Journal of Fractional Calculus and Nonlinear Systems
jfcns.sabapub.com
ISSN : 2709-9547

J Frac Calc & Nonlinear Sys
(2023)4(1) : 17-34
doi:10.48185/jfcns.v4i1.753

Gohar Fractional Derivative: Theory and Applications

A. A. Gohar [†,*], M. S. Younes [†], S. B. Doma [†]

[†] Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria, Egypt

• *Received: 30 April 2023* • *Accepted: 12 May 2023* • *Published Online: 30 June 2023*

Abstract

The local fractional derivatives marked the beginning of a new era in fractional calculus. Due to their properties that have never been observed before in the field, they are able to fill in the gaps left by the nonlocal fractional derivatives and substantially increase the field's theoretical and applied potential. In this article, we introduce a new local fractional derivative that possesses some classical properties of the integer-order calculus, such as the product rule, the quotient rule, the linearity, and the chain rule. It meets the fractional extensions of Rolle's theorem and the mean value theorem and has more properties beyond those of previously defined local fractional derivatives. We reveal its geometric interpretation and physical meaning. We prove that a function can be differentiable in its sense without being classically differentiable. Moreover, we apply it to solve the Riccati fractional differential equations to demonstrate that it provides more accurate results with less error in comparison with the previously defined local fractional derivatives when applied to solve fractional differential equations. The numerical results obtained in this work by our local fractional derivative are shown to be in excellent agreement with those produced by other analytical and numerical methods such as the enhanced homotopy perturbation method (EHPM), the improved Adams-Bashforth-Moulton method (IABMM), the modified homotopy perturbation method (MHPM), the Bernstein polynomial method (BPM), the fractional Taylor basis method (FTBM), and the reproducing kernel method (RKM).

Keywords: Gohar Fractional Calculus. Gohar Fractional Derivative. Gohar Fractional Curves.

1. Introduction

It is generally agreed that fractional calculus can be traced back to a question that was asked by L'Hospital in his letters to Leibniz about the validity of extending the derivative of an integer order to assume fractional orders. Despite its pure origins, this field has lately come to be recognized for its great potential for accurately modelling a wide variety of physical phenomena [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] beyond what was previously possible with the classical integer-order calculus. As time goes on, however, the complexity of natural systems increases, necessitating the formulation and analysis of more general and accurate definitions of the fractional derivative. In this regard, several proposals

*Corresponding author: sci.AbdelrahmanAdel42194@alexu.edu.eg

for fractional derivatives have enriched the theoretical and applied potential of fractional calculus. Among the fractional derivatives introduced thus far, we report on those introduced by Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, and Riesz. The most commonly used definitions of the fractional derivative are those of Riemann-Liouville and Caputo [13], and they are defined as follows:

Definition 1.1. [13] The Riemann-Liouville derivative of fractional order α of a function $f(x)$ is defined as

$$ {}_{RL}\mathfrak{D}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x \frac{f(\tau)}{(x-\tau)^{\alpha-m+1}} d\tau, \tag{1.1} $$

where $m-1 \leqslant \alpha < m \in \mathbb{Z}^{+}$.

Definition 1.2. [13] The Caputo derivative of fractional order α of a function $f(x)$ is defined as

$$ {}_{C}\mathfrak{D}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha-m+1}} d\tau, \tag{1.2} $$

where $m-1 \leqslant \alpha < m \in \mathbb{Z}^{+}$.

Since fractional integrals are used to define the aforementioned fractional derivatives, they exhibit non-local behavior, the non-locality criterion creates what is known as the "memory effect," [14] in which the non-local fractional derivative of a function takes into account the entire function history, where the future state of a fractional-order system depends not only on its present state but also on all of its previous states.

Although the non-local fractional derivatives have many practical uses, they have their limitations, as they do not satisfy the product rule, quotient rule, chain rule, Rolle's theorem, and the mean value theorem. The property $\mathfrak{D}^{\alpha}\mathfrak{D}^{\beta} f = \mathfrak{D}^{\alpha+\beta} f$ is not generally valid for the non-local fractional derivatives. The Riemann-Liouville fractional derivative does not meet ${}_{RL}\mathfrak{D}^{\alpha}(1.1) = 0$ for $\alpha \in (n, n+1), n \in \mathbb{N}$, and Caputo fractional derivative is defined only for the classically differentiable functions, which, in turn, limits its applicability range.

Beyond these limitations, in 2014 R. Khalil, et al. [15] introduced a local definition of the fractional derivative known as a "conformable fractional derivative" (CFD), extending the classical limit definition of the derivative and is defined as follows:

Definition 1.3. [15] Given a function $f : [0, \infty) \to \mathbb{R}$. The CFD of f of order α is defined by

$$ T_{\alpha} f(x) = \lim_{\to 0} \frac{f\left(x + \varepsilon x^{1-\alpha}\right) - f(x)}{\varepsilon}, \tag{1.3} $$

for $x > 0, \alpha \in (0, 1)$.

The CFD meets the product, quotient, and chain rules for two α-differentiable functions, it also provides theorems analogous to Rolle's theorem and the mean value theorem in classical integer-order calculus. However, when the CFD is applied to solve fractional differential equations, the resulting error is substantially greater than that of the Caputo fractional derivative. Motivated by this observation, our work aims to introduce a new local fractional derivative, the "Gohar fractional derivative" (GFD), that is more accurate than the CFD in a way that results in less error when applied to solve fractional differential

equations and possesses more properties than both the CFD and the previously defined non-local fractional derivatives.
For the sake of preserving the nonlocality criterion of fractional calculus, we will introduce a nonlocal formulation of the GFD in a forthcoming study. However, the nonlocality comes with its own limitations and our future work will focus on overcoming them.

2. The Gohar Fractional Derivative (GFD)

In this section, we introduce the basic definition and classical features of our new local fractional derivative. We reveal its geometric interpretation and physical meaning, and highlight some of its additional properties that are not satisfied by the CFD. To this end, we proceed with the following definition, which generalizes the classical limit definition of the derivative.

