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Implicit solution for logistic Caputo-Fabrizio fractional differential equation with Allee effect

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Abstract

In the current research work, the fractional version of logistic differential model is studied with Allee effect which includes a threshold population. It is also studied with initial conditions. The definition of the fractional derivative utilised in this research is the Caputo-Fabrizio. In order to give the solution of the posed problem by an implicit representation, a method based on fractional derivative properties is used.

Keywords: logistic fractional differential equation, Implicit solution, Caputo-Fabrizio fractional derivative, Caputo-Fabrizio integral.

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1. Introduction

The Allee effect is a principle named after Walter Clyde Allee. This principle is based on the idea that the individuals are correlated within their society in order to survive and reproduce [1]. Therefore, there cannot be a positive growth rate of any population if its size is too small. For instance, some examples are presented by animals that hunt or defend themselves as a group [2]-[5].

The classic general population growth model equations studied with Allee effect are formalised by the generalisation of the known Verhulst's logistic ordinary differential equation [1].

$$x'(t) = -rx(t) \left[1 - \frac{x(t)}{\mathcal{T}}\right] \left[1 - \frac{x(t)}{\mathcal{K}}\right],$$

with $x(t)$ is the population size at time $t \geq 0$, \mathcal{T} is the critical threshold, \mathcal{K} is the carrying capacity and r is called the intrinsic growth rate.

The generalizations of integer order of differential equations to an arbitrary order are

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known as fractional differential equations [6]-[10]. The fractional calculus theory has been applied in different scientific fields. Among these applications we mention the fractional versions of the logistic differential equation which is presented below see [11]:

$${}^{\text{CF}}\partial_t^\alpha x(t) = x(t)[1 - x(t)],$$

where ${}^{\text{CF}}\partial_t^\alpha$ is the Caputo-Fabrizio fractional derivative operator of order $\alpha \in (0, 1)$.

This paper studies the fractional Verhulst logistic differential equation with Allet effect :

$${}^{\text{CF}}\partial_t^\alpha X(t) = -rX(t)\left[1 - \frac{X(t)}{\mathcal{J}}\right]\left[1 - \frac{X(t)}{\mathcal{K}}\right], \quad (1.1)$$

with initial condition

$$X(0) = X_0, \quad (1.2)$$

where $r, \mathcal{J}, \mathcal{K}$ and X_0 are positive constants with $\mathcal{J} < \mathcal{K}$, and ${}^{\text{CF}}\partial_t^\alpha$ is the operator of Caputo-Fabrizio derivative with $\alpha \in (0, 1)$.

2. Preliminaries

In this section we recall some results that we need.

Definition 2.1. [12] The Caputo-Fabrizio fractional derivative of $\varphi \in L^2([0, T])$ can be expressed as

$${}^{\text{CF}}\partial_t^{\alpha+1}\varphi(t) = \frac{1}{1-\alpha} \int_0^t e^{\frac{-\alpha(t-s)}{1-\alpha}} \varphi'(s) ds, \alpha \in [0, 1].$$

Definition 2.2. [12] For a function $\varphi : [a, +\infty) \rightarrow \mathcal{R}$ smooth, the Caputo-Fabrizio integral of order α , is given by

$${}^{\text{CF}}I_t^\alpha \varphi(t) = (1-\alpha)[\varphi(t) - \varphi(0)] + \alpha \int_0^t \varphi(s) ds,$$

with

$$\varphi(0) = \int_a^0 e^{\frac{-\alpha}{1-\alpha}s} \varphi'(s) ds.$$

Lemma 1. [11] Let $\varphi \in L^2([0, T])$. The solutions of differential equation

$${}^{\text{CF}}I_t^\alpha \varphi(t) = 0,$$

are only the constant functions.

Lemma 2. [11] The Caputo-Fabrizio fractional derivative relationship and the corresponding integral is given by

$${}^{\text{CF}}I_t^\alpha ({}^{\text{CF}}\partial_t^{\alpha+1}\varphi(t)) = \varphi(t) - \varphi(0), \quad t > 0, \alpha \in (0, 1).$$

Lemma 3. [13] Let us consider the following fractional differential equation:

$${}^{\text{CF}}I_t^\alpha \varphi(t) = \xi(t), \quad t > 0, \alpha \in (0, 1).$$

Then we have

$$\varphi(t) = (1-\alpha)[\xi(t) - \xi(0)] + \alpha \int_0^t \xi(s) ds + \varphi(0).$$

3. Solution of the fractional logistic differential equation

in this section we will start by proving this result about expressing the algebraic fraction as the sum of its partial fractions.

