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Existence theory and stability analysis to a coupled nonlinear fractional mixed boundary value problem

Shahid Saifullah [a], Sumbel Shahid [a], Akbar Zada [a]

[a] Department of Mathematics, University of Peshawar, Peshawar, Pakistan

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Abstract

In this manuscript, we conclude a comprehensive approach to a class of nonlinear coupled system of fractional differential equations with mixed type boundary value conditions. Subsequently, the solution of coupled system exists and unique under mixed type boundary value conditions with the reference of Schaefer and Banach fixed-point theorems. Further, we developed the Hyers- Ulam stability for the considered problem. Finally, we set an example for the support of our results.

Keywords: Green's function, Coupled system, Riemann–Liouville fractional order derivative, Existence theory, Ulam stability.
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1. Introduction

Fractional calculus is three centuries old calculus but not very popular calculus in the field of engineering or science. The generalization of ordinary differential equations are known as fractional differential equations (FDEs) with an arbitrary (non-integer) order. In other words, may be this article interprets the facts of nature. Therefore, aiming to make this idea of differential equations available as a popular area for the engineering and science, another aspect has been added to better comprehend or explain the basic nature. Perhaps the calculation of the fraction is what nature understands and the communication with nature in this language is therefore effective. For the last three centuries, it has been a topic with mathematicians, and only in the last few years has it been extended to many applied fields of engineering and science and economics see in [1, 2, 3, 4, 5, 6, 7]. The next decade will see a number of applications based on this three hundred-year old new article. In fractional calculus, the basic idea of fractional derivative was introduced by Liouville and Riemann.

The research area, that has received the best and most attention from scholars is dedicated to the existence of solution for different models. Some different researchers have

*Corresponding author: shahidsaif78@gmail.com

been set up interesting results of the existence of solutions of FDEs in view of different fixed points theorems. For detail study, see [8, 9, 10]. Although, the study of coupled systems of the FDEs is also very significant because this kind of systems seem to be used in different variety of problems of applied nature, see [11, 12, 13].

Cabada and Kaithoum [14], presented their work in which the considerable model is all concerned about the existence and uniqueness $(\mathcal{EU})$ of positive solution of implicit nonlinear fractional problem with mixed type boundary conditions, which is as follows;

$$\begin{cases} \mathcal{D}^{\beta}u(x) - \lambda u(x) + f(x, x^{2-\beta}u(x)) = 0, \\ \lim_{x\to 0} x^{2-\beta}u(x) = 0 \quad \text{and} \quad u'(1) = 0. \ \forall\, x \in \mathcal{I} = [0,1]. \end{cases}$$

Where $\lambda \in R$, $1 < \beta \leqslant 2$, $\mathcal{D}^{\beta}$ is Riemann-Liouville $(\mathcal{R}-\mathcal{L})$ fractional derivative and f is continuous function.

Cabada and Kaithoum in [15] also studied the existence and non-existence results for the solutions of nonlinear FDE with three parameter family under mixed type integral boundary conditions.

$$\begin{cases} \mathcal{D}^{\beta}u(x) - \lambda u(x) + y(x) = 0, \\ \lim_{x\to 0} x^{2-\beta}u(x) = \mu\int_0^1 u(y)dy \quad \text{and} \quad u'(1) = \eta\int_0^1 u(y)dy. \ \forall\, x \in \mathcal{I} = [0,1]. \end{cases}$$

Where $\lambda \in R$ and $\mu, \eta \geqslant 0$, $1 < \beta \leqslant 2$, $\mathcal{D}^{\beta}$ is $\mathcal{R}-\mathcal{L}$ fractional derivative and y is continuous function. For more refinements of above model see [16, 17].

FDEs have been studied for different angles. Among these results in stability analysis,Hyers Ulam $(\mathcal{HU})$ stability was introduced by Hyers and Ulam in 1941. The sense of $\mathcal{HU}$ stability is an important which gained a great interest from many researcher. In addition to above investigations, different researchers have been examined the $\mathcal{HU}$ stability for differential equations having different orders, see [18, 19, 20, 21, 13, 22, 23].

In this paper, we discuss the $\mathcal{EU}$ of the solution for nonlinear coupled FDEs with boundary conditions

$$\begin{cases} \mathcal{D}^{p}u(x) - \lambda_1 u(x) + g(x, x^{2-p}u(x), x^{2-p}v(x)) = 0, \forall\, x \in \mathcal{I}, \\ \mathcal{D}^{q}v(x) - \lambda_2 v(x) + h(x, x^{2-q}u(x), x^{2-q}v(x)) = 0, \forall\, x \in \mathcal{I}, \\ \lim\limits_{x\to 0^+} x^{2-p}u(x) = u'(1) = 0, \\ \lim\limits_{x\to 0^+} x^{2-q}v(x) = v'(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda_1, \lambda_2 \in R$ and $1 < p, q \leqslant 2$, $\mathcal{D}^{p}$ and $\mathcal{D}^{q}$ are $\mathcal{R}-\mathcal{L}$ fractional derivatives and g, h are continuous functions.

This article is comprised as follows; our Section 2, we study, some basic and relative introductory result and definitions. Next section is totally devoted to deduce Green functions for our considered problem by using classical theory of fractional calculus. Moreover, we also construct some main properties of the Green functions. Finally in the last section we ensuring the $\mathcal{EU}$ of positive solutions of implicit coupled nonlinear problem under mixed type boundary conditions and we also investigate that under adequate conditions the implicit coupled model is $\mathcal{HU}$ stable. At last, we give an example to support our results.

2. Preliminaries and some basic definitions

In the following section, we recollect preliminary result and some basic definitions, which will be used throughout this paper. These definitions and result are taken from [24].

Definition 2.1. [24] The $Riemann\ -Liouville\ \mathcal{R}-\mathcal{L}\ fractional\ integral$ of fractional order $\beta > 0$ for a measurable function $u(s)$, is defined as

$$\mathcal{I}^{\beta}u(s) = \frac{1}{\Gamma(\beta)}\int_0^s (s-t)^{\beta-1}u(s)ds,$$

where Γ is $Euler\ Gamma\ function$.

Definition 2.2. [24] The $R-L\ fractional\ derivative$ of order $\beta > 0$ for measurable function $u(s)$ is given by

$$\mathcal{D}^{\beta}u(s) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^s (s-t)^{n-\beta-1}u(s)ds.$$

Here $n = [\beta]+1$, where $[\beta]$ denote the integer part of β.

Definition 2.3. [24] Mittag Leffler is played a very key role in the theory of FDEs. It is defined with two parameter function as;

$$E_{\mu,\eta}(\zeta) = \sum_{i=0}^{\infty}\frac{\zeta^i}{\Gamma(\mu i+\eta)} \quad \mu,\eta > 0\ ,\zeta \in R.$$

Theorem 2.4. *[24] The general solution of* FDE *of order* $n-1 < \beta \leqslant n$, $n \in \mathcal{N}$ *and* $\lambda \in \mathcal{R}$

$$\mathcal{D}^{\beta}u(s) - \lambda u(s) + y(s) = 0,$$

has solution given by

$$u(s) = \sum_{k=1}^{m} c_k s^{\beta-k}E_{\beta,\beta+1-k}(\lambda s^{\beta}) + \int_0^s (s-t)^{\beta-1}E_{\beta,\beta}[\lambda(s-t)^{\beta}]y(t)dt,$$

with $c_k \in R$, $i = 0,1,2,\ldots,m$, *chosen arbitrary and where* y *be a given real function defined on* $\mathcal{R}$.