Definition 2.1. Given a function $f : [0, \infty) \to \mathbb{R}$, the GFD of f of order α, denoted by G_α, is defined by

$$G_\alpha f(x) = \lim_{h\to 0} \frac{f\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right)-f(x)}{h}. \tag{2.1}$$

for $x > 0$, $\alpha \in (0,1)$, $\eta \in \mathbb{R}^+$.

Remark 2.1.

(I) If the limit in the definition above exists, then f is said to be a Gohar differentiable function (G_α-differentiable function).

(II) For $x > 0$, $\alpha \in (0,1)$, $\eta \in \mathbb{R}^+$. If f is a G_α- differentiable function in an open interval $(0,\delta)$, $\delta > 0$, and $\lim_{x\to 0^+} G_\alpha f(x)$ exists, then $G_\alpha f(0) = \lim_{x\to 0^+} G_\alpha f(x)$.

(III) In contrast to the fractional derivatives introduced by Riemann- Liouville and Caputo, which possess a delay impact due to the existence of a kernel within their integral form, the GFD does not possess a delay effect because its definition does not depend on a kernel.

Before we start our discussion, it is worth mentioning the Maclaurin series expansion

$$\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)^k = h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}+O(h^2), \tag{2.2}$$

as it will be used frequently to construct the arguments in this work.

Theorem 2.1. The GFD is a fractal derivative or a generalization of the q-derivative.

Proof. With the aid of the Maclaurin series expansion in Eq. (2.2), we have

$$G_\alpha f(x) = \lim_{h\to 0} \frac{f\left(x\left[1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}+O(h^2)\right]\right)-f(x)}{h}.$$

With the following substitution

$$q = 1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}+O(h^2),\ q\longrightarrow 1 \text{ as } h\longrightarrow 0,$$

we have

$$G_{\alpha}f(x) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} \lim_{q\to 1} \frac{f(qx)-f(x)}{qx^{\alpha}-x^{\alpha}}.$$

Theorem 2.2. If $f : [0,\infty) \to \mathbb{R}$ is G_{α}- differentiable at $x_0 > 0$, with $\alpha \in (0,1]$, $\eta \in \mathbb{R}^{+}$, then f is continuous at x_0.

Proof. Consider the following equality

$$f\left(x_0\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}{x_0}^{-\alpha}\right)\right]\right) - f(x_0) = \frac{f\left(x_0\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}{x_0}^{-\alpha}\right)\right]\right) - f(x_0)}{h}\cdot h.$$

Taking the limit of both sides as $h \to 0$, we have

$$\lim_{h\to 0} f\left(x_0\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}{x_0}^{-\alpha}\right)\right]\right) - f(x_0) = \lim_{h\to 0}\frac{f\left(x_0\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}{x_0}^{-\alpha}\right)\right]\right) - f(x_0)}{h} . \lim_{h\to 0} h .$$

By using Eq. (2.2), we have $x_0\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}{x_0}^{-\alpha}\right)\right] = x_0 + h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}{x_0}^{1-\alpha} + O(h^2)$.

With the following incremental change

$$\Delta x = h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}{x_0}^{1-\alpha} + O(h^2), \text{ with } \Delta x \longrightarrow 0, \text{ as } h \longrightarrow 0,$$

we have

$$\lim_{\Delta x\to 0}[f(x_0+\Delta x) - f(x_0)] = G_{\alpha}f(x_0).0 = 0,$$

which implies that

$$\lim_{\Delta x\to 0} f(x_0+\Delta x) = f(x_0).$$

Theorem 2.3. A G_{α}-differentiable function at a point need not be classically differentiable there.

Proof. Consider the function $f : [0,\infty) \to \mathbb{R}$, defined by $f(x) = \sqrt[3]{x^2}$. Obviously, it is $G_{\frac{2}{3}}$-differentiable at $x = 0$, and for $\alpha = \frac{2}{3}$, in view of Remark 2.1, the value of its GFD at $x = 0$ is $G_{\frac{2}{3}}(f(0)) = \lim_{x\to 0^+} G_{\frac{2}{3}}f(x) = \frac{2}{3}\frac{\Gamma(\eta)}{\Gamma(\eta+\frac{1}{3})}$, where $\eta \in \mathbb{R}^{+}$. However, $f'(0)$ does not exist.

Lemma 2.1. If $f : [0,\infty) \to \mathbb{R}$ is a G_{α}- differentiable function at $x > 0$, then

$$G_{\alpha}f(x) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{1-\alpha}\frac{df(x)}{dx}, \tag{2.3}$$

for $\alpha \in (0,1]$, $\eta \in \mathbb{R}^+$.

Proof. With the aid of the Maclaurin series expansion in Eq. (2.2), we have

$$G_\alpha f(x) = \lim_{h\to 0} \frac{f\left(x\left[1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}+O(h^2)\right]\right)-f(x)}{h}$$

with the incremental change

$$\Delta x = h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{1-\alpha}+O(h^2), \Delta x \longrightarrow 0 \text{ as } h \longrightarrow 0,$$

we deduce that

$$G_\alpha f(x) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{1-\alpha}\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{1-\alpha}\frac{df(x)}{dx}.$$

Remark 2.2 Given that the classical derivative $\frac{df(t)}{dt}$ is the instantaneous velocity of a moving particle at $t > 0$, Lemma 2.1 provides the physical meaning of GFD as a deviation from the classical instantaneous velocity in both direction and magnitude.
While the range $\alpha \in (0,1]$ is by far the most important and often used in applications, how do we define the GFD for $\alpha \in (n, n+1]$, $n \in \mathbb{N}$?

Definition 2.2. For $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $\eta \in \mathbb{R}^+$. The GFD of a function $f : [0,\infty] \to \mathbb{R}$ of order α, denoted by $G_{\alpha;n}$, is defined by

$$G_{\alpha;n} f(x) = \lim_{h\to 0}\frac{f^{(n)}\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right)-f^{(n)}(x)}{h}.$$

From Definition 2.2 and Lemma 2.1, via mathematical induction on n, we can show that

$$G_\alpha f(x) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{n+1-\alpha}f^{(n+1)}(x),$$

provided that f is $(n+1)$-differentiable function at $x > 0$.