Lemma 4. Let $X(t) : D_X \subset \mathcal{R} \rightarrow \mathcal{R}$. For all constants \mathcal{T} and \mathcal{K} , we have

$$\frac{1}{X(t)\left[1 - \frac{X(t)}{\mathcal{T}}\right]\left[1 - \frac{X(t)}{\mathcal{K}}\right]} = \frac{1}{X(t)} + \frac{\mathcal{T}\mathcal{B}}{\mathcal{T} - X(t)} + \frac{\mathcal{K}\mathcal{C}}{\mathcal{K} - X(t)},$$

with

$$\mathcal{B} = \frac{\mathcal{K}}{\mathcal{K}\mathcal{T} - \mathcal{T}^2},$$

and

$$\mathcal{C} = \frac{1}{\mathcal{K}} + \frac{1}{\mathcal{T}} - \frac{\mathcal{K}}{\mathcal{K}\mathcal{T} - \mathcal{T}^2}.$$

Proof. Let A, B and C be constants, then the following equation

$$\frac{1}{X(t)\left[1 - \frac{X(t)}{\mathcal{T}}\right]\left[1 - \frac{X(t)}{\mathcal{K}}\right]} = \frac{A}{X(t)} + \frac{B}{1 - \frac{X(t)}{\mathcal{T}}} + \frac{C}{1 - \frac{X(t)}{\mathcal{K}}},$$

is equivalent to the systeme :

$$\begin{cases} A = 1, \\ B + C - \frac{1}{\mathcal{K}} - \frac{1}{\mathcal{T}} = 0, \\ \frac{A}{\mathcal{K}\mathcal{T}} - \frac{B}{\mathcal{K}} - \frac{C}{\mathcal{T}} = 0. \end{cases}$$

□

Therefore, we concluded this theorem :

Theorem 3.1. For all $r, \mathcal{T}, \mathcal{K}$ and X_0 positive constants with $\mathcal{T} < \mathcal{K}$, the Verhults logistic differential equation with Allee effect (1 – 2) have a solution given by an implicit form :

$$\frac{(X(t))^{\frac{1}{\alpha r}} (\mathcal{T}\mathcal{K}X(t) - (\mathcal{K} + \mathcal{T})X^2(t) + X^3(t))^{\frac{1-\alpha}{\alpha}}}{(\mathcal{T} - X(t))^{\frac{\mathcal{T}\mathcal{B}}{\alpha r}} (\mathcal{K} - X(t))^{\frac{\mathcal{K}\mathcal{C}}{\alpha r}}} = ke^{-t}, \quad (3.1)$$

with

$$k = \frac{(X_0^{\frac{1}{\alpha r}}) (\mathcal{T}\mathcal{K}X_0 - (\mathcal{K} + \mathcal{T})X_0^2 + X_0^3)^{\frac{1-\alpha}{\alpha}}}{(\mathcal{T} - X_0)^{\frac{\mathcal{T}\mathcal{B}}{\alpha r}} (\mathcal{K} - X_0)^{\frac{\mathcal{K}\mathcal{C}}{\alpha r}}}.$$

Proof. We first applying the Caputo-Fabrizio integral (${}^{CF}I_\alpha$) to (1), we obtain

$${}^{CF}I_t^\alpha ({}^{CF}\partial_t^\alpha X(t)) = ({}^{CF}I_t^\alpha \xi(t)), \quad (3.2)$$

with

$$\xi(t) = -rX(t)\left[1 - \frac{X(t)}{\mathcal{T}}\right]\left[1 - \frac{X(t)}{\mathcal{K}}\right].$$

By using lemme (3), we have

$$X(t) - X(0) = (1 - \alpha) [\xi(t) - \xi(0)] + \alpha \int_0^t \xi(s) ds. \quad (3.3)$$

Taking the first derivative of equation (5), we obtain

$$X'(t) = (1 - \alpha)\xi'(t) + \alpha\xi(t). \quad (3.4)$$

Replacing $\xi(t)$ in (6), then

$$\begin{aligned} X'(t) = & -(1 - \alpha)r \left[X'(t) - \frac{2X'(t)X(t)}{\mathcal{J}} - \frac{2X'(t)X(t)}{\mathcal{K}} + \frac{3X'(t)X^2(t)}{\mathcal{J}\mathcal{K}} \right] \\ & - \alpha r \left[X(t) - \frac{X^2(t)}{\mathcal{J}} - \frac{X^2(t)}{\mathcal{K}} + \frac{X^3(t)}{\mathcal{J}\mathcal{K}} \right], \end{aligned}$$

then

$$\begin{aligned} X'(t) + (1 - \alpha)r \left[X'(t) - \frac{2X'(t)X(t)}{\mathcal{J}} - \frac{2X'(t)X(t)}{\mathcal{K}} + \frac{3X'(t)X^2(t)}{\mathcal{J}\mathcal{K}} \right] \\ = -\alpha r \left[X(t) - \frac{X^2(t)}{\mathcal{J}} - \frac{X^2(t)}{\mathcal{K}} + \frac{X^3(t)}{\mathcal{J}\mathcal{K}} \right], \end{aligned}$$

which refers to

$$\begin{aligned} & \frac{X'(t)}{\alpha r \left[X(t) - \frac{X^2(t)}{\mathcal{J}} - \frac{X^2(t)}{\mathcal{K}} + \frac{X^3(t)}{\mathcal{J}\mathcal{K}} \right]} \\ & + \frac{1 - \alpha}{\alpha} \left(\frac{\mathcal{J}\mathcal{K}X'(t) - 2\mathcal{K}X'(t)X(t) - 2\mathcal{J}X'(t)X(t) + 3X'(t)X^2(t)}{\mathcal{J}\mathcal{K}X(t) - \mathcal{K}X^2(t) - \mathcal{J}X^2(t) + X^3(t)} \right) = -1. \quad (3.5) \end{aligned}$$