3. Green's function of coupled problem

In this section, we obtain Green's functions in the equivalent integral equations of the associated proposed system (1.1). Moreover, we give some properties of Green's functions.

Theorem 3.1. *Consider* $p, q \in (1,2]$ *and* $g, h : \mathcal{I} \times \mathcal{R} \times \mathcal{R} \to R$, *then system*

$$\begin{cases} \mathcal{D}^p u(x) - \lambda_1 u(x) + g(x) = 0, \forall\ t \in \mathcal{I}, \\ \mathcal{D}^q v(x) - \lambda_2 v(x) + h(x) = 0, \forall\ t \in \mathcal{I}, \\ \lim\limits_{x \to 0^+} x^{2-p} u(x) = u'(1) = 0, \\ \lim\limits_{x \to 0^+} x^{2-q} v(x) = v'(1) = 0, \end{cases}$$

where

$$\begin{aligned} g(x) &= g(x, x^{2-p}u(x), x^{2-p}v(x)) \\ h(x) &= h(x, x^{2-q}u(x), x^{2-q}v(x)), \end{aligned}$$

has an equivalent integral

$$\begin{cases} u(x) = \int_0^1 G_{\lambda_1}(x,y) g(y) dy, \forall\ x \in \mathcal{I}, \\ v(x) = \int_0^1 G_{\lambda_2}(x,y) h(y) dy, \forall\ x \in \mathcal{I}, \end{cases} \tag{3.1}$$

where $G_{\lambda_1}(x,y)$ *and* $G_{\lambda_2}(x,y)$ *are the Green's functions given as*

$$G_{\lambda_1}(x,y) = \begin{cases} \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}[\lambda_1(1-y)^p]}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} - (x-y)^{p-1}E_{p,p}(\lambda_1(x-y)^p), & 0 \leqslant y \leqslant x \leqslant 1 \\ \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)}, & 0 \leqslant x < y < 1. \end{cases} \tag{3.2}$$

$$G_{\lambda_2}(x,y) = \begin{cases} \frac{x^{q-1}E_{q,q}(\lambda_2 x^q)E_{q,q-1}(\lambda_2(1-y)^q)}{(1-y)^{2-q}E_{q,q-1}(\lambda_2)} - (x-y)^{q-1}E_{q,q}(\lambda_2(x-y)^q), & 0 \leqslant y \leqslant x \leqslant 1 \\ \frac{x^{q-1}E_{q,q}(\lambda_2 x^q)E_{q,q-1}(\lambda_2(1-y)^q)}{(1-y)^{2-q}E_{q,q-1}(\lambda_2)}, & 0 \leqslant x < y < 1. \end{cases} \tag{3.3}$$

Proof. Let

$$\mathcal{D}^p u(x) - \lambda_1 u(x) + g(x) = 0, \tag{3.4}$$

by using Theorem (2.4), we get

$$\begin{aligned} u(x) = {} & C_1 x^{p-1}E_{p,p}(\lambda_1 x^p) + C_2 x^{p-2}E_{p,p-1}(\lambda_1 x^p) \\ & - \int_0^x (x-y)^{p-1}E_{p,p}(\lambda_1(x-y)^p) g(y) dy. \end{aligned} \tag{3.5}$$

Since $\lim_{x\to 0^+} x^{2-p}u(x) = 0$, it is clear that $C_2 = 0$. Now, we take the derivative of (3.5) for $x > 0$, we have

$$u'(x) = C_1 x^{p-2} E_{p,p-1}(\lambda_1 x^p) - \int_0^x (x-y)^{p-2} E_{p,p-1}(\lambda_1 (x-y)^p) g(y) dy.$$

Using condition $u'(1) = 0$ implies that

$$C_1 = \int_0^1 \frac{(1-y)^{p-2} E_{p,p-1}(\lambda_1 (1-y)^p)}{E_{p,p-1}(\lambda_1)} g(y) dy.$$

As consequence, the unique solution of (3.4) is given by

$$u(x) = \frac{1}{E_{p,p-1}(\lambda_1)} \int_0^1 (1-y)^{p-2} x^{p-1} E_{p,p}(\lambda_1 x^p) E_{p,p-1}(\lambda_1 (1-y)^p) g(y) dy \\ - \int_0^x (x-y)^{p-1} E_{p,p}(\lambda_1 (x-y)^p) g(y) dy.$$

Hence, (3.4) has the unique solution given as

$$\begin{aligned} u(x) &= \int_0^x \Big(\frac{x^{p-1} E_{p,p}(\lambda_1 x^p) E_{p,p-1}(\lambda_1 (1-y)^p)}{(1-y)^{2-p} E_{p,p-1}(\lambda_1)} - (x-y)^{p-1} E_{p,p}(\lambda_1 (x-y)^p) \Big) g(y) dy \\ &+ \int_x^1 \frac{x^{p-1} E_{p,p}(\lambda_1 x^p) E_{p,p-1}(\lambda_1 (1-y)^p)}{(1-y)^{2-p} E_{p,p-1}(\lambda_1)} g(y) dy. \end{aligned}$$

Further we can write $u(x)$ in term of Green's function as:

$$u(x) = \int_0^1 G_{\lambda_1}(x,y) g(y) dy.$$

where $G_{\lambda_1}(x,y)$ is given by (3.2).

Similarly, one can follow all the above steps and show Green's function $G_{\lambda_2}(x,y)$ for the differential equation given as;

$$\mathcal{D}^q v(x) - \lambda_2 v(x) + h(x) = 0,$$

gives equivalent integral equation, i.e.,

$$v(x) = \int_0^1 G_{\lambda_2}(x,y) h(y) dy,$$

where, $G_{\lambda_2}(x,y)$ is given by (3.3). □

In the following lemma, we describe λ a set of parameters, for which the Green's function has a constant sign. Now, we introduce λ_1^* and λ_2^* as the biggest negative zero of $E_{p,p-1}(\lambda_1) = 0$ and $E_{q,q-1}(\lambda_2) = 0$.

Lemma 3.2. *Let G_{λ_1} and G_{λ_2} be the Green's functions coupled system (1.1). Also λ_1^* and λ_2^* be the first negative zero if $E_{p,p-1}(\lambda_1) = 0$ and $E_{q,q-1}(\lambda_2) = 0$, then for $1 < p, q \leqslant 2$, it is satisfies that*

$$\begin{cases} G_{\lambda_1}(x,y) > 0 \quad \forall \quad x,y \in (0,1), \\ G_{\lambda_2}(x,y) > 0 \quad \forall \quad x,y \in (0,1), \end{cases}$$

provided that $\lambda_1 > \lambda_1^*$ *and* $\lambda_2 > \lambda_2^*$.