Theorem 2.4. If $f(x) = x^\eta$, $\eta \in \mathbb{R}^+$, then the following relation holds

$$G_\alpha f(x) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)}x^{\eta-\alpha}. \tag{2.4}$$

Proof. With the aid of Eq. (2.3), we have

$$G_\alpha f(x) = G_\alpha x^\eta = \frac{\eta\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{\eta-\alpha} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)}x^{\eta-\alpha}.$$

Remark 2.3. The relation defined by Eq. (2.4) coincides with the relation for the derivative of monomials in the Caputo sense. That is,

$$G_\alpha x^n = {}_C\mathfrak{D}^\alpha x^n.$$

In what follows, we introduce the theorem that captures the standard classical features of the integer-order derivatives.

Theorem 2.5. For $\alpha \in (0,1], \eta \in \mathbb{R}^+$. Suppose that $f, g : [0,\infty) \to \mathbb{R}$ are G_α-differentiable functions at $x > 0$. Then they satisfy the following properties:

(I) Linearity:

$$G_\alpha (\lambda f + \mu g)(x) = \lambda G_\alpha f(x) + \mu G_\alpha g(x);\ \lambda, \mu \in \mathbb{R}.$$

(II) Product rule:

$$G_\alpha (fg)(x) = f(x) G_\alpha g(x) + g(x) G_\alpha f(x).$$

(III) Quotient rule:

$$G_\alpha \left(\frac{f}{g}\right)(x) = \frac{g(x) G_\alpha f(x) - f(x) G_\alpha g(x)}{[g(x)]^2},\ g(x) \neq 0.$$

(IV) Chain rule:

$$G_\alpha (f \circ g)(x) = G_\alpha (g(x)) f'(g(x)).$$

(V) Derivative of a constant:

$$G_\alpha (c) = 0, c \in \mathbb{R}.$$

Proof. With the aid of Eq. (2.3), we have

$$\begin{aligned}
G_\alpha (\lambda f + \mu g)(x) &= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{d}{dx} (\lambda f(x) + \mu g(x)) \\
&= \lambda \left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{df(x)}{dx} \right) + \mu \left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{dg(x)}{dx} \right) \\
&= \lambda G_\alpha f(x) + \mu G_\alpha g(x). \\
G_\alpha (fg)(x) &= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{d}{dx} (f(x) g(x)) \\
&= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \left(f(x) \frac{dg(x)}{dx} + g(x) \frac{df(x)}{dx} \right) \\
&= f(x) \left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{dg(x)}{dx} \right) + g(x) \left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{df(x)}{dx} \right) \\
&= f(x) G_\alpha g(x) + g(x) G_\alpha f(x). \\
G_\alpha \left(\frac{f}{g}\right)(x) &= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) \\
&= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \left(\frac{g(x) \frac{df(x)}{dx} - f(x) \frac{dg(x)}{dx}}{(g(x))^2} \right) \\
&= \frac{g(x) \left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{df(x)}{dx} \right) - f(x) \left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{dg(x)}{dx} \right)}{(g(x))^2} \\
&= \frac{g(x) G_\alpha f(x) - f(x) G_\alpha g(x)}{(g(x))^2}.
\end{aligned}$$

$$G_\alpha(c) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{d}{dx}(c) = \left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \right) \cdot 0 = 0.$$

Since f, g are G_α-differentiable functions, their composite function $f \circ g$ is G_α-differentiable and satisfies Definition 2.1 as follows

$$\begin{aligned} G_\alpha(f \circ g)(x) &= \lim_{h \to 0} \frac{f\left(g\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right)\right) - f(g(x))}{h} \\ &= \lim_{h \to 0} \frac{f\left(g\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right)\right) - f(g(x))}{g\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right) - g(x)} \\ &\quad \cdot \frac{g\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right) - g(x)}{h}. \end{aligned}$$

Since g is G_α-differentiable at $x > 0$, it is continuous there by Theorem 2.2, therefore, with the aid of Eq. (2.2), we have

$$\begin{aligned} \Delta g(x) &= g\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right) - g(x) \\ &= g\left(x + h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{1-\alpha} + O(h^2)\right) - g(x) = g(x+\Delta x) - g(x) \longrightarrow 0, \text{ as } h \longrightarrow 0, \end{aligned}$$

where we take $\Delta x = h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} + O(h^2)$. Consequently,

$$\begin{aligned} G_\alpha(f \circ g)(x) &= \lim_{h \to 0} \frac{g\left(x\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}x^{-\alpha}\right)\right]\right) - g(x)}{h} \\ &\quad \cdot \lim_{\Delta g(x) \to 0} \frac{f(g(x)+\Delta g(x)) - f(g(x))}{\Delta g(x)} = G_\alpha g(x) \cdot f'(g(x)); \; h, \; \Delta g(x) \neq 0. \end{aligned}$$

In view of the chain rule included in Theorem 2.4, we introduce the following corollary.

Corollary 2.1. For $\alpha \in (0,1]$, $\eta \in \mathbb{R}^+$, we have the following results:

(I)

$$G_\alpha\left(\frac{x^\alpha}{\alpha}\right) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}.$$

(II)

$$G_\alpha\left(\sin\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)}\frac{x^\alpha}{\alpha}\right)\right) = \cos\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)}\frac{x^\alpha}{\alpha}\right).$$

(III)

$$G_\alpha\left(\cos\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)}\frac{x^\alpha}{\alpha}\right)\right) = -\sin\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)}\frac{x^\alpha}{\alpha}\right).$$

(IV)

$$G_\alpha\left(\exp\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)}\frac{x^\alpha}{\alpha}\right)\right) = \exp\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)}\frac{x^\alpha}{\alpha}\right).$$

Remark 2.4. Relations II, III, and IV of corollary 2.1 imply the existence of a pseudo-invariant space corresponds to the GFD on which the natural exponential, sine, and cosine functions are G_α-invariant and possess their classical integer-order derivatives.

Now, we shall demonstrate that Rolle's theorem and the mean value theorem can be generalized to include G_α-differentiable functions.

