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On Fractional Kinetic Equation Relating to the Generalized k-Bessel Function by the Mellin Transform

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Abstract

This paper aims to investigate the solutions of a new generalized form of the fractional Kinetic equation involving Hadamard fractional integral operator and generalized k-Bessel function. The graphics interpretation of the solutions for fractional Kinetic equations for various values are presented. These results are very useful for research in several issues in the applied sciences.

Keywords: Fractional Kinetic equation; k-Bassel function, Gamma function; Hadamard fractional integral operator; Mellin transform.
2010 MSC: 26A33, 74A25, 33C45, 33C60.

1. Introduction and Preliminaries

The subject of fractional calculus is the calculus of derivatives and integrals of any arbitrary order. In view of its scientific applications in the last three decades, it has gain a significant and attractiveness to the researchers of this area. For more details about fractional calculus and its applications in engineering and science, we refer the readers to these works [1, 2, 3, 4, 5, 6, 40, 8, 9, 10, 11, 12]. Very recently, during COVID-19 pandemic appeared many research papers studied transmission dynamics of COVID-19 mathematical model involving fractional derivative, for examples [13, 14, 15].

The Mellin's integral transform is regarded as one of the most important useful tools in mathematics and numerous applications. Indeed, the Mellin integral transform is strongly linked to the Fourier transform, however, it is more suitable for certain applications, such as the theory special functions of the hypergeometric type, integral transforms with the functions of the hypergeometric type in the kernels and fractional calculus.

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The authors in [3], defined the Mellin integral transform of the function $\varphi(t)$ as follows:

$$\mathcal{M}\{\varphi(t)\}(\tau) = \int_0^\infty t^{\tau-1}\varphi(t)dt, \quad (\tau \in \mathbb{C}), t \in \mathbb{R}^+, \tag{1.1}$$

and the inverse Mellin integral transform is given by

$$\mathcal{M}^{-1}\{\varphi(\tau)\}(t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} t^{-\tau}\varphi(\tau)d\tau, \quad (\varepsilon = \Re(\tau)). \tag{1.2}$$

Furthermore, consider $\vartheta(t)$ be a function, then the Mellin convolution of $\vartheta(t)$ and $\varphi(t)$ is given by

$$\vartheta * \varphi = (\vartheta * \varphi)(\tau) := \int_0^\tau \vartheta\left(\frac{\tau}{t}\right)\varphi(t)\frac{dt}{t}. \tag{1.3}$$

In 2012, Mathur and Poonia [16], defined the Mellin transformation of $\varphi(t)$ in $[0, a]$ as follows:

$$\mathcal{M}\left[\varphi(t), \tau, 0, a\right] = \Phi(\tau) := \int_0^a a^{-\tau}t^{\tau-1}\varphi(t)dt, t \in [0, a]. \tag{1.4}$$

Then,

$$\varphi(t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{t^{-\tau}}{\tau}\mathcal{M}\left[\varphi(t), \tau, 0, c\right]d\tau. \tag{1.5}$$

Also, the first kind of k-Bessel function given by [17], as

$$J_{k,l}^{y,\lambda}(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(y)r,k}{\Gamma_k(r\lambda + l + 1)}\frac{1}{(r!)^2}\left(\frac{z}{2}\right)^r, \tag{1.6}$$

where $k \in \mathbb{R}, \lambda, l, y \in \mathbb{C}, \Re(l) > 0$, and

$$(y)_{r,k} := \begin{cases} \frac{\Gamma_k(y+rk)}{\Gamma_k(y)}, & y \in \mathbb{C}\setminus\{0\}, k \in \mathbb{R}, \\ y(y+k)...(y+(r-1)k), & (r \in \mathbb{N}; y \in \mathbb{C}), \end{cases} \tag{1.7}$$

and the classical Euler's Gamma function is the following:

$$\Gamma_k(y) = \int_0^\infty t^{y-1}e^{-\frac{t^k}{k}}dt := k^{\frac{y}{k}-1}\Gamma\left(\frac{y}{k}\right), \quad (\Re(y) > 0). \tag{1.8}$$

In addition, the authors [18], generalized the k-Bessel function $J_{k,l}^{y,\lambda}(z)$ as follows:

$$W_{l,c}^{k}(t) := \sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk + l + k)r!}\left(\frac{t}{2}\right)^{2r+\frac{l}{k}}. \tag{1.9}$$

The kinetic equations are fundamental in mathematical physics and natural sciences that interpretation the continuity of motion of the materials. In the present work, the function (1.9) was taken into consideration and we try to get solutions of fractional Kinetic equations. For more Details about fractional Kinetic equations and its solutions, the k-Bessel functions and the k