Proof. On contrary supposed that there are $\lambda_3 > \lambda_1^*$ and $\lambda_4 > \lambda_2^*$ and $(x_0, y_0) \in (0,1) \times (0,1)$, such that $G_{\lambda_1}(x_0, y_0) = 0$ and $G_{\lambda_2}(x_0, y_0) = 0$, where G_{λ_1} and G_{λ_2} be the Green's functions for coupled system (1.1).

Now, we define functions $u, v : [0,1] \to \mathcal{R}$,

$$u(x) = G_{\lambda_3}(x, y_0) \ \text{ and } \ v(x) = G_{\lambda_4}(x, y_0).$$

Let suppose $y_0 \geqslant x_0$ from Green's functions we know that $u(x) \neq 0$ and $v(x) \neq 0$ on interval $[0, x_0]$. So, from the definition of Green's function it follows that u is a solution of the problem:

$$\mathcal{D}^p u(x) - \lambda_3 u(x) = 0, \quad 0 < x < x_0, \quad \lim_{x \to 0} x^{2-p} u(x) = u'(x) = 0. \tag{3.6}$$

Similarly, v is the solution of the problem:

$$\mathcal{D}^q v(x) - \lambda_4 v(x) = 0, \quad 0 < x < x_0, \quad \lim_{x \to 0} x^{2-q} v(x) = v'(x) = 0. \tag{3.7}$$

In particular, λ_3 and λ_4 are the eigenvalue for (3.6) and (3.7) respectively.

Arguing as in the beginning of this section, one can easily verify that $\tilde{\lambda_m}$ and $\tilde{\lambda_n}$ are the eigenvalues for the problem (3.6) and (3.7) given by the expression:

$$E_{p,p}(\tilde{\lambda_m} x_0^p) = 0 \ \text{ and } \ E_{q,q}(\tilde{\lambda_n} x_0^q) = 0. \tag{3.8}$$

Since, λ_1^* is the biggest negative zero of $E_{p,p}(\lambda_1)$ and λ_2^* is the biggest negative zero of $E_{q,q}(\lambda_2)$, so we deduce from (3.8) that; $\tilde{\lambda_1} := \frac{{\lambda_1}^*}{x_0^p}$ and $\tilde{\lambda_2} := \frac{{\lambda_2}^*}{x_0^q}$ are the first eigenvalue of the problem (3.6) and (3.7). Thus, we conclude that (λ_3, u) and (λ_4, v) are the eigenvalue eigenvector pair of the problem (3.6) and (3.7), and

$$\lambda_3 > \lambda_1^* > \tilde{\lambda_1} \quad \text{and} \quad \lambda_4 > \lambda_2^* > \tilde{\lambda_2},$$

which contradicts the fact that λ_1^* and λ_2^* are the first eigenvalue of the problem (3.6) and (3.7).

As consequence, we have proved that if $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, then the Green's function is positive on $\mathcal{I} \times \mathcal{I}$. Now, consider $\lambda_1 < \lambda_1^*$ and $\lambda_2 < \lambda_2^*$, such that $E_{p,p}(\lambda_1) \neq 0$ and $E_{q,q}(\lambda_2) \neq 0$, so Green's function exists and y_0 be small enough such that $\lambda_1(1-y_0)^p < \lambda_1^*$ and $\lambda_2(1-y_0)^q < \lambda_2^*$, then, we have that $E_{p,p}(\lambda_1(1-y_0)^p)$ and $E_{q,q}(\lambda_2(1-y_0)^q)$ change its sign on $[y_0, 1)$. Therefore, $G_{\lambda_1}(x_0, y)$ and $G_{\lambda_2}(x_0, y)$ change its sign on $[y_0, 1]$. □

Now, let us prove the following inequalities for Green's function, which will be fundamental to existence of solutions of our main result.

Lemma 3.3. *Let* G_{λ_1} *and* G_{λ_2} *be the Green's functions associated to coupled problem* (1.1), $1 < p, q \leqslant 2$, *and* $\lambda_1 > \lambda_1^*$. *Then* $\exists$ M_1 *and* M_2 *positive constants and* $m_1(x)$ *and* $m_2(x)$ *continuous functions,* $m_1(0) = m_2(0) = 0$ *and* $m_1(x) \geqslant 0$, $m_2(x) \geqslant 0$. *Then the following inequalities are satisfied:*

$$m_1(x) \leqslant \frac{x^{2-p} G_{\lambda_1}(x,y)}{y(1-y)^{p-2}} \leqslant M_1 \quad \forall \ x, y \in (0,1)$$

and

$$m_2(x) \leqslant \frac{x^{2-q} G_{\lambda_2}(x,y)}{y(1-y)^{q-2}} \leqslant M_2 \quad \forall\ x,y \in (0,1).$$

Proof. Define

$$h_1(x,y) = \frac{x^{2-p} G_{\lambda_1}(x,y)}{y(1-y)^{p-2}} \quad \text{and} \quad h_2(x,y) = \frac{x^{2-q} G_{\lambda_2}(x,y)}{y(1-y)^{q-2}},$$

which are continues on $\mathcal{J} \times (0,1]$.

In addition, for $x \in \mathcal{J}$, we have

$$\begin{aligned} \lim_{y\to 0} h_1(x,y) &= \lim_{y\to 0} \frac{x^{2-p} e_p^{\lambda_1 x} E_{p,p}(\lambda_1(1-y)^p) - (1-y)^{2-p} E_{p,p-1}(\lambda_1) e_p^{\lambda_1(x-y)}}{y E_{p,p-1}(\lambda_1)} \\ &= H(x). \end{aligned}$$

So, $H(x)$ exists and finite and $H(x) > 0$ for all $x \in (0,1]$. As direct consequence, we get that

$$m_1(x) = \min_{y\in\mathcal{J}} h_1(x,y) \quad \text{and} \quad M_1 = \max_{(x,y)\in\mathcal{J}\times\mathcal{J}} h_1(x,y),$$

is continuous function on $\mathcal{J}$, $m(0) = 0$ and $m(x) > 0$ for all $x \in (0,1]$. Similarly, we can get $m_2(x)$ and M_2 for $h_2(x,y)$:

$$m_2(x) = \min_{y\in\mathcal{J}} h_2(x,y) \quad \text{and} \quad M_2 = \max_{(x,y)\in\mathcal{J}\times\mathcal{J}} h_2(x,y).$$

□

Theorem 3.4. *Let* $G_{\lambda 1}$ *and* $G_{\lambda 2}$ *be the Green's functions. Then* $G_{\lambda 1}$ *and* $G_{\lambda 2}$ *satisfies the following properties:*

(C_1) $G_{\lambda 1}$ *and* $G_{\lambda 2}$ *are continuous.*

(C_2) $G_{\lambda 1}(0,s) = G_{\lambda 2}(0,s) = G_{\lambda 1}(x,0) = G_{\lambda 2}(x,0) = \frac{d}{dx} G_{\lambda 1}(x,y)\Big|_{x=1} = \frac{d}{dx} G_{\lambda 2}(x,y)\Big|_{x=1} = 0.$

(C_3) $|G_{\lambda 1}(x,y)| = +\infty$ and $|G_{\lambda 2}(x,y)| = +\infty$.