Theorem 2.6. (Rolle's theorem for $\mathbf{G}_\alpha$-differentiable functions). Given that $\alpha \in (0,1)$, $a > 0, \eta \in \mathbb{R}^+$. If $f : [a,b] \to \mathbb{R}$ is a function that meets the following criteria:

(I) f is G_α-differentiable on (a,b),
(II) f is continuous on $[a,b]$,
(III) $f(a) = f(b)$.

Then, there exists $c \in (a,b)$, such that $G_\alpha(f(c)) = 0$.

Proof. Given that f is continuous on $[a,b]$, and $f(a) = f(b)$, there must be a local extreme point $c \in (a,b)$ at which

$$G_\alpha f\left(c^+\right) = \lim_{h\to 0^+} \frac{f\left(c\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}c^{-\alpha}\right)\right]\right)-f(c)}{h},$$

$$G_\alpha f\left(c^-\right) = \lim_{h\to 0^+} \frac{f\left(c\left[1+\ln\left(1+h\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}c^{-\alpha}\right)\right]\right)-f(c)}{h}.$$

Since the two limits have opposite signs, we deduce that

$$G_\alpha f(c) = 0.$$

Theorem 2.7. (Mean value theorem for $\mathbf{G}_\alpha$-differentiable functions). Given that $\alpha \in (0,1)$, $a > 0, \eta \in \mathbb{R}^+$. If $f : [a,b] \to \mathbb{R}$ is a function that meets the following criteria:

(I) f is G_α-differentiable on (a,b),
(II) f is continuous on $[a,b]$.

Then, there exists $c \in (a,b)$, such that

$$G_\alpha f(c) = \frac{\alpha\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}\left(\frac{f(b)-f(a)}{b^\alpha-a^\alpha}\right).$$

Proof. Let us introduce the function $\phi(x)$ to be

$$\phi(x) = f(x) - f(a) - \alpha\left(\frac{f(b)-f(a)}{b^\alpha-a^\alpha}\right)\left(\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right).$$

Applying GFD to both sides of the equation above, we have

$$G_\alpha\phi(x) = G_\alpha f(x) - \alpha\left(\frac{f(b)-f(a)}{b^\alpha-a^\alpha}\right)G_\alpha\left(\frac{x^\alpha}{\alpha}\right).$$

The function $\phi(x)$ satisfies Rolle's theorem for G_α-differentiable functions, and so there must be a point $c \in (a,b)$ at which $G_\alpha(\phi(c)) = 0$. With the aid of the relation I in corollary 2.1, we deduce that

$$G_\alpha f(c) = \frac{\alpha\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}\left(\frac{f(b)-f(a)}{b^\alpha - a^\alpha}\right).$$

Theorem 2.8. (Extended mean value theorem for $\mathbf{G}_\alpha$-differentiable functions). Given that $\alpha \in (0,1)$, $a > 0$, $\eta \in \mathbb{R}^+$. If $f, g : [a,b] \to \mathbb{R}$ are two functions that meet the following criteria:

(I) f, g are G_α-differentiable on (a,b),
(II) f, g are continuous on $[a,b]$.

Then, there exists a point $c \in (a,b)$ such that

$$\frac{G_\alpha f(c)}{G_\alpha g(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

The following proposition is a direct consequence of the mean value theorem.

Proposition 2.1. Given the G_α-differentiable function $f : [a,b] \to \mathbb{R}$, whose GFD is bounded on $[a,b]$ for $\alpha \in (0,1)$, $a > 0, \eta \in \mathbb{R}^+$. If $G_\alpha f$ is continuous at either a or b, then f is a uniformly continuous function on $[a,b]$, and therefore f is bounded over $[a,b]$.

Theorem 2.9. For $\alpha \in (0,1]$, $a > 0$, let $f : [a,b] \to \mathbb{R}$ be a function that has the following properties:

(I) f is G_α- differentiable on (a,b),

(II) f is continuous on $[a,b]$.

Then, we have:

(I) If $G_\alpha f(x) > 0$, $\forall x \in (a,b)$, then f is a strictly increasing function on $[a,b]$.

(II) If $G_\alpha f(x) < 0$, $\forall x \in (a,b)$, then f is a strictly decreasing function on $[a,b]$.

Proof. Suppose that $x_0, x_f \in [a,b]$, with $x_0 < x_f$. Then $[x_0,x_f] \subseteq [a,b]$, $(x_0,x_f) \subseteq (a,b)$; thus, f is G_α- differentiable on (x_0,x_f) and continuous on $[x_0,x_f]$. The mean value theorem, Theorem 2.7, implies the existence of $c \in (x_0,x_f)$ with

$$G_\alpha f(c) = \frac{\alpha\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}\left(\frac{f(x_f)-f(x_0)}{{x_f}^\alpha - {x_0}^\alpha}\right).$$

(I) If $G_\alpha(f(c)) > 0$, then $f(x_f) > f(x_0)$, $\forall x_0 < x_f$, and so f is a strictly increasing function on $[a,b]$.

(II) If $G_\alpha(f(c)) < 0$, then $f(x_f) < f(x_0)$, $\forall x_0 < x_f$, and so f is a strictly decreasing function on $[a,b]$.

Theorem 2.10. For $\alpha \in (0,1]$, $a > 0$, let $f : [a,b] \to \mathbb{R}$ be a function that has the following properties:

(I) f is G_α- differentiable on (a,b),

(II) f is continuous on $[a, b]$.

If $G_\alpha f(x) = 0, \ \forall x \in (a, b)$, then $f(x) = C, C \in \mathbb{R}$.

Proof. Using the same argument as that of Theorem 2.9, the mean value theorem implies the existence of $c \in (x_0, x_f)$ with

$$G_\alpha f(c) = \frac{\alpha\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} \left(\frac{f(x_f) - f(x_0)}{x_f{}^\alpha - x_0{}^\alpha} \right) = 0.$$

Therefore, $f(x_f) - f(x_0) = 0$, or $f(x_0) = f(x_f)$. Given that $x_0$and x_f are arbitrary numbers in $[a, b]$ with $x_0 < x_f$, then $f(x) = C, C \in \mathbb{R}$.