We use the lemme 4, we have

$$\begin{aligned} & \frac{1}{\alpha r} \left[\frac{X'(t)}{X(t)} + \frac{\mathcal{J}BX'(t)}{\mathcal{J} - X(t)} + \frac{\mathcal{K}CX'(t)}{\mathcal{K} - X(t)} \right] \\ & + \frac{1 - \alpha}{\alpha} \left(\frac{\mathcal{J}\mathcal{K}X'(t) - 2\mathcal{K}X'(t)X(t) - 2\mathcal{J}X'(t)X(t) + 3X'(t)X^2(t)}{\mathcal{J}\mathcal{K}X(t) - (\mathcal{K} + \mathcal{J})X^2(t) + X^3(t)} \right) = -1. \quad (3.6) \end{aligned}$$

Integrating both sides of equation (8), we have

$$\begin{aligned} & \frac{1}{\alpha r} [\ln|X(t)| - \mathcal{J}B \ln|\mathcal{J} - X(t)| - \mathcal{K}C \ln|\mathcal{K} - X(t)|] \\ & + \frac{1 - \alpha}{\alpha} (\ln|\mathcal{J}\mathcal{K}X(t) - \mathcal{K}X^2(t) - \mathcal{J}X^2(t) + X^3(t)|) = -t + c, \quad (3.7) \end{aligned}$$

then

$$\ln\left(|X(t)|^{\frac{1}{\alpha r}}\right) - \ln\left(|\mathcal{J} - X(t)|^{\frac{\mathcal{J}B}{\alpha r}}\right) - \ln\left(|\mathcal{K} - X(t)|^{\frac{\mathcal{K}C}{\alpha r}}\right)$$

$$+\ln(|\mathcal{J}\mathcal{K}\mathcal{X}(t) - (\mathcal{K} + \mathcal{J})\mathcal{X}^2(t) + \mathcal{X}^3(t)|^{\frac{1-\alpha}{\alpha}}) = -t + c,$$

or

$$\ln\left[\left(|\mathcal{X}(t)|^{\frac{1}{\alpha r}}\right)\left(|\mathcal{J}\mathcal{K}\mathcal{X}(t) - (\mathcal{K} + \mathcal{J})\mathcal{X}^2(t) + \mathcal{X}^3(t)|^{\frac{1-\alpha}{\alpha}}\right)\right]$$

$$-\ln\left[\left(|\mathcal{J} - \mathcal{X}(t)|^{\frac{\mathcal{J}\mathcal{B}}{\alpha r}}\right)\left(|\mathcal{K} - \mathcal{X}(t)|^{\frac{\mathcal{K}\mathcal{C}}{\alpha r}}\right)\right] = -t + c,$$

we obtain

$$\ln \frac{\left(|\mathcal{X}(t)|^{\frac{1}{\alpha r}}\right)\left(|\mathcal{J}\mathcal{K}\mathcal{X}(t) - (\mathcal{K} + \mathcal{J})\mathcal{X}^2(t) + \mathcal{X}^3(t)|^{\frac{1-\alpha}{\alpha}}\right)}{\left(|\mathcal{J} - \mathcal{X}(t)|^{\frac{\mathcal{J}\mathcal{B}}{\alpha r}}\right)\left(|\mathcal{K} - \mathcal{X}(t)|^{\frac{\mathcal{K}\mathcal{C}}{\alpha r}}\right)} = -t + c. \quad (3.8)$$

Hence, we have obtained the solution (3.1). \square

Example. We fixed the following parameters :

$$r = 1, \mathcal{J} = 2, \mathcal{K} = 3, \alpha = \frac{5}{2}.$$

Then, the problem(1-2) become as the following :

$${}^{\text{CF}}\partial_t^{\frac{5}{2}}\mathcal{X}(t) = -\mathcal{X}(t)\left[1 - \frac{\mathcal{X}(t)}{2}\right]\left[1 - \frac{\mathcal{X}(t)}{3}\right], \quad (3.9)$$

with initial condition :

$$\mathcal{X}(0) = 1. \quad (3.10)$$

Here, the values of B and C, wich is calculated in the lemma (4), are :

$$B = \frac{3}{2}, C = \frac{-2}{3}.$$

Finally, the implicit solution of this example is the next :

$$\mathcal{X}(t)^{\frac{2}{5}}(6\mathcal{X}(t) - 5\mathcal{X}^2(t) + \mathcal{X}^3(t)) = 2^{\frac{1}{5}}e^{-t}.$$

4. Conclusion

We have studied the fractional logistic model with Allee effect. The goal of this work was to prove that the technique of integration using partial fractions which is applicable to classical logistic model with Allee effect, is applicable to fractional model. Finally, we obtained implicit solution of fractional logistic model with Allee effect.

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