\gamma_{r_{n+1}} = \Gamma\gamma_{r_n}$ is a Cauchy sequence in (W_r, σ_{ml}), as a result there exist $\eta_r \in W_r$ such that $\gamma_{r_n} \to \eta_r$. It is enough to show that $\eta_r \in W_r$ is the fixed point of Γ.

On contrary we suppose that $(\Gamma\eta_r, \eta_r) > 0$. Then from condition (iii) there exists a subsequence $\gamma_{r_{n(i)}}$ of $\{\gamma_{r_n}\}$ such that $\alpha(\gamma_{r_{n(i)}}, \eta_r) \geqslant 1$ for all i. By Using given contractive condition (2.1) for $\gamma_r = \gamma_{r_{n(i)}}$ and $\zeta_r = \eta_r$ and property of F we have

$$\begin{aligned}\tau + F\left(\int_0^{\sigma_{ml}(\gamma_{r_{n(i)+1}}, \Gamma\eta_r)} \varphi(t)\, dt\right) &= \tau + F\left(\int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n(i)}}, \Gamma\eta_r)} \varphi(t)\, dt\right) \\ &\leqslant \tau + F\left(\alpha(\gamma_{r_{n(i)}}, \eta_r)\int_0^{\sigma_{ml}(\Gamma\gamma_{r_{n(i)}}, \Gamma\eta_r)} \varphi(t)\, dt\right) \\ &\leqslant F\left(\int_0^{M(\gamma_{r_{n(i)}}, \eta_r)} \varphi(t)\, dt\right), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} M(\gamma_{r_{n(i)}}, \eta_r) &= \max\{\sigma_{ml}(\gamma_{r_{n(i)}}, \eta_r), \sigma_{ml}(\gamma_{r_{n(i)}}, \Gamma\gamma_{r_{n(i)}}), \sigma_{ml}(\eta_r, \Gamma\eta_r)\} \\ &= \max\{\sigma_{ml}(\gamma_{r_{n(i)}}, \eta_r), \sigma_{ml}(\gamma_{r_{n(i)}}, \gamma_{r_{n(i)+1}}), \sigma_{ml}(\eta_r, \Gamma\eta_r)\}. \end{aligned} \tag{2.20}$$

Letting $i \to \infty$ in (2.20) and taking (2.18) into account we get that

$$\lim_{i\to\infty} M(\gamma_{r_{n(i)}}, \eta_r) = \sigma_{ml}(\eta_r, \Gamma\eta_r). \tag{2.21}$$

Now, Letting $i \to \infty$ in (2.19) and using (2.21) and the continuity of F we get that

$$\tau + F\left(\int_0^{\sigma_{ml}(\eta_r, \Gamma\eta_r)} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{\sigma_{ml}(\eta_r, \Gamma\eta_r)} \varphi(t)\, dt\right),$$

which is a contradiction since $\tau > 0$, Thus we have $\Gamma\eta_r = \eta_r$. This shows that η_r is a fixed point of Γ. Further, suppose η_r and η_s be two fixed points of Γ such that $\sigma_{ml}(\eta_r, \eta_s) > 0$. From (2.1) we have

$$\begin{aligned}\tau + F\left(\int_0^{\sigma_{ml}(\eta_r, \eta_s)} \varphi(t)\, dt\right) &= \tau + F\left(\int_0^{\sigma_{ml}(\Gamma\eta_r, \Gamma\eta_s)} \varphi(t)\, dt\right) \\ &\leqslant \tau + F\left(\alpha(\eta_r, \eta_s)\int_0^{\sigma_{ml}(\Gamma\eta_r, \Gamma\eta_s)} \varphi(t)\, dt\right) \\ &\leqslant F\left(\int_0^{M(\eta_r, \eta_s)} \varphi(t)\, dt\right), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} M(\eta_r, \eta_s) &= \max\{\sigma_{ml}(\eta_r, \eta_s), \sigma_{ml}(\eta_r, \Gamma\eta_r), \sigma_{ml}(\eta_s, \Gamma\eta_s)\} \\ &= \max\{\sigma_{ml}(\eta_r, \eta_s), \sigma_{ml}(\eta_r, \eta_r), \sigma_{ml}(\eta_s, \eta_s)\} \\ &= \sigma_{ml}(\eta_r, \eta_s). \end{aligned} \tag{2.23}$$

putting (2.23) in (2.22) we get

$$\tau + F\left(\int_0^{\sigma_{ml}(\eta_r, \eta_s)} \varphi(t)\, dt\right) \leqslant F\left(\int_0^{\sigma_{ml}(\eta_r, \eta_s)} \varphi(t)\, dt\right), \tag{2.24}$$

which is a contradiction. Hence Γ has a unique fixed point. This completes the proof. □

Following are consequences of the theorems.