Corollary 2.2. For $\alpha \in (0, 1]$, $a > 0$, let $f, g : [a, b] \to \mathbb{R}$ be G_α-differentiable functions with
$G_\alpha f(x) = G_\alpha g(x), \ \forall x \in (a, b)$. Then there exists a constant $C \in \mathbb{R}$ such that $f(x) = g(x) + C$.

Proof. Consider the difference $\psi(x) = f(x) - g(x)$. Since $G_\alpha \psi(x) = 0, \forall x \in (a, b)$, Theorem 2.10 implies that $\psi(x) = C, C \in \mathbb{R}$.

It is worth mentioning that $G_{\frac{1}{2}}\left(\sqrt{x}\right) = \frac{\Gamma(\eta)}{2\Gamma(\eta-\alpha+1)}$, rather than being equal to $\frac{\sqrt{\pi}}{2}$ as in the case of the Riemann-Liouville nonlocal fractional derivative, which suggests that the GFD has its own geometric interpretation, which we shall introduce in the following Theorem.

Theorem 2.11. (The Geometric interpretation of GFD). Given a G_α-differentiable function $f : [0, \infty) \to \mathbb{R}$, the GFD of f at a point $x_0 \geqslant 0$ is the slope of the Gohar fractional curve (GFC) that intersects the graph of f at $(x_0, f(x_0))$, and is defined by

$$f(x) = \left(f^\alpha(x_0) + \left(\frac{f^{\alpha-1}(x_0)\left(x^\alpha - x_0{}^\alpha\right)}{x_0{}^{\alpha-1}} \right) G_\alpha f(x_0)\right)^{\frac{1}{\alpha}},$$

for $\alpha \in (0, 1], \eta \in \mathbb{R}^+$.

Remark 2.5. For the special case in which $\alpha = 1$, the GFC passing through a point on the graph of f reduces to the tangent line to the graph of f at that point. and therefore, the GFD of f reduces to the classical derivative of f at that point.

In the next section, we will define the "Gohar fractional integral" (GFI) corresponding to the GFD and introduce the Gohar fractional extension of the fundamental theorem of calculus.

3. The Gohar Fractional Integral (GFI)

Definition 3.1. For $x \geqslant 0$, if f is a function defined on $(0, x]$, then the GFI of f, of order α, denoted by $\mathfrak{T}^\alpha$, is defined by

$$\mathfrak{T}^\alpha f(x) = \frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)} \int_0^x \frac{f(t)}{t^{1-\alpha}} dt, \tag{3.1}$$

where $\alpha \in (0, 1), \eta \in \mathbb{R}^+$.

Remark 3.1.
(I) For $\alpha = 1$, the GFI coincides with the classical Riemannian integral.

(II) If the Riemann improper integral in Eq. (3.1) exists, then f is said to be a Gohar integrable function ($\mathfrak{T}^{\alpha}$-integrable function).

Theorem 3.1 (Fundamental Theorem of Gohar Fractional Calculus). For $x \geqslant 0$, If f is a continuous $\mathfrak{T}^{\alpha}$-integrable function, then

$$G_{\alpha}\left(\mathfrak{T}^{\alpha} f(x)\right) = f(x),$$

$$\mathfrak{T}^{\alpha}\left(G_{\alpha}(f(x))\right) = f(x) - f(0),$$

where $\alpha \in (0,1)$, $\eta \in \mathbb{R}^{+}$.

Proof.

$$\begin{aligned}
G_{\alpha}\left(\mathfrak{T}^{\alpha} f(x)\right) &= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{d}{dx}\left(\mathfrak{T}^{\alpha} f(x)\right) \\
&= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha} \frac{d}{dx}\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)} \int_0^x \frac{f(t)}{t^{1-\alpha}} dt\right) \\
&= \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} x^{1-\alpha}\left(\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)} \frac{f(x)}{x^{1-\alpha}}\right) = f(x). \\
\mathfrak{T}^{\alpha}\left(G_{\alpha}(f(x))\right) &= \frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)} \int_0^x \frac{G_{\alpha}(f(t))}{t^{1-\alpha}} dt \\
&= \frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)} \int_0^x \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)} t^{1-\alpha} \frac{df(t)}{dt} \frac{dt}{t^{1-\alpha}} \\
&= \frac{\Gamma(\eta-\alpha+1)}{\Gamma(\eta)}\left(\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha+1)}(f(x)-f(0))\right) = f(x) - f(0).
\end{aligned}$$

4. Applications

4.1. Gohar fractional differential equations

Consider the following fractional differential equation in view of the GFD

$$G_{\frac{1}{2}} f(x) = (1+x)^{n},\ f(0) = 0. \tag{4.1}$$

By using Eq. (2.3), and the Maclaurin series expansion of the binomial $(1+x)^{n}$, we have

$$G_{\frac{1}{2}} f(x) = (1+x)^{n},$$

$$\frac{\Gamma(\eta)}{\Gamma\left(\eta+\frac{1}{2}\right)} x^{\frac{1}{2}} \frac{df(x)}{dx} = \sum_{k=0}^{\infty} \binom{n}{k} x^{k},$$

$$\frac{df(x)}{dx} = \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{\Gamma(\eta)} \sum_{k=0}^{\infty} \binom{n}{k} x^{k-\frac{1}{2}},$$

$$\int df(x) = \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{\Gamma(\eta)} \sum_{k=0}^{\infty} \binom{n}{k} \int x^{k-\frac{1}{2}} dx,$$

$$f(x) = \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{\Gamma(\eta)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{x^{k+\frac{1}{2}}}{k+\frac{1}{2}} + C = \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{(k+\frac{1}{2})\Gamma(\eta)} x^{k+\frac{1}{2}} + C.$$

For $\eta = k + \frac{1}{2}$, we have

$$f(x) = \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(k+1)}{(k+\frac{1}{2})\Gamma\left(k+\frac{1}{2}\right)} x^{k+\frac{1}{2}} + C,$$

Since $f(0) = 0$, the series solution of Eq. (4.1) is given by

$$f(x) = \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(k+1)}{\Gamma\left(k+\frac{3}{2}\right)} x^{k+\frac{1}{2}}, \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This solution is compatible with that of Caputo non-local fractional derivative given by Eq. (1.2).