(C_4) $\int_0^1 |G_{\lambda 1}(x,y)| dy < \infty$ and $\int_0^1 |G_{\lambda 2}(x,y)| dy < \infty$.

(C_5) $\int_0^1 \left|\frac{d x^{2-p} G_{\lambda 1}(x,y)}{dx}\right| dy < \infty$ *and* $\int_0^1 \left|\frac{d x^{2-q} G_{\lambda 2}(x,y)}{dx}\right| dy < \infty$.

Proof. It is very easy to prove $(C_1) - (C_3)$, so we leave it and for $(C_4), (C_5)$ we can use below theorems. □

4. Existence and uniqueness of a solution of coupled system

The following section is devoted to the $\mathcal{EU}$ of positive solution of nonlinear coupled system (1.1). For fixed point result we apply a Schaefer's fixed-point theorem.

Let $\mathcal{B}$ be real Banach space with cone $\mathcal{B}_+$. Consider $x \in \mathcal{B}_+$ with $\|u_0\| \leqslant 1$ and also define the sub-cone $\mathcal{P}_{u_0}$ and $\mathcal{P}_{v_0}$ on Banach space $\mathcal{B}$ i.e., $\mathcal{P}_{u_0} = \{x \in \mathcal{B}_+,\ x \geqslant \|x\| u_0\}$ and $\mathcal{P}_{v_0} = \{v \in C_{2-q}(\mathcal{J}) :\ v(x) \geqslant v_0(t)\|v\|_{2-q}\}$, where $C_{2-p}(\mathcal{J}) = \{u : \mathcal{J} \to R; x^{2-p}u(x) \in C^1(\mathcal{J})\}$ and $C_{2-q}(\mathcal{J}) = \{v : \mathcal{J} \to R; x^{2-q}v(x) \in C^1(\mathcal{J})\}$ such that $C^1(\mathcal{J})$ be the Banach space of all continues functions. Thus, It clearly shows that $\mathcal{P}_{u_0} \subset C_{2-p}(\mathcal{J})$ and $\mathcal{P}_{v_0} \subset C_{2-q}(\mathcal{J})$.

Consider sub-cone $\mathcal{P}_{u_0} \subset C_{2-p}(\mathcal{J})$ and $\mathcal{P}_{v_0} \subset C_{2-q}(\mathcal{J})$ of a Banach space $\mathcal{B}$, which is given as

$$\mathcal{P}_{u_0} = \{u \in C_{2-p}(\mathcal{J}) :\ u(x) \geqslant u_0(x)\|u\|_{2-p}\} \text{ and } \mathcal{P}_{v_0} = \{v \in C_{2-q}(\mathcal{J}) :\ v(x) \geqslant v_0(t)\|v\|_{2-q}\},$$

with $u_0(x)$ and $v_0(x)$ are defined as;

$$u_0(x) = x^{p-2}\frac{m_1(x)}{M_1},\ v_0(x) = x^{q-2}\frac{m_2(x)}{M_2},\ \ x \in \mathcal{J},$$

where $m_1(x)$, M_1, $m_2(x)$ and M_2 were defined in (3.3). So, it is clear that $u_0(x) \in \mathcal{B}$ and $v_0(x) \in \mathcal{B}$ with $\|u_0\|_{2-p} \leqslant 1$ and $\|v_0\|_{2-q} \leqslant 1$.

Define an operator $T : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ as

$$T(u,v)(x) = \begin{pmatrix} \int_0^1 G_{\lambda_1}(x,y)g(y)dy \\ \\ \int_0^1 G_{\lambda_2}(x,y)h(y)dy \end{pmatrix},$$

$$T(u,v)(x) = \begin{pmatrix} T_p(u,g)(x) \\ \\ T_q(v,h)(x) \end{pmatrix}.$$

The solution of the proposed system (1.1) coincides with the fixed point of the operator T, where

$$T_p u(x) = \int_0^1 G_{\lambda_1}(x,y)g(y, y^{2-p}u(y), s^{2-p}v(y))dy,$$

and

$$T_q v(x) = \int_0^1 G_{\lambda 2}(x,y)h(y, y^{2-q}u(y), y^{2-q}v(y))dy.$$

where $G_{\lambda 1}(x,y)$ and $G_{\lambda 2}(x,y)$ are defined in (3.2) and (3.3).

For rest of paper, we assume the following hypothesis:

(H_1) $g, h : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \to \mathcal{J}$ are continuous functions.

(H_2) There exists constants M_o and $M_o^{'}$, such that

$$M_o = \max_{x \in \mathcal{J}} |g(x,u,v)| \quad \text{and} \quad M_o^{'} = \max_{x \in \mathcal{J}} |h(x,u,v)|.$$

(H_3) $\frac{M_0}{M_1 p(p-1)} < 1$ and $\frac{M_0^{'}}{M_2 q(q-1)} < 1$.

(H_4) There exists a constant $0 < K_1 < 1$, $\forall\, u,, v, s, t \in R$, such that

$$\begin{aligned}|g(x, x^{2-p}u, x^{2-p}v) - g(x, x^{2-p}s, x^{2-p}t)| &\leqslant K_1\{|x^{2-p}(u-s)| + |x^{2-p}(v-t)|\}\\ &\leqslant K_1\{\|u-s\|_{2-p} + \|v-t\|_{2-p}\}.\end{aligned}$$

Similarly, there exists a constant $0 < K_2 < 1$, $\forall\, u, v, s, t \in R$, such that

$$\begin{aligned}|h(x, x^{2-q}u, x^{2-q}v) - h(x, x^{2-q}s, x^{2-q}t)| &\leqslant K_2\{|x^{2-q}(u-s)| + |x^{2-q}(v-t)|\}\\ &\leqslant K_2\{\|u-s\|_{2-q} + \|v-t\|_{2-q}\}.\end{aligned}$$

Theorem 4.1. *Assume that* (H_1) *and* (H_2) *hold and* $\lambda > \lambda_1^*$, $\lambda > \lambda_2^*$. *Then* $T : \mathcal{P}_{u_0} \times \mathcal{P}_{v_0} \to \mathcal{P}_{u_0} \times \mathcal{P}_{v_0}$ *is completely continuous operator.*