Corollary 2.4. *Let (W_r, σ_{ml}) be a complete metric-like space and $\Gamma : W_r \to W_r$ be a self map. Suppose there exist two functions $\alpha : W_r \times W_r \to [0, \infty)$ and $F \in \Delta_{\mathfrak{F}}$ such that for $\tau > 0$ with $\sigma_{ml}(\Gamma\gamma_r.\Gamma\zeta_r) > 0$ and satisfying the conditions:*

(i) *Γ is triangular α-admissible mapping;*
(ii) *Γ is generalized $\alpha - F$-contractive mapping i.e.*

$$\tau + F\Big(\alpha(\gamma_r, \zeta_r)(\Gamma\gamma_r, \Gamma\zeta_r)\Big) \leqslant F\Big(M(\gamma_r, \zeta_r)\Big), \tag{2.25}$$

where

$$M(\gamma_r, \zeta_r) = \max\{\sigma_{ml}(\gamma_r, \zeta_r), \sigma_{ml}(\gamma_r, \Gamma\gamma_r), \sigma_{ml}(\zeta_r, \Gamma\zeta_r)\};$$

(iii) *There exists $\gamma_{r_0} \in W_r$ such that $\alpha(\gamma_{r_0}, \Gamma\gamma_{r_0}) \geqslant 1$;*
(iv) *Γ is continuous or if $\{\gamma_{r_n}\}$ is a sequence in W_r such that $\alpha(\gamma_{r_n}, \gamma_{r_{n+1}}) \geqslant 1$ for all n and $\gamma_{r_n} \to \eta_r \in W_r$ as $n \to \infty$, then there exists a subsequence $\gamma_{r_{n(i)}}$ of $\{\gamma_{r_n}\}$ such that $\alpha(\gamma_{r_{n(i)}}, \eta_r) \geqslant 1$ for all i.*

Then Γ has a fixed point in W_r.

Corollary 2.5. *Let (W_r, σ_{ml}) be a complete metric-like space and let $\Gamma : W_r \to W_r$ be a continuous self map. Suppose that there exists $k \in (0, 1)$ such that*

$$\int_0^{\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r)} \varphi(t)\, dt \leqslant k \int_0^{\sigma_{ml}(\gamma_r, \zeta_r)} \varphi(t)\, dt \tag{2.26}$$

and $\varphi \in \Phi$. Then Γ has a unique fixed point in W_r.

Example 2.6. Let $W_r = [0, 1]$ and define $\sigma_{ml} : W_r \times W_r \to \mathbb{R}^+$ by $\sigma_{ml}(\gamma_r, \zeta_r) = \max\{\gamma_r, \zeta_r\}$. Then (W_r, σ_{ml}) is a complete metric-like space. Consider the mapping $\Gamma : W_r \to W_r$ defined by $T(\eta_r) = \frac{\eta_r}{4}$. Suppose that $\varphi(t) = 2t$. Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(a) = \ln a$ for all $a \in \mathbb{R}^+ > 0$ and $\alpha : W_r \times W_r \to [0, \infty)$ by $\alpha(\gamma_r, \zeta_r) = 4$ for all $\gamma_r, \zeta_r \in W_r$.
We show that contractive conditions of Theorem 2.2 are satisfied. Let $\gamma_r, \zeta_r \in W_r$, without loss of generality we assume that $\gamma_r \geqslant \zeta_r$. Suppose that $\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r) > 0$ and let $\tau = \ln(2)$, then

$$\begin{aligned}
\tau + F\Big(\alpha(\gamma_r, \zeta_r) \int_0^{\sigma_{ml}(\Gamma\gamma_r, \Gamma\zeta_r)} \varphi(t)\, dt\Big) &= \tau + F\Big(4 \int_0^{\sigma_{ml}(\frac{\gamma_r}{4}, \frac{\zeta_r}{4})} 2t\, dt\Big) \\
&= \tau + F(\frac{\gamma_r^2}{4}) \\
&= \ln(2) + \ln(\frac{\gamma_r^2}{4}) = \ln(\frac{\gamma_r^2}{2}) \\
&\leqslant \ln(\gamma_r^2) = F(\gamma_r^2) = F\Big(\int_0^{M(\gamma_r, \zeta_r)} \varphi(t)\, dt\Big).
\end{aligned} \tag{2.27}$$

Hence Γ has a fixed point, which in this case is 0.