Now, let us solve another Gohar fractional differential equation of the form

$$G_{\frac{1}{2}} f(x) = x^2 \cos(x), \quad f(0) = 0. \tag{4.2}$$

With the aid of Eq. (2.3) and the Maclaurin series expansion of the cosine function, we proceed as follows:

$$G_{\frac{1}{2}} f(x) = x^2 \cos(x),$$

$$\frac{\Gamma(\eta)}{\Gamma\left(\eta+\frac{1}{2}\right)} x^{\frac{1}{2}} \frac{df(x)}{dx} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2(k+1)},$$

$$\frac{df(x)}{dx} = \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{\Gamma(\eta)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+\frac{3}{2}},$$

$$\int df(x) = \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{\Gamma(\eta)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int x^{2k+\frac{3}{2}} dx,$$

$$f(x) = \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{\Gamma(\eta)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{x^{2k+\frac{5}{2}}}{2k+\frac{5}{2}} + C = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\Gamma\left(\eta+\frac{1}{2}\right)}{(2k+\frac{5}{2})\Gamma(\eta)} x^{2k+\frac{5}{2}} + C,$$

for $\eta = 2k + \frac{5}{2}$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\Gamma(2k+3)}{(2k+\frac{5}{2})\Gamma\left(2k+\frac{5}{2}\right)} x^{2k+\frac{5}{2}} + C = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+3)(2k+2)(2k+1)}{\Gamma\left(2k+\frac{7}{2}\right)} x^{2k+\frac{5}{2}} + C,$$

since $f(0) = 0$, the series solution of Eq. (4.2) takes the form

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+3)(2k+2)(2k+1)}{\Gamma\left(2k+\frac{7}{2}\right)} x^{2k+\frac{5}{2}}.$$

This solution is compatible with that of Caputo non-local fractional derivative given by Eq. (1.2).

4.2. The Gohar fractional Riccati differential equations (GFRDEs)

Consider the nonlinear Riccati fractional differential equation in the GFD sense:

$$G_{\alpha} f(x) + (f(x))^2 = 1, f(0) = 0, \tag{4.3}$$

where $\alpha \in (0, 1]$, $x \geqslant 0, \eta \in \mathbb{R}^+$. With the aid of Eq. (2.3), the MATLAB software is utilized to obtain the following solutions for $\alpha = 0.75$ and $\alpha = 0.9$:

$$f(x) = \frac{\exp\left(\frac{8x^{3/4}}{3\lambda}\right) - 1}{\exp\left(\frac{8x^{3/4}}{3\lambda}\right) + 1}, \ \alpha = \eta = 0.75, \tag{4.4}$$

$$f(x) = \frac{\exp\left(\frac{20x^{9/10}}{9\lambda}\right) - 1}{\exp\left(\frac{20x^{9/10}}{9\lambda}\right) + 1}, \ \alpha = \eta = 0.9. \tag{4.5}$$

Another nonlinear Riccati fractional differential equation is expressed in view of the GFD as

$$G_{\alpha} f(x) = 2f(x) - (f(x))^2 + 1, f(0) = 0, \tag{4.6}$$

where $\alpha \in (0, 1]$, $x \geqslant 0, \eta \in \mathbb{R}^+$. With the aid of Eq. (2.3), the MATLAB software is utilized to produce the following solution for $\alpha = 0.9$:

$$f(x) = \frac{e^{\nu} - \sqrt{2}e^{\nu} + \sqrt{2} + 1}{e^{\nu} + 1}, \ \alpha = \eta = 0.9, \tag{4.7}$$

where

$$\nu = \frac{2\sqrt{2}}{9\lambda}\left(\frac{9\lambda \ln\left(\sqrt{2}+1\right)}{\sqrt{2}} - 10x^{9/10}\right), \ \lambda = \frac{\Gamma(\eta)}{\Gamma\left(\eta + \frac{1}{4}\right)}.$$

In a forthcoming article, we will go further into the wide range of potential applications of the GFD.

5. Results and Discussion

In this section, we lay out the main objective of this work and provide numerical and graphical evidence to prove that the GFD yields more accurate results than the CFD when applied to solve fractional differential equations. Some numerical results of the solutions given by Eqs. (4.4), (4.5), and (4.7) that are obtained by applying the GFD to the GFRDEs defined by Eqs. (4.3) and (4.6) are included in Tables 1-3 for various values of the parameters α and η, where we take $\alpha = \eta$, along with the corresponding results obtained by using the CFD [15], EHPM [16], IABMM [16], MHPM [17], BPM [18], FTBM [19], and RKM [20] for comparison purposes. A good agreement can be observed in Tables 1-3 between the numerical solutions obtained with the GFD and those obtained with the EHPM, IABMM, MHPM, BPM, FTBM, and RKM methods. On the other hand, the results obtained by the CFD do not coincide with those from the aforementioned methods or with our own, which, in turn, proves that the GFD yields more accurate results with less error in comparison with the CFD. In Figures 1-3, we graphically demonstrate the absolute relative error for the numerical solutions to the Riccati fractional differential equations obtained by our newly proposed GFD and those obtained by the CFD. The absolute relative error for the two local fractional differential operators is obtained by subtracting the exact results for the Riccati fractional differential equations obtained at $\alpha = 1$ in Ref. [18] from the corresponding results obtained by the GFD and CFD at $\alpha = 0.75$, $\alpha = 0.9$. Once again, the GFD reveals more accuracy with less error compared to the CFD, as suggested by their absolute relative error. Figure 4 reveals the geometric interpretation of the GFD with the quadratic function $Q\ (x) = x^2$. The GFC is depicted for α =0.4, α =0.6, α =0.8, and $\alpha = 1$, where it crosses the graph of $Q(x)$ at the point (2,4), for $\eta = 0.5$. Figure 4 confirms what we have mentioned earlier in Remark 2.5 about the GFC, as it converges to the tangent line as α approaches 1.