Proof. We have that $Tu(x) \geqslant 0\ \forall x \in \mathcal{J}$ and $\|Tu(x)\|_{2-p} < \infty$. Now let $u \in P_{u_0}$ and $v \in P_{v_0}$, then

$$\begin{aligned}x^{2-p}T_p(u) &= \int_0^1 x^{2-p}G_{\lambda_1}(x,y)g(y, y^{2-p}u(y), y^{2-p}v(y))dy\\ &\geqslant m_1(t)\int_0^1 y(1-y)^{p-2}g(y, y^{2-p}u(y), y^{2-p}v(y))dy\\ &\geqslant \frac{m_1(x)}{M_1}\int_0^1 \max_{x\in\mathcal{J}}\{x^{2-p}G_{\lambda_1}(x,y)\}g(y, y^{2-p}u(y), y^{2-p}v(y))dy\\ &\geqslant \frac{m_1(x)}{M_1}\max_{x\in\mathcal{J}}\{\int_0^1 x^{2-p}G_{\lambda_1}(x,y)g(y, y^{2-p}u(y), y^{2-p}v(y))dy\}\\ &= \frac{m_1(x)}{M_1}\|Tu\|_{2-p}.\end{aligned}$$

Similarly, one can also write

$$x^{2-q}T_q(v) \geqslant \frac{m_2(x)}{M_2}\|Tv\|_{2-q}.$$

Next, we will show that T is uniformly bounded. Let $\Omega_1 \subset P_{u_0}$, $\Omega_2 \subset P_{v_0}$ be bounded sets in P_{u_0}, P_{v_0}, then $\exists$ two positive numbers M and L such that

$$\|u\|_{2-p} \leqslant L \quad \text{and} \quad \|v\|_{2-q} \leqslant M.$$

Using condition (H_2), we have $\forall\, (u, v) \in \Omega$, where $\Omega = (\Omega_1, \Omega_2)$.

$$\begin{aligned}|x^{2-p}T_p u(x)| &\leqslant M_0\int_0^1 x^{2-p}G_{\lambda_1}(x,y)dy\\ &\leqslant M_0 M_1\int_0^1 y(1-y)^{p-2}dy\\ &\leqslant \frac{M_0 M_1}{p(p-1)}\end{aligned}$$

Similarly,

$$\begin{aligned} |x^{2-q}T_q v(x)| &\leqslant M_0' \int_0^1 x^{2-q} G_{\lambda_2}(x,y)dy \\ &\leqslant M_0' M_2 \int_0^1 y(1-y)^{q-2} dy \\ &\leqslant \frac{M_0' M_2}{q(q-1)}. \end{aligned}$$

Hence, $T(\Omega)$ is bounded.

Now, we are showing that $T(\Omega)$ is equi-continuous in $C_{2-p}(\mathcal{J})$ and $C_{2-q}(\mathcal{J})$. Consider, $0 < x_1 < x_2 < 1$ and $u \in \mathcal{B}$, then

$$\begin{aligned} & |x_2^{2-p} T_p u(x_2) - x_1^{2-p} T_p u(x_1)| \\ = & \Big| \int_0^1 x_2^{2-p} G_{\lambda_1}(x_2,y) g(y, y^{2-p}u(y), y^{2-p}v(y))dy \\ - & \int_0^1 x_1^{2-p} G_{\lambda_1}(x_1,y) g(y, y^{2-p}u(y), y^{2-p}v(y))\Big| dy \\ \leqslant & \int_0^{x_1} (1-y)^{p-2} \Big| \frac{x_2 E_{p,p}(\lambda_1 x_2^p) E_{p,p-1}[\lambda_1(1-y)^p]}{E_{p,p-1}(\lambda_1)} \\ - & \frac{x_1 E_{p,p}(\lambda_1 x_1^p) E_{p,p-1}[\lambda_1(1-y)^p]}{E_{p,p-1}(\lambda_1)} \Big| g(y, y^{2-p}u(y), y^{2-p}v(y))dy \\ + & \int_0^{x_1} \Big| x_1^{2-p}(x_1-y)^{p-1} E_{p,p-1}(\lambda_1(1-y)^p) \\ - & x_2^{2-p}(x_2-y)^{p-1} E_{p,p-1}(\lambda_1(1-y)^p) \Big| g(y, y^{2-p}u(y), y^{2-p}v(y))dy \\ + & \int_{x_1}^{x_2} (1-y)^{p-2} \Big| \frac{x_2 E_{p,p}(\lambda_1 x_2^p) E_{p,p-1}[\lambda_1(1-y)^p]}{E_{p,p-1}(\lambda_1)} \\ - & \frac{x_1 E_{p,p}(\lambda_1 x_1^p) E_{p,p-1}[\lambda_1(1-y)^p]}{E_{p,p-1}(\lambda_1)} \Big| g(y, y^{2-p}u(y), y^{2-p}v(y))dy \\ + & x_2^{2-p} \int_{x_1}^{x_2} \Big| (x_2-y)^{p-1} E_{p,p}(\lambda_1(x_2-y))^p \Big| g(y, y^{2-p}u(y) y^{2-p}v(y))dy \\ \leqslant & \int_{x_2}^1 (1-y)^{p-2} \Big| \frac{x_2 E_{p,p}(\lambda_1 x_2^p) E_{p,p-1}[\lambda_1(1-y)^p]}{E_{p,p-1}(\lambda_1)} \\ - & \frac{x_1 E_{p,p}(\lambda_1 x_1^p) E_{p,p-1}[\lambda_1(1-y)^p]}{E_{p,p-1}(\lambda_1)} \Big| g(y, y^{2-p}u(y), y^{2-p}v(y))dy \end{aligned}$$

Since, the functions $\frac{x_1 E_{p,p}(\lambda_1 x_1^p) E_{p,p-1}[\lambda_1(1-y)^p]}{E_{p,p-1}(\lambda_1)}$ and $\frac{x_2 E_{q,q}(\lambda_1 x_2^q) E_{q,q-1}[\lambda_1(1-y)^q]}{E_{q,q-1}(\lambda_1)}$ are uniformly continuous in given intervals. Then, for $\epsilon > 0 \ \exists \ \delta > 0 \ni$ if $|x_1 - x_2| < \delta$, we obtain that first integral is bounded.

$$M_0 \epsilon \int_0^{x_1} (1-y)^{p-2} dy \leqslant \frac{M_0 \epsilon}{(p-1)},$$

third and last integrals are bounded

$$M_0\epsilon\int_{x_1}^{x_2}(1-y)^{p-2}dy \leqslant \frac{M_0\epsilon}{(p-1)},$$

and

$$M_0\epsilon\int_{x_2}^{1}(1-y)^{p-2}dy \leqslant \frac{M_0\epsilon}{(p-1)}.$$

The function in second integral is uniformly continuous on $\mathcal{J}\times R^2$. So, we deduce that second integral is bounded by $M_0\epsilon$. Also, the function $\frac{t_1E_{p,p}(\lambda x_1^p)E_{p,p-1}[\lambda(1-y)^p]}{E_{p,p-1}(\lambda)}$ is continuous, then the fourth integral is

$$|E_{p,p}(\lambda_1(x-y)^p)| \leqslant M_1$$

So,

$$\begin{aligned} S &= |\int_{x_1}^{x_2}(t_2-y)^{p-1}E_{p,p}[\lambda_1(x_2-y)^p]g(y,y^{2-p}u(y),y^{2-p}v(y))dy)| \\ &\leqslant \frac{M_0M_1}{2}(x_2-x_1)^p. \end{aligned}$$

Finally, for every $\epsilon>0\ \exists\ \delta>0 \ni$ if $|x_2-x_1|<\delta$, we deduce that

$$|x_2^{2-p}T_pu(x_2)-x_1^{2-p}T_pv(x_1)| \leqslant \Big(\frac{3M_0}{p-1}+M_0+1\Big)\epsilon.$$

Similarly, we can also find for

$$|x_2^{2-q}T_qu(x_2)-x_1^{2-q}T_qv(x_1)| \leqslant \Big(\frac{3M_0^{'}}{q-1}+M_0^{'}+1\Big)\epsilon.$$

Then, operators $x^{2-p}Tu(x)$ and $x^{2-q}Tv(x)$ are equi-continuous in $C(\mathcal{J})$ and also $T(\Omega)$ is equi-continuous in $C_{2-p}(\mathcal{J})$ and $C_{2-q}(\mathcal{J})$.