Example 2.7. Let $W_r = \{0,1,2\}$ and let $\sigma_{ml} : W_r \times W_r \to \mathbb{R}^+$ be a metric-like function defined by

$$\begin{aligned}
&\sigma_{ml}(0,0) = \sigma_{ml}(2,2) = 0, \sigma_{ml}(1,1) = 1\\
&\sigma_{ml}(1,2) = \sigma_{ml}(2,1) = 2\\
&\sigma_{ml}(2,0) = \sigma_{ml}(0,2) = 3\\
&\sigma_{ml}(0,1) = \sigma_{ml}(1,0) = \frac{3}{2}.
\end{aligned}$$

Then (W_r, σ_{ml}) is a complete metric-like space. Let $\Gamma : W_r \to W_r$ be defined by $\Gamma 0 = \Gamma 1 = 0$ and $\Gamma 2 = 1$. Define $\alpha : W_r \times W_r \to [0,\infty)$ by

$$\alpha(\gamma_r, \zeta_r) = \begin{cases} 1 & \gamma_r, \zeta_r \in \{0,1,2\} \\ 0 & \text{otherwise} \end{cases} \tag{2.28}$$

Suppose that $F(t) = e^t$, $\varphi(t) = t$ and $\tau = \frac{1}{16}$. We show that conditions of Corollary 2.4 are satisfied. We have the following cases:

Case 1 $\gamma_r = 0, \zeta_r = 2$
Then $\sigma_{ml}(\Gamma 0, \Gamma 2) = \sigma_{ml}(0,1) = \frac{3}{2} > 0$, and
$M(0,2) = \max\{\sigma_{ml}(0,2), \sigma_{ml}(0,\Gamma 0), \sigma_{ml}(2,\Gamma 2)\} = \max\{\sigma_{ml}(0,2), \sigma_{ml}(0,0), \sigma_{ml}(2,1)\} = 3$

$$\begin{aligned}
\tau + F(\alpha(0,2)\sigma_{ml}(\Gamma 0, \Gamma 2)) &= \frac{1}{16} + F\left(\frac{3}{2}\right)\\
&= \frac{1}{16} + e^{\frac{3}{2}}\\
&\leqslant e^3 = F(M(0,2))
\end{aligned}$$

Case 2 $\gamma_r = 1, \zeta_r = 2$
Then $\sigma_{ml}(\Gamma 1, \Gamma 2) = \sigma_{ml}(0,1) = \frac{3}{2} > 0$, and
$M(1,2) = \max\{\sigma_{ml}(1,2), \sigma_{ml}(1,\Gamma 1), \sigma_{ml}(2,\Gamma 2)\} = \max\{\sigma_{ml}(1,2), \sigma_{ml}(1,0), \sigma_{ml}(2,1)\} = 2$

$$\begin{aligned}
\tau + F(\alpha(1,2)\sigma_{ml}(\Gamma 1, \Gamma 2)) &= \frac{1}{16} + F\left(\frac{3}{2}\right)\\
&= \frac{1}{16} + e^{\frac{3}{2}}\\
&\leqslant e^2 = F(M(1,2))
\end{aligned}$$

Case 3 $\gamma_r = 2, \zeta_r = 2$
Then $\sigma_{ml}(\Gamma 2, \Gamma 2) = \sigma_{ml}(1,1) = 1 > 0$, and

$M(2,2) = \max\{\sigma_{ml}(2,2), \sigma_{ml}(2,\Gamma 2), \sigma_{ml}(2,\Gamma 2)\} = \max\{\sigma_{ml}(2,2), \sigma_{ml}(2,1), \sigma_{ml}(2,1)\} = 2$

$$\begin{aligned}\tau + F(\alpha(2,2)\sigma_{ml}(\Gamma 2, \Gamma 2)) &= \frac{1}{16} + F(1)\\ &= \frac{1}{16} + e\\ &\leqslant e^2 = F(M(2,2)).\end{aligned}$$

Therefore, it satisfies the condition of Corollary 2.4. Hence Γ has a fixed point, which in this case is 0.

3. Conclusion

In this article, we presented the generalized integral type $\alpha - F$ contraction mappings in complete metric-like spaces and established some fixed point results for such mappings. We also provided some consequences of established results and examples.

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