Table 1: Comparison of the numerical solutions to Eq. (4.3) with the other methods for $\alpha = 0.75$.

x	0	0.2	0.4	0.6	0.8	1
GFD	0	0.31439	0.49848	0.63022	0.72609	0.79618
CFD [13]	0	0.37889	0.58539	0.72064	0.81029	0.87006
EHPM [14]	0	0.3214	0.5077	0.6259	0.7028	0.7542
IABMM [14]	0	0.3117	0.4855	0.6045	0.688	0.7478
MHPM [15]	0	0.3138	0.4929	0.5974	0.6604	0.7183
BPM [16]	0	0.30996891	0.48162749	0.59777979	0.67884745	0.73684181
FTBM [17]	0	0.30997528	0.48163169	0.59778267	0.67884949	0.73683667
RKM [18]	0	0.307359	0.480346	0.597542	0.679657	0.738213

Table 2: Comparison of the numerical solutions to Eq. (4.3) with the other methods for $\alpha = 0.9$.

x	0	0.2	0.4	0.6	0.8	1
GFD	0	0.23952	0.42667	0.57607	0.69138	0.7778
CFD [13]	0	0.25526	0.45191	0.60539	0.72063	0.80445
EHPM [14]	0	0.2647	0.4591	0.6031	0.7068	0.7806
IABMM [14]	0	0.2393	0.4234	0.5679	0.6774	0.7584
MHPM [15]	0	0.2391	0.4229	0.5653	0.674	0.7569
BPM [16]	0	0.23878798	0.42258214	0.56617082	0.67462642	0.75460256
FTBM [17]	0	0.23878913	0.42258308	0.56617156	0.67462699	0.7545888
RKM [18]	0	0.237652	0.421766	0.565673	0.674467	0.754632

Table 3: Comparison of the numerical solutions to Eq. (4.6) with the other methods for $\alpha = 0.9$.

x	0	0.2	0.4	0.6	0.8	1
GFD	0	0.30718	0.67131	1.0666	1.4397	1.7485
CFD [13]	0	0.33295	0.73105	1.1561	1.5422	1.8457
EHPM [14]	0	—	—	—	—	2.0697
IABMM [14]	0	—	—	—	—	1.7356
MHPM [15]	0	—	—	—	—	1.872
BPM [16]	0	0.31488815	0.69756771	1.10789047	1.47772823	1.76542008
FTBM [17]	0	0.31485423	0.69751826	0.90364539	1.47768008	1.76525852
RKM [18]	0	—	—	—	—	—

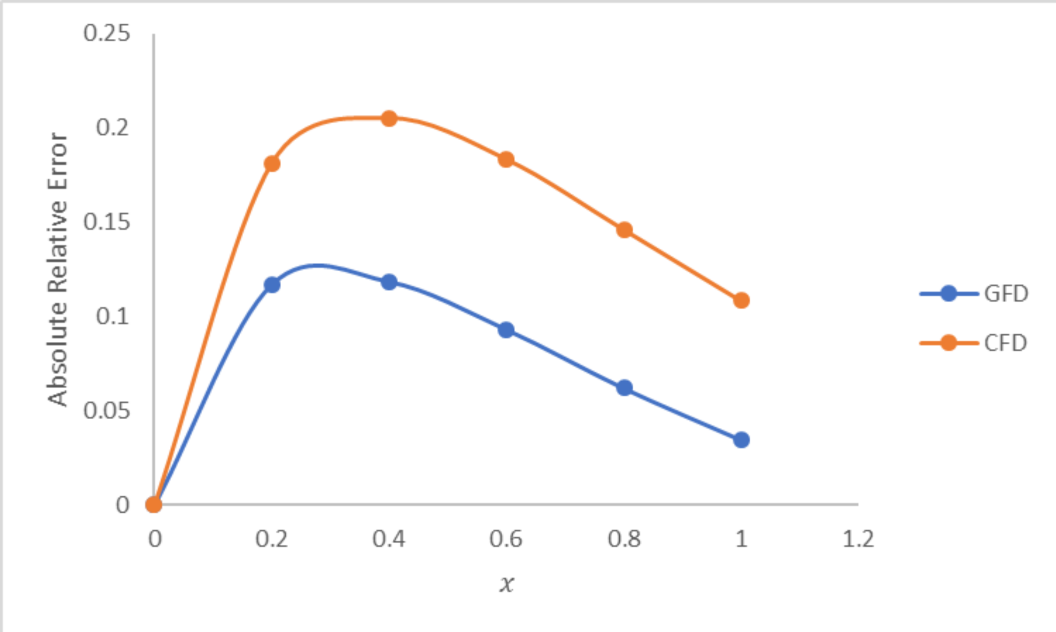

Figure 1: The absolute relative error for the GFD and CFD for Eq. (4.3) at $\alpha = 0.75$.