By Arzilā Ascoli's theorem, $T(\Omega)$ is a relatively compact in $C_{2-p}(\mathcal{J})\times C_{2-q}(\mathcal{J})$. As consequence, $T:\mathcal{P}_{u_0}\times\mathcal{P}_{v_0}\to\mathcal{P}_{u_0}\times\mathcal{P}_{v_0}$ is completely continuous operator. □

Theorem 4.2. *Suppose the functions* $g,h:\mathcal{J}\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ *are continuous. If the hypothesis* (H_1)-(H_3) *holds. Then the coupled system* (1.1)*, has at least one solution.*

Proof. Since, the functions $G_{\lambda_1}, G_{\lambda_2}, g$ and h are continuous functions, so the operator T is also continuous. For each $\epsilon\ \exists\ \mathcal{R}_{\mathcal{B}}>0$, such that

$$\mathcal{R}_{\mathcal{B}} \leqslant S_pM_0+S_qM_0^{'}.$$

To show set

$$B=\{(u,v)\in\mathcal{B}\times\mathcal{B}:\ \|(u,v)\|_{\mathcal{B}\times\mathcal{B}}\leqslant\Gamma\}.$$

For $u \in B$, we have

$$\begin{aligned}|x^{2-p}T_p(u)(x)| &\leqslant |x^{2-p}\int_0^1 G_1(x,y)g(y,y^{2-p}u(y),y^{2-p}v(y))dy| \\ &\leqslant M_0|\int_0^1 x^{2-p}G_1(x,y)dy| \\ &\leqslant \frac{M_0}{M_1}|\int_0^1 y(1-y)^{p-2}dy| \\ &\leqslant \frac{M_0}{M_1p(p-1)},\end{aligned}$$

implies that

$$\|T_p(u)\|_{\mathcal{B}} \leqslant \frac{M_0}{M_1p(p-1)}. \tag{4.1}$$

On the same way, we get

$$\|T_q(v)\|_{\mathcal{B}} \leqslant \frac{M_0^{'}}{M_2q(q-1)}. \tag{4.2}$$

From (4.1) and (4.2), we have

$$\|T_p(u)\|_{\mathcal{B}} + \|T_q(v)\|_{\mathcal{B}} \leqslant \frac{M_0}{M_1p(p-1)} + \frac{M_0^{'}}{M_2q(q-1)} = S_pM_0 + S_qM_0^{'}.$$

Thus,

$$\|T(u,v)\|_{\mathcal{B}\times\mathcal{B}} \leqslant \mathcal{R}_{\mathcal{B}}.$$

Hence, by Theorem (4.1) the operator T is completely continuous and bounded. Thus, T has at least one solution corresponding to coupled problem (1.1). □

Theorem 4.3. *Assume that* (H_1)-(H_4) *holds, then proposed coupled problem* (1.1) *has a unique solution in* P_{u_0,v_0} *provided that*

$$\mathcal{Z}_g + \mathcal{Z}_h < 1$$

Proof. We apply the Banach fixed point theorem. So, let us show that T is a contraction operator in P_{u_0}, P_{v_0}. Suppose, $(u_1,v_1),(u_2,v_2) \in P_{u_0}, P_{v_0}$

$$\begin{aligned}&\left|x^{2-p}T(u_1,v_1) - x^{2-p}T(u_2,v_2)\right| \\ \leqslant\ & x^{2-p}\int_0^1 G_{\lambda_1}(x,y)\left|g(y,y^{2-p}u_1(y),y^{2-p}v_1(y)) - g(y,y^{2-p}u_2(y),y^{2-p}v_2(y))\right|dy \\ \leqslant\ & K_1M_1\int_0^1 y(1-y)^{p-2}(|y^{2-p}u_1 - y^{2-p}u_2| + |y^{2-p}v_1 - y^{2-p}v_2|)dy \\ \leqslant\ & K_1M_1\int_0^1 \left\{\|u_1-u_2\|_{2-p} + \|v_1-v_2\|_{2-p}\right\}y(1-y)^{p-2}dy \\ \leqslant\ & \frac{K_1M_1}{p(p-1)}\left\{\|u_1-u_2\|_{2-p} + \|v_1-v_2\|_{2-p}\right\}.\end{aligned} \tag{4.3}$$

By this way one can also show for

$$\left|x^{2-q}T(u_1,v_1)-x^{2-q}T(u_2,v_2)\right| \leqslant \frac{K_2M_2}{q(q-1)}\left\{\|u_1-u_2\|_{2-q}+\|v_1-v_2\|_{2-q}\right\}. \quad (4.4)$$

So, from (4.3) and (4.4), we have

$$\begin{aligned}\|T(u_1,v_1)-T(u_2,v_2)\|_{\mathcal{B}\times\mathcal{B}} &\leqslant \left\{\frac{K_1M_1}{p(p-1)}+\frac{K_2M_2}{q(q-1)}\right\}\left\{\|(u_1,v_1)-(u_2,v_2)\|\right\} \\ &\leqslant (\mathcal{Z}_g+\mathcal{Z}_h)\left\{\|(u_1,v_1)-(u_2,v_2)\|\right\}\end{aligned}$$

Hence, T is a contraction operator and has a fixed-point corresponding to unique solution of coupled problem (1.1). □

5. Hyers-Ulam stability analysis for coupled system

In the following section, we deduced the $\mathcal{HU}$ stability for FDE with given boundary conditions. For some $\delta_i > 0$, $i = 1, 2, \ldots$ consider the system given by

$$\begin{cases}|\mathcal{D}^p u(x)-\lambda_1 u(x)+g(x,x^{2-p}u(x),x^{2-p}v(x))| \leqslant \delta_1, \\ \\ |\mathcal{D}^q v(x)-\lambda_2 v(x)+h(x,x^{2-q}u(x),x^{2-q}v(x))| \leqslant \delta_2.\end{cases} \quad (5.1)$$

