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Investigation of the solution of incomplete fractional integrals and derivatives associated with an incomplete Mittag-Leffler function

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Abstract

This paper is based upon incomplete fractional calculus and with the help of this, derived the fractional calculus formula for the incomplete Mittag-Leffler function. The results obtained are found in the form of incomplete Wright function and hypergeometric function.

Keywords: Incomplete Mittag-Leffler function, Incomplete Wright Function, Incomplete Fractional Fractional Integrals, Incomplete Fractional Derivatives, Hypergeometric function.
2010 MSC: 26A33, 33E12, 33E60.

1. Introduction

Incomplete Fractional Order Integrals and derivatives:
Fractional calculus has long been an essential part of core curricula in most branches of physical sciences and engineering. The classical theory of fractional integrals and derivatives occupies an important place in modern science and engineering. It forms one of the foundations of bioengineering, mechanical engineering, Chemical engineering, marine engineering, aeronautics, astronautics, and in fact, just about every scientific or engineering field. There are numerous studies focused in this direction [1, 2, 3, 4, 5]. All these discoveries simulated our interest not only in applications of the notions of the derivatives and integrals of arbitrary order but also in the basic mathematical properties of the fascinating operators. Singh [6] introduced fractional calculus as a new tool using incomplete hypergeometric function.
Let $\alpha, \beta, \gamma \in C$, and let $x \in R_+$, the incomplete fractional integral and incomplete fractional derivative of a function $f(x)$ on R_+ are defined in the following forms:

$$({}_{\Gamma}I_{0,+}^{\alpha,\beta,\gamma})f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2\Gamma_1((\alpha+\beta;x),-\gamma;\alpha;1-\frac{t}{x})f(t)dt, \Re(\alpha) > 0. \quad (1.1)$$

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$$= \frac{d^n}{dx^n} {}_\Gamma I_{0,+}{}^{\alpha+n,\beta-n,\gamma-n} f(x), 0 < \Re(\alpha) + n \leqslant 1 (n \in N_0). \tag{1.2}$$

$$({}_\Gamma I_{-}^{\alpha,\beta,\gamma}) f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{-\alpha-\beta} {}_2\Gamma_1((\alpha+\beta;x), -\gamma; \alpha; 1-\frac{x}{t}) f(t) dt, \Re(\alpha) > 0. \tag{1.3}$$

$$= (-1)^n \frac{d^n}{dx^n} {}_\Gamma I_{-}{}^{\alpha+n,\beta-n,\gamma} f(x), 0 < \Re(\alpha) + n \leqslant 1 (n \in N_0). \tag{1.4}$$

$$({}_\Gamma D_{0,+}^{\alpha,\beta,\gamma}) f(x) = ({}_\Gamma I_{0,+}^{-\alpha,-\beta,-\alpha+\gamma}) f(x)$$

$$= \left(\frac{d}{dx}\right)^n ({}_\Gamma I_{0,+}^{-\alpha+n,-\beta-n,-\alpha+\gamma-n}) f(x), \quad R(\alpha) > 0; n = [R(\alpha)] + 1 \tag{1.5}$$

$$({}_\Gamma D_{-}^{\alpha,\beta,\gamma}) f(x) = ({}_\Gamma I_{-}^{-\alpha,-\beta,\alpha+\gamma}) f(x).$$

$$= \left(\frac{d}{dx}\right)^n ({}_\Gamma I_{0,+}^{-\alpha+n,-\beta-n,\alpha+\gamma}) f(x), \quad R(\alpha) > 0; n = [R(\alpha)] + 1. \tag{1.6}$$

Operators of the incomplete fractional calculus are defined as follows

$$({}_\Gamma I_{0,x}^{\alpha,\beta,\gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\alpha+\rho+\gamma)} x^{\rho-\beta-1} - \frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)} {}_2F_2 \left[\begin{matrix} \alpha+\rho+\gamma, \alpha+\beta \\ \alpha+\rho, \alpha+\beta+1 \end{matrix} ; -x \right], \tag{1.7}$$

where $\alpha, \beta, \gamma, \rho \in C, \Re(\alpha) > 0$, and $\Re(\rho) > max[0, \Re(\beta-\gamma)]$.