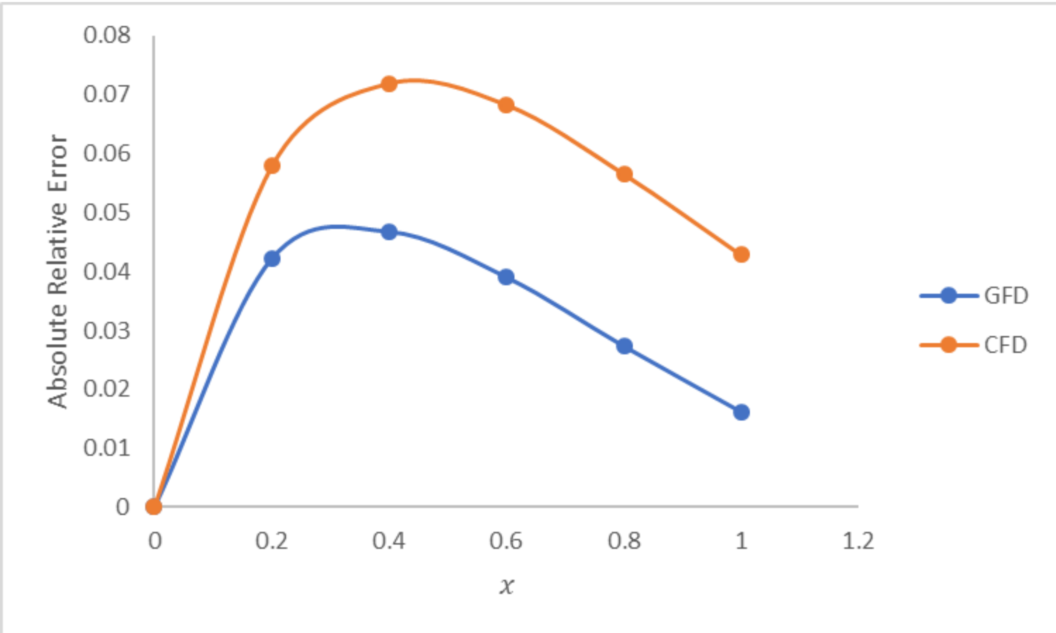

Figure 2: The absolute relative error for the GFD and CFD for Eq. (4.3) at $\alpha = 0.9$.

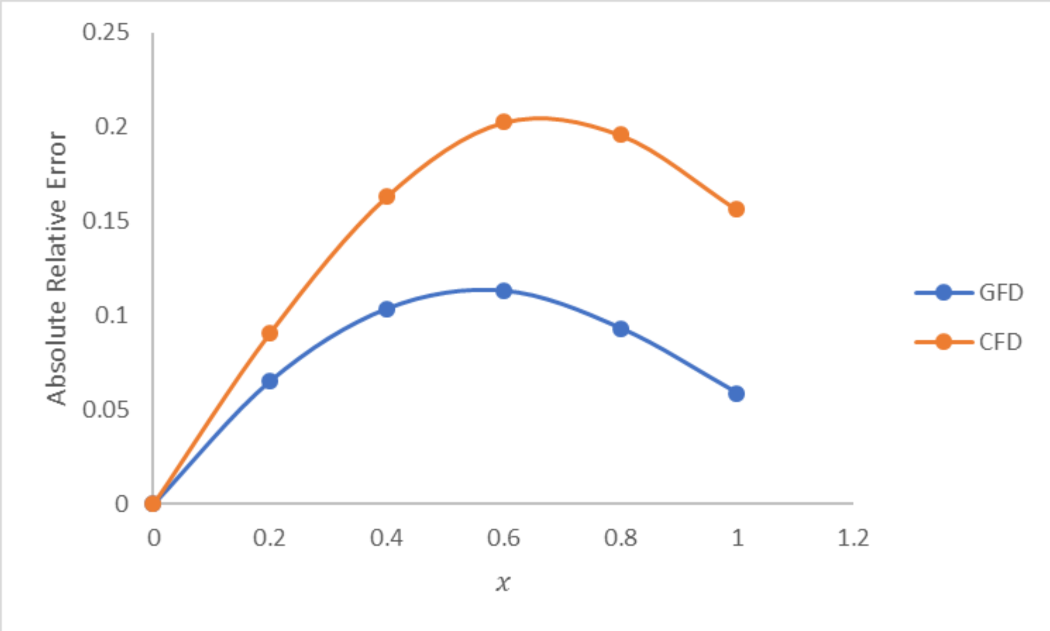

Figure 3: The absolute relative error for the GFD and CFD for Eq. (4.6) at $\alpha = 0.9$.

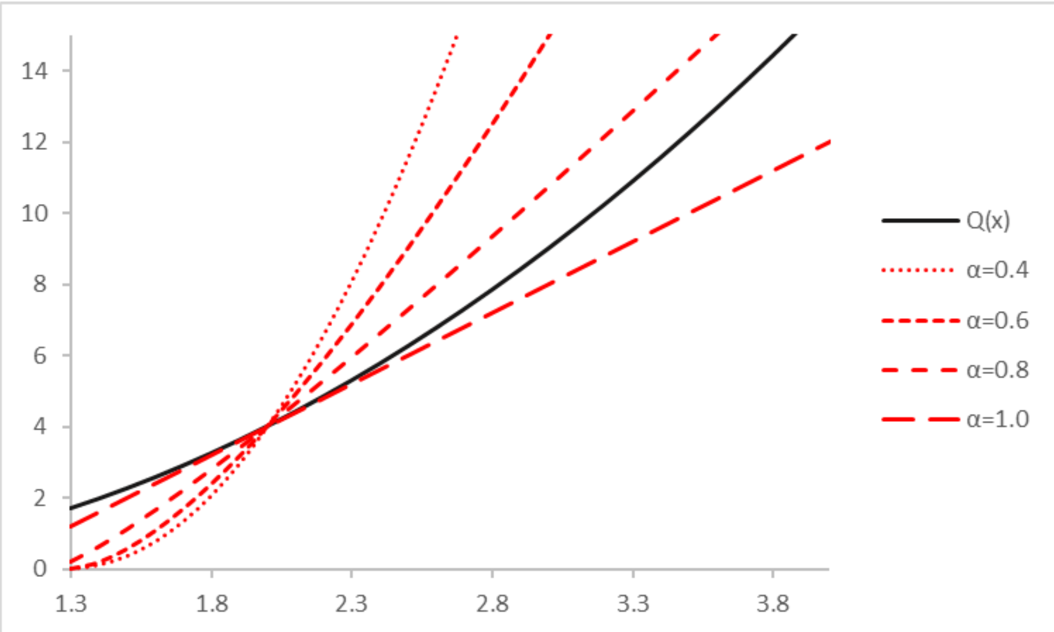

Figure 4: The graph of the quadratic function $Q\ (x) = x^2$ intersecting its GFCs at (2,4).

Statements and Declarations

Availability of data and materials: The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.
Authors's contributions: A. A. Gohar: Conceptualization, Methodology, Writing — Origi-

nal draft preparation. M. S. Younes: Software, Visualization, Investigation, Data Curation. S. B. Doma: Suggesting the topic of research, Supervision, Writing-Review & Editing. All authors read and approved the final manuscript.

Acknowledgements: We are grateful to M. T. Tajuddin for for his valuable revision and editing.

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