Definition 5.1. System (1.1) is $\mathcal{HU}$- stable, if there is $\mathcal{C} = (\mathcal{C}_p, \mathcal{C}_q) > 0$ such that for some $\delta = (\delta_1, \delta_2) > 0$ and for each solution $(u, v) \in \mathcal{B}\times\mathcal{B}$ of (5.1) there is a solution $(s, t) \in \mathcal{B}\times\mathcal{B}$ of (1.1) with

$$|(u,v)(x)-(s,t)(x)| \quad \leqslant \quad \mathcal{C}\delta \quad \text{for all } x \in \mathcal{J}. \quad (5.2)$$

$(u, v) \in \mathcal{B}\times\mathcal{B}$ is a solution of (5.1), if there are functions $\rightarrow_g$ and $\leftarrow_h$ which depends upon u, v respectively, such that

$(\mathcal{R}_1)$: $\left|\omega_g(x)\right| \leqslant \delta_1,\ \left|\psi_h(x)\right| \leqslant \delta_2, \forall\, x \in \mathcal{J}$;

$(\mathcal{R}_2)$:

$$\begin{cases}\mathcal{D}^p u(x) = \lambda_1 u(x) - g(x,x^{2-p}u(x),x^{2-p}v(x)) + \omega_g(x), \forall\, x \in \mathcal{J}, \\ \\ \mathcal{D}^q v(x) = \lambda_2 u(x) - h(x,x^{2-q}u(x),x^{2-q}v(x)) + \psi_h(x), \forall\, x \in \mathcal{J}.\end{cases}$$

Theorem 5.2. *Let $(u, v) \in \mathcal{B}\times\mathcal{B}$ be the solution of (5.1), then the following inequalities hold:*

$$\begin{cases}|u(x)-m(x)| & \leqslant \mathcal{X}_p\delta_1 \\ \\ |v(x)-n(x)| & \leqslant \mathcal{X}_q\delta_2.\end{cases}$$

Proof. Using $(\mathcal{R}_2)$ from Remark (5), we have

$$
\begin{cases}
\mathcal{D}^p u(x) = \lambda_1 u(x) - g(x, x^{2-p}u(x), x^{2-p}v(x)) + \omega_g(x), \forall\, x \in \mathcal{J}, \\
\mathcal{D}^q v(x) = \lambda_2 v(x) - h(x, x^{2-q}u(x), x^{2-q}v(x)) + \psi_h(x), \forall\, x \in \mathcal{J}, \\
\lim\limits_{x \to 0^+} x^{2-p}u(x) = u'(1) = 0, \\
\lim\limits_{x \to 0^+} x^{2-q}v(x) = v'(1) = 0,
\end{cases}
\tag{5.3}
$$

We can get the solution of (5.3) by Theorem (3.1)

$$u(x) = \int_0^1 G_{\lambda_1}(x,y)g(y)dy + \int_0^1 G_{\lambda_1}(x,y)\omega_g(y)dy.$$

and

$$v(x) = \int_0^1 G_{\lambda_2}(x,y)h(y)dy + \int_0^1 G_{\lambda_2}(x,y)\psi_h(y)dy.$$

From the first equation we have

$$
\begin{aligned}
u(x) &= \int_0^x \Big(\frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} - (x-y)^{p-1}E_{p,p}(\lambda_1(x-y)^p)\Big) g(y)dy \\
&+ \int_x^1 \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} g(y)dy \\
&+ \int_0^x \Big(\frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} - (x-y)^{p-1}E_{p,p}(\lambda_1(x-y)^p)\Big) \omega_g(y)dy \\
&+ \int_x^1 \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} \omega_g(y)dy.
\end{aligned}
$$

So, the above equation becomes

$$
\begin{aligned}
&|u(x) - m(x)| \\
\leqslant & \int_0^x \Big(\frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} - (x-y)^{p-1}E_{p,p}(\lambda_1(x-y)^p)\Big) \omega_g(y)dy \\
+ & \int_x^1 \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} \omega_g(y)dy,
\end{aligned}
$$

where

$$
\begin{aligned}
m(x) &= \int_0^x \Big(\frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} - (x-y)^{p-1}E_{p,p}(\lambda_1(x-y)^p)\Big) g(y)dy \\
&+ \int_x^1 \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)} g(y)dy.
\end{aligned}
$$

Using (5), we obtain

$$|u(x) - m(x)| \leqslant \mathcal{X}_p \delta_1.$$

Similarly, for $v(x)$ we obtain

$$|v(x) - n(x)| \leqslant \mathcal{X}_q \delta_2.$$

□

Theorem 5.3. *Under the hypothesis* (H_1)-(H_4), *the proposed coupled system is* $\mathcal{HU}$ *stable if*

$$1 - \mathcal{Z}_g^{'} \mathcal{Z}_h^{'} > 0.$$

Proof. The solution $(u, v) \in \mathcal{B} \times \mathcal{B}$ of the system (1.1) is given

$$\begin{cases} \mathcal{D}^p s(x) - \lambda_1 s(x) + g(x, x^{2-p} s(x), x^{2-p} t(x)) = 0 \\ \mathcal{D}^q t(x) - \lambda_2 t(x) + h(x, x^{2-q} s(x), x^{2-q} t(x)) = 0 \\ \lim_{x \to 0} x^{2-p} s(x) = s'(1) = 0 \\ \lim_{x \to 0} x^{2-q} t(x) = t'(1) = 0. \end{cases} \tag{5.4}$$

Then by Theorem (3.1), the solution of (5.4) is given by

$$\begin{cases} s(x) = \int_0^1 G_{\lambda_1}(x, y) g(y, y^{2-p} s(x), y^{2-p} t(x)) dy \\ t(x) = \int_0^1 G_{\lambda_2}(x, y) h(y, y^{2-q} s(x), y^{2-q} t(x)) dy. \end{cases} \tag{5.5}$$

Now consider

$$\begin{aligned}
|u(x)-s(x)| &\leqslant |u(x)-m(x)|+|m(x)-s(x)| \\
&\leqslant \mathcal{X}_p\delta_1+\Big(\int_0^x \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)}-(x-y)^{p-1}E_{p,p}(\lambda_1(x-y)^p) \\
&+ \int_x^1 \frac{x^{p-1}E_{p,p}(\lambda_1 x^p)E_{p,p-1}(\lambda_1(1-y)^p)}{(1-y)^{2-p}E_{p,p-1}(\lambda_1)}\Big)|g(y)-g_s(y)|dy, \\
&\leqslant \mathcal{X}_p\delta_1+\int_0^1 y^{2-p}G_{\lambda_1}(x,y)|g(y)-g_s(y)|dy \\
&\leqslant \mathcal{X}_p\delta_1+\int_0^1 M_1 y(y-1)^{p-2}|g(y)-g_s(y)|dy \\
&\leqslant \mathcal{X}_p\delta_1+\frac{M_1}{p(p-1)}|g(y)-g_s(y)|dy
\end{aligned} \tag{5.6}$$