And

$$({}_\Gamma I_{-}^{\alpha,\beta,\gamma} t^{\rho-1})(x) = \frac{\Gamma(1+\beta-\rho)\Gamma(1+\gamma-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\beta-\rho+\gamma)} x^{\rho-\beta-1} - \frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(1+\beta-\rho)}{\Gamma(1+\alpha+\beta-\rho)}$$

$$\times {}_2F_2 \left[\begin{matrix} 1+\alpha+\beta-\rho+\gamma, \alpha+\beta \\ 1+\alpha+\beta-\rho, \alpha+\beta+1 \end{matrix} ; -x \right], \tag{1.8}$$

where $\alpha, \beta, \gamma, \rho \in C, \Re(\alpha) > 0$, and $\Re(\rho) > max[\Re(-\beta), \Re(-\gamma)]$.

Incomplete Mittag-Leffler Function:

In recent years some extensions of the Mittag-Leffler function [7, 8] considered by several authors [9, 10, 11] and [2, 3, 12, 13, 14, 15, 16, 17, 18]. In 2013, Singh and Porwal [19] introduced the following incomplete Mittag-Leffler

$$E_{\alpha,\beta}^{[\delta,x]} = \sum_{k=0}^{\infty} \frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} \tag{1.9}$$

$$E_{\alpha,\beta}^{(\delta,x)} = \sum_{k=0}^{\infty} \frac{(\delta,x)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \tag{1.10}$$

where $\alpha,\beta,\delta \in C;\ \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0$ and $[\delta, x]_k$ and $(\delta, x)_k$ represent incomplete Pocchammer symbol and these incomplete Pocchammer symbols satisfy the following decomposition

$$(\lambda;x)_\nu + [\lambda;x]_\nu = (\lambda)_\nu \qquad (\lambda,\nu \in C; x \geqslant 0). \tag{1.11}$$

Incomplete Wright Function:
An incomplete generalized hypergeometric function [22] definded as

$${}_q\overline{\Psi}_q(z) = {}_q\overline{\Psi}_q \left[\begin{array}{c} [a_1,\alpha_1,x]...(a_p,\alpha_p) \\ (b_1,\beta_1,x]...(b_q,\beta_q) \end{array} |z \right]$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 k, x)\Gamma(a_2+\alpha_2 k)...\Gamma(a_p+\alpha_p k)}{\Gamma(b_1+\beta_1 k, x)\Gamma(b_2+\beta_2 k)...\Gamma(b_q+\beta_q k)} \frac{z^k}{k!} \tag{1.12}$$

$${}_q\underline{\Psi}_q(z) = {}_q\underline{\Psi}_q \left[\begin{array}{c} (a_1,\alpha_1,x)...(a_p,\alpha_p) \\ (b_1,\beta_1,x]...(b_q,\beta_q) \end{array} |z \right]$$

$$\sum_{k=0}^{\infty} \frac{\gamma(a_1+\alpha_1 k, x)\Gamma(a_2+\alpha_2 k)...\Gamma(a_p+\alpha_p k)}{\gamma(b_1+\beta_1 k, x)\Gamma(b_2+\beta_2 k)...\Gamma(b_q+\beta_q k)} \frac{z^k}{k!}. \tag{1.13}$$

With decomposition formula [20] incomplete Wright function (1.12) and (1.13) as classical Wright function ${}_p\Psi_q(z)$[21, 22, 23, 24] and this particular function is an entire function if there hold the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_j > 1. \tag{1.14}$$

2. Incomplete Fractional Integration For Incomplete Mittag-Leffler Function

If $\alpha,\beta,\gamma,\rho,\mu,\sigma,a,b \in C;\ \Re(\rho) > \max[0, \Re(\beta-\gamma)]$ and condition (1.14) exist, then

$$ {}_\Gamma I_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x) = \frac{x^{\rho-\beta-1}}{\Gamma(\delta)} {}_3\overline{\Psi}_3\left[\begin{array}{c} [\delta,1,x]\ (\rho,\sigma)(\rho+\gamma-\beta,\sigma) \\ (b,a)(\rho-\beta,\sigma)(\rho+\alpha+\gamma,\sigma) \end{array} |\mu x^\sigma \right]$$

$$-\frac{x^{\alpha+\rho-1}}{\Gamma(1+\alpha+\beta)}\frac{1}{\Gamma(\delta)} {}_1F_1\left[\alpha+\beta;1+\alpha+\beta;-x\right] {}_3\overline{\Psi}_3\left[\begin{array}{c} [\delta,1,x]\ (\rho,\sigma)(\rho+\alpha+\gamma+n,\sigma) \\ (b,a)(\rho+\alpha+\gamma,\sigma)(\rho+\alpha+n,\sigma) \end{array} |\mu x^\sigma \right]. \tag{2.1}$$

Proof. With (1.1) and (1.9)

$$ {}_\Gamma I_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} {}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;\left(1-\frac{t}{x}\right)\right)$$

$$\times t^{\rho-1}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(ak+b)}\frac{\mu^k t^{\sigma k}}{k!}dt. \tag{2.2}$$

Interchanging the order of integration and summation which is permissible under the condition

$$= \sum_{k=0}^{\infty}\frac{\mu^k}{\Gamma(ak+b)k!} {}_\Gamma I_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho+\sigma k-1}\right)(x). \tag{2.3}$$

Using (1.7), we arrive at

$$= \sum_{k=0}^{\infty} \frac{[\delta, x]_k}{\Gamma(ak+b)} \frac{\mu^k}{k!} \frac{\Gamma(\rho+\sigma k)\Gamma(\rho+\gamma-\beta+\sigma k)}{\Gamma(\rho-\beta+\sigma k)\Gamma(\rho+\alpha+\gamma+\sigma k)} x^{\rho+\sigma k-\beta-1}$$

$$-\frac{x^{\alpha+\rho-1}}{\Gamma(1+\alpha+\beta)} \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{(\alpha+\beta)_n}{(1+\alpha+\beta)_n} \frac{(-x)^n}{n!}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\delta+k, x)\Gamma(\rho+\sigma k)\Gamma(\alpha+\gamma+\rho+n+\sigma k)}{\Gamma(ak+b)\Gamma(\alpha+\gamma+\rho+\sigma k)\Gamma(\alpha+\rho+\sigma k+n)} \frac{(\mu x^\sigma)^k}{k!}. \tag{2.4}$$

Which is the required result. □

Corollary 2.1. *Now if we set $\beta = -\alpha$ in (2.4), then with the formula ([20], p. 104) equation (2.4) as Rieman-Liouville fractional calculus operator*

$${}_{\Gamma}I_{0,x}^{\alpha,-\alpha,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x) = I_{0,x}^{\alpha}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x) = \frac{x^{\rho+\alpha+1}}{\Gamma(\delta)} {}_2\overline{\Psi}_2\left[\begin{array}{c}[\delta,1,x]\,(\rho,\sigma)\\(b,a)(\rho+\alpha,\sigma)\end{array}\Big|\mu x^\sigma\right]. \tag{2.5}$$

If we take $\beta = 0$, in (2.4), we get

$${}_{\Gamma}I_{0,x}^{\alpha,0,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x) = {}_{\Gamma}E_{0,x}^{\alpha,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x) = \frac{x^{\rho-1}}{\Gamma(\delta)} {}_2\overline{\Psi}_2\left[\begin{array}{c}[\delta,1,x]\,(\rho+\gamma,\sigma)\\(b,a)(\rho+\alpha+\gamma,\sigma)\end{array}\Big|\mu x^\sigma\right]$$

$$-\frac{x^{\alpha+\rho-1}}{\Gamma(1+\alpha)} \frac{1}{\Gamma(\delta)} {}_1F_1[\alpha; 1+\alpha; -x]\, {}_3\overline{\Psi}_3\left[\begin{array}{c}[\delta,1,x]\,(\rho,\sigma)(\rho+\alpha+\gamma+n,\sigma)\\(b,a)(\rho+\alpha+\gamma,\sigma)(\rho+\alpha+n,\rho)\end{array}\Big|\mu x^\sigma\right]. \tag{2.6}$$

Equation (2.6) as incomplete Erdélyi-Kober operator.
If $\alpha, \beta, \gamma, \rho, \mu, \sigma, a, b \in C$, $\Re(\alpha) > 0$; $\Re(\rho) > \max[\Re(-\beta), \Re(-\gamma)]$ and condition (1.14) exist, then

$${}_{\Gamma}I_{-}^{\alpha,\beta,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x)$$

$$= \frac{x^{\rho-\beta-1}}{\Gamma(\delta)} {}_3\overline{\Psi}_3\left[\begin{array}{c}[\delta,1,x]\,(1+\beta-\rho,-\sigma)(1+\gamma-\rho,-\sigma)\\(b,a)(1-\rho,-\sigma)(1+\alpha+\beta+\gamma-\rho,-\sigma)\end{array}\Big|\mu x^\sigma\right]$$

$$-\frac{x^{\alpha+\rho-1}}{\Gamma(1+\alpha+\beta)} \frac{1}{\Gamma(\delta)} {}_1F_1[\alpha+\beta; 1+\alpha+\beta; -x]$$

$${}_3\overline{\Psi}_3\left[\begin{array}{c}[\delta,1,x]\,(1+\beta-\rho,-\sigma)(1+\alpha+\beta+\gamma+n-\rho,-\sigma)\\(b,a)(1+\alpha+\beta+\gamma-\rho,-\sigma)(1+\alpha+\beta-\rho+n,-\rho)\end{array}\Big|\mu x^\sigma\right]. \tag{2.7}$$

Proof. With (1.3) and (1.9)

$${}_{\Gamma}I_{-}^{\alpha,\beta,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^\sigma)\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2\Gamma_1\left((\alpha+\beta,x), -\gamma; \alpha; \left(1-\frac{x}{t}\right)\right)$$

$$\times t^{\rho-1} \sum_{k=0}^{\infty} \frac{[\delta,x]_k}{\Gamma(ak+b)} \frac{\mu^k t^{\sigma k}}{k!} dt. \tag{2.8}$$

Interchanging the order of integration and summation which is permissible under the condition

$$=\sum_{k=0}^{\infty}\frac{\mu^k}{\Gamma(ak+b)k!}\,{}_{\Gamma}I_{-}^{\alpha,\beta,\gamma}\left(t^{\rho+\sigma k-1}\right)(x). \tag{2.9}$$

Using (1.8), we arrive at

$$=\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(ak+b)}\frac{\mu^k}{k!}\frac{\Gamma(1+\beta-\rho-\sigma k)\Gamma(1+\gamma-\rho-\sigma k)}{\Gamma(1-\rho-\sigma k)\Gamma(1+\alpha+\beta+\gamma-\rho-\sigma k)}x^{\rho+\sigma k-\beta-1}$$

$$-\frac{x^{\alpha+\rho-1}}{\Gamma(1+\alpha+\beta)}\frac{1}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_n}{(1+\alpha+\beta)_n}\frac{(-x)^n}{n!}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\delta+k,x)\Gamma(1+\beta-\rho-\sigma k)\Gamma(1+\alpha+\beta+\gamma+n-\rho-\sigma k)}{\Gamma(ak+b)\Gamma(1+\alpha+\beta+\gamma-\rho-\sigma k)\Gamma(1+\alpha+\beta+n-\rho-\sigma k)}\frac{(\mu x^{\sigma})^k}{k!}. \tag{2.10}$$

Which is the required result □

Corollary 2.2. *Now if we set $\beta=-\alpha$ in (2.10), then with the formula ([20], p. 104) equation (2.10) as Weyl operator*

$$\begin{aligned}{}_{\Gamma}I_{-}^{\alpha,-\alpha,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x) &= I_{-}^{\alpha}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x)\\ &= \frac{x^{\rho+\alpha+1}}{\Gamma(\delta)}{}_2\overline{\Psi}_2\left[\begin{matrix}[\delta,1,x]\,(1-\alpha-\rho,-\sigma)\\ (b,a)(1-\rho,-\sigma)\end{matrix}\ |\mu x^{\sigma}\right].\end{aligned}$$

If we take $\beta=0$, in (2.10), we get

$$\begin{aligned}{}_{\Gamma}I_{-}^{\alpha,0,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x) &= {}_{\Gamma}K_{x,\infty}^{\alpha,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x)\\ &= \frac{x^{\rho-1}}{\Gamma(\delta)}{}_2\overline{\Psi}_2\left[\begin{matrix}[\delta,1,x]\,(1+\gamma-\rho,-\sigma)\\ (b,a)(1+\alpha+\gamma-\rho,-\sigma)\end{matrix}\ |\mu x^{\sigma}\right]\end{aligned}$$

$$-\frac{x^{\alpha+\rho-1}}{\Gamma(1+\alpha)}\frac{1}{\Gamma(\delta)}{}_1F_1\left[\alpha;1+\alpha;-x\right]{}_3\overline{\Psi}_3\left[\begin{matrix}[\delta,1,x]\,(1-\rho,-\sigma)(1+\alpha+\gamma+n-\rho,-\sigma)\\ (b,a)(1+\alpha+\gamma-\rho,-\sigma)(1+\alpha+n-\rho,-\sigma)\end{matrix}\ |\mu x^{\sigma}\right]. \tag{2.11}$$

Here (2.12) as incomplete Erdélyi-Kober operator.

3. Incomplete Fractional Differentiation for Incomplete Mittag-Leffler Function

If $\alpha,\beta,\gamma,\rho,\mu,\sigma,a,b\in C$; $\Re(\rho)>\max[0,\Re(\beta-\gamma)]$ and condition (1.14) exist, then

$${}_{\Gamma}D_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x)=\frac{x^{\rho+\beta-1}}{\Gamma(\delta)}{}_3\overline{\Psi}_3\left[\begin{matrix}[\delta,1,x]\,(\rho,\sigma)(\rho+\alpha+\beta+\gamma,\sigma)\\ (b,a)(\rho+\beta,\sigma)(\rho+\gamma,\sigma)\end{matrix}\ |\mu x^{\sigma}\right]$$

$$-\frac{x^{\rho-\alpha-1}}{\Gamma(1-\alpha-\beta)}\frac{1}{\Gamma(\delta)}{}_1F_1\left[-\alpha-\beta;1-\alpha-\beta;-x\right]{}_3\overline{\Psi}_3\left[\begin{matrix}[\delta,1,x]\,(\rho+\gamma+k,\sigma)(\rho-\alpha+n,\sigma)\\ (b,a)(\rho+\gamma,\sigma)(\rho-\alpha+k,\sigma)\end{matrix}\ |\mu x^{\sigma}\right]. \tag{3.1}$$

Proof. With (1.5) and (1.9)

$$
{}_{\Gamma}D_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x)=\left(\frac{d}{dx}\right)^{n}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}{}_{2}\Gamma_{1}\left((\alpha+\beta,x),-\gamma;\alpha;\left(1-\frac{t}{x}\right)\right)
$$

$$
\times t^{\rho-1}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(ak+b)}\frac{\mu^k t^{\sigma k}}{k!}dt. \tag{3.2}
$$

Interchanging the order of integration and summation which is permissible under the condition

$$
=\sum_{k=0}^{\infty}\frac{\mu^k}{\Gamma(ak+b)k!}{}_{\Gamma}I_{0,x}^{-\alpha+n,-\beta-n,\alpha+\gamma-n}\left(t^{\rho+\sigma k-1}\right)(x). \tag{3.3}
$$

Using (1.7), we arrive at

$$
=\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(ak+b)}\frac{\mu^k}{k!}\frac{\Gamma(\rho+\sigma k)\Gamma(\rho+\alpha+\beta+\gamma+\sigma k)}{\Gamma(\rho+\beta+n+\sigma k)\Gamma(\rho+\gamma+\sigma k)}\left(\frac{d}{dx}\right)^{n}x^{\rho+\sigma k+\beta+n-1}
$$

$$
-\sum_{m=0}^{\infty}\frac{(\rho+\gamma+\sigma k)_m(-\alpha-\beta)_m}{(\rho-\alpha+n+\sigma k)_m(1-\alpha-\beta)_m}\frac{(-1)^m}{m!}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(ak+b)}\frac{\mu^k}{k!}\left(\frac{d}{dx}\right)^{n}x^{\rho-\alpha+n+m+\sigma k-1}
$$

$$
=\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(ak+b)}\frac{\mu^k}{k!}\frac{\Gamma(\rho+\sigma k)\Gamma(\rho+\alpha+\beta+\gamma+\sigma k)}{\Gamma(\rho+\beta+\sigma k)\Gamma(\rho+\gamma+\sigma k)}x^{\rho+\sigma k+\beta-1}
$$

$$
-\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(ak+b)}\frac{\mu^k}{k!}\frac{\Gamma(\rho-\alpha+n+m+\sigma k)}{\Gamma(\rho-\alpha+k+\sigma k)}
$$

$$
\sum_{m=0}^{\infty}\frac{(\rho+\gamma+\sigma k)_m(-\alpha-\beta)_m}{(\rho-\alpha+n+\sigma k)_m(1-\alpha-\beta)_m}\frac{(-1)^m}{m!}x^{\rho-\alpha+m+\sigma k-1}.
$$

Which is the required result □

Corollary 3.1. *Now if we set $\beta=-\alpha$ in (3.4), then with the formula ([20], p. 104)*

$$
{}_{\Gamma}D_{0,x}^{\alpha,-\alpha,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x)=\frac{x^{\rho-\alpha-1}}{\Gamma(\delta)}{}_{3}\Psi_{3}\left[\begin{array}{c}(\delta,1,x)(\rho,\sigma)\\(b,a)(\rho-\alpha,\sigma)\end{array}|\mu x^{\sigma}\right]. \tag{3.4}
$$

If we take $\beta=0$, in (3.4), we get

$$
\begin{aligned}
{}_{\Gamma}D_{0,x}^{\alpha,0,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x) &= {}_{\Gamma}D_{0,x}^{\alpha,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x)\\
&= \frac{x^{\rho-1}}{\Gamma(\delta)}{}_{2}\overline{\Psi}_{2}\left[\begin{array}{c}[\delta,1,x]\,(\rho+\alpha+\gamma,\sigma)\\(b,a)(\rho+\gamma,\sigma)\end{array}|\mu x^{\sigma}\right]
\end{aligned}
$$

$$
-\frac{x^{\rho-\alpha-1}}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\delta)}{}_{1}F_{1}\left[-\alpha;1-\alpha;-x\right]{}_{3}\overline{\Psi}_{3}\left[\begin{array}{c}[\delta,1,x]\,(\rho+\gamma+m,\sigma)(\rho-\alpha+n,\sigma)\\(b,a)(\rho+\gamma,\sigma)(\rho-\alpha+m,\rho)\end{array}|\mu x^{\sigma}\right]. \tag{3.5}
$$

If $\alpha, \beta, \gamma, \rho, \mu, \sigma, a, b \in C$; $\Re(\rho) > \max[0, \Re(\beta-\gamma)]$ *and condition (1.14) exist, then*

$$ {}_\Gamma D_{-}^{\alpha,\beta,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x) = \frac{x^{\rho+\beta-1}}{\Gamma(\delta)} {}_3\overline{\Psi}_3\left[\begin{array}{c} [\delta,1,x]\,(1-\beta-\rho,-\sigma)(1+\alpha+\gamma-\rho,-\sigma) \\ (b,a)(1-\rho,-\sigma)(1-\beta+\gamma,-\sigma) \end{array} \,\middle|\, \mu x^{\sigma}\right] $$

$$ -\frac{x^{\rho-\alpha-1}}{\Gamma(1-\alpha-\beta)}\frac{1}{\Gamma(\delta)}{}_1F_1\left[-\alpha-\beta;1-\alpha-\beta;-x\right] $$

$$ {}_3\overline{\Psi}_3\left[\begin{array}{c} [\delta,1,x]\,(1-\beta-\rho,-\sigma)(1-\beta+\gamma+m-\rho,-\sigma) \\ (b,a)(1-\beta+\gamma-\rho,-\sigma)(1-\alpha-\beta-\rho+m,-\sigma) \end{array} \,\middle|\, \mu x^{\sigma}\right]. \tag{3.6} $$

Proof. As previous calculation □

Corollary 3.2. *Now if we set* $\beta = -\alpha$ *in (3.6), then with the formula ([20], p. 104)*

$$ {}_\Gamma D_{-}^{\alpha,-\alpha,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x) = \frac{x^{\rho-\alpha-1}}{\Gamma(\delta)} {}_2\Psi_2\left[\begin{array}{c} (\delta,1,x)(1+\alpha-\rho,-\sigma) \\ (b,a)(1-\rho,-\sigma) \end{array} \,\middle|\, \mu x^{\sigma}\right]. \tag{3.7} $$

If we take $\beta = 0$, *in (3.6), we get*

$$ {}_\Gamma D_{-}^{\alpha,0,\gamma}\left(t^{\rho-1}E_{a,b}^{[\delta,x]}(\mu t^{\sigma})\right)(x) = \frac{x^{\rho-1}}{\Gamma(\delta)} {}_2\overline{\Psi}_2\left[\begin{array}{c} [\delta,1,x]\,(1+\alpha+\gamma-\rho,-\sigma) \\ (b,a)(1+\gamma-\rho,-\sigma) \end{array} \,\middle|\, \mu x^{\sigma}\right] $$

$$ -\frac{x^{\rho-\alpha-1}}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\delta)}{}_1F_1\left[-\alpha;1-\alpha;-x\right] {}_3\overline{\Psi}_3\left[\begin{array}{c} [\delta,1,x]\,(1-\rho,-\sigma)(1-\gamma+m-\rho,-\sigma) \\ (b,a)(1+\gamma-\rho,-\sigma)(1-\alpha+m-\rho,-\sigma) \end{array} \,\middle|\, \mu x^{\sigma}\right]. \tag{3.8} $$

Conclusion. *The aim of this paper is to present a wide variety of interesting fractional calculus formulas of incomplete Mittag-Leffler function in a flexible and accessible format and by using which in the field of science and engineering, even more, important results can be achieved.*

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