Using (H_4), we obtain

$$\begin{aligned}
|g(y)-g_s(y)| &\leqslant |g(y,y^{2-p}u(x),y^{2-p}v(x))-g_s(y,y^{2-p}s(x),y^{2-p}t(x))| \\
&\leqslant |y^{2-p}(u(x)-s(x))|+|y^{2-p}(v(x)-t(x)| \\
&\leqslant K_1\{|u(x)-s(x)|+|v(x)-t(x)|\}.
\end{aligned} \tag{5.7}$$

Now, (5.6) implies that

$$\begin{aligned}
|u(x)-s(x)| &\leqslant \mathcal{X}_p\delta_1+\frac{M_1}{p(p-1)}\{K_1\{|u(x)-s(x)|+|v(x)-t(x)|\} \\
&\leqslant \mathcal{X}_p\frac{\delta_1}{\left(1-\frac{K_1M_1}{p(p-1)}\right)}+\frac{K_1M_1}{p(p-1)-K_1M_1}|v(x)-t(x)|
\end{aligned}$$

implies that

$$|u(x)-s(x)| \leqslant \mathcal{X}_p\delta_1^{'}+\mathcal{Z}_g^{'}|v(x)-t(x)|,$$

where

$$\delta_1^{'}=\frac{\delta_1}{\left(1-\frac{K_1M_1}{p(p-1)}\right)} \quad \text{and} \quad \mathcal{Z}_g^{'}=\frac{K_1M_1}{p(p-1)-K_1M_1}|v(x)-t(x)|$$

$$\|u-s\| \leqslant \mathcal{X}_p\delta_1^{'}+\mathcal{Z}_g^{'}\|v-t\|. \tag{5.8}$$

Similarly for second equation, we have

$$\|v-t\| \leqslant \mathcal{X}_q\delta_2^{'}+\mathcal{Z}_h^{'}\|u-s\|. \tag{5.9}$$

From (5.8) and (5.9), we get

$$\begin{bmatrix} 1 & -\mathcal{Z}_g^{'} \\ -\mathcal{Z}_h^{'} & 1 \end{bmatrix}\begin{bmatrix} \|u-s\|_{\mathcal{B}} \\ \|v-t\|_{\mathcal{B}} \end{bmatrix} \leqslant \begin{bmatrix} \mathcal{X}_p\delta_1^{'} \\ \mathcal{X}_q\delta_2^{'} \end{bmatrix}.$$

$$\begin{bmatrix} \|u-s\|_{\mathcal{B}} \\ \\ \|v-t\|_{\mathcal{B}} \end{bmatrix} \leqslant \begin{bmatrix} \frac{1}{\Theta} & \frac{\mathcal{Z}_g'}{\Theta} \\ \\ \frac{\mathcal{Z}_h'}{\Theta} & \frac{1}{\Theta} \end{bmatrix} \begin{bmatrix} \mathcal{X}_p\delta_1' \\ \\ \mathcal{X}_q\delta_2' \end{bmatrix},$$

where

$$\Theta = 1-\mathcal{Z}_g'\mathcal{Z}_h' > 0.$$

Further simplification, we have

$$\|u-s\| \leqslant \frac{\mathcal{X}_p\delta_1'}{\Theta}+\frac{\mathcal{Z}_g'\mathcal{X}_q\delta_2'}{\Theta}$$

and

$$\|v-t\| \leqslant \frac{\mathcal{X}_q\delta_2'}{\Theta}+\frac{\mathcal{Z}_h'\mathcal{X}_q\delta_2'}{\Theta}$$

$$\|u-s\|+\|v-t\| \leqslant \frac{\mathcal{X}_p\delta_1'}{\Theta}+\frac{\mathcal{X}_q\delta_2'}{\Theta}+\frac{\mathcal{Z}_g'\mathcal{X}_q\delta_2'}{\Theta}+\frac{\mathcal{Z}_h'\mathcal{X}_q\delta_2'}{\Theta},$$

implies that

$$\|(u,v)-(s,t)\| \leqslant \mathcal{C}\delta,$$

where

$$\mathcal{C} = \frac{\mathcal{X}_p}{\Theta}+\frac{\mathcal{X}_q}{\Theta}+\frac{\mathcal{Z}_g'\mathcal{X}_q}{\Theta}+\frac{\mathcal{Z}_h'\mathcal{X}_q}{\Theta}.$$

Thus the coupled system (1.1) is $\mathcal{HU}$ stable. □

6. Example

Consider, the coupled system (1.1) for particular values as given below

$$\begin{cases} D^p u(x)-\lambda_1 u(x)+g(x,x^{2-p}u(x),x^{2-p}v(x))=0, \\ \\ D^q v(x)-\lambda_2 v(x)+h(x,x^{2-q}u(x),x^{2-q}v(x))=0, \end{cases} \tag{6.1}$$

where $p=\frac{7}{6}$, $q=\frac{9}{8}$, $\lambda_1,\lambda_2\in R$ and

$$g(x,x^{2-p}u(x),x^{2-p}v(x)) = \frac{x+x^{\frac{5}{6}}u(x)}{e^{50x}}+\frac{x^5 6v(x)}{e^{100x}},$$

$$h(x,x^{2-q}u(x),x^{2-q}v(x)) = \frac{\sin u(x)}{75-x^{2-q}\cos u(x)}+\frac{x^{2-q}\sin v(x)}{100x}.$$

By (H_2), there exists constants M_o and M_o', such that

$$M_o = \max|g(x,u,v)| = \max\left|\frac{x+x^{\frac{5}{6}}u(x)}{e^{50x}}+\frac{x^5 6v(x)}{e^{100x}}\right| = \frac{1}{e^{25}}$$

$$M_o' = \max|h(x,\tilde{u},\tilde{v}) = \max\left|\frac{\sin(u(x))}{75-x^{2-q}\cos u(x)}+\frac{x^{2-q}\sin v(x)}{100x}\right| = 0.08.$$

Using Theorem (4.2) and hypothesis $(H_1) - (H_3)$, the system (6.1) has at least one solution. Also, using the following inequalities from Theorem (4.3), we have

$$\mathcal{Z}_p = \frac{K_1 M_1}{p(p-1)} \approx 7.90e^{-50} < 1 \quad \text{and} \quad \mathcal{Z}_q = \frac{K_2 M_2}{q(q-1)} \approx 0.8746 < 1,$$

this implies that

$$\mathcal{Z}_g + \mathcal{Z}_h < 1.$$

then the proposed model has a unique solution. Moreover, the following inequality

$$1 - \mathcal{Z}_g' \mathcal{Z}_h' \approx 132.2 > 0,$$

where

$$\mathcal{Z}_g' = 1.53 \times 10^{-21} \quad \text{and} \quad \mathcal{Z}_h' = 6.9744$$

also satisfied. Thus, the system (6.1) is $\mathcal{HU}$-stable.

7. Conclusion

In this paper, we have explained the required conditions for the existence and uniqueness of the proposed system (1.1). The desired results are obtained by using fixed-point theorems i.e., Banach and Schaefer's fixed-point. Further, we also investigate Hyers-Ulam stability for our proposed model. At last, we set an example for support of our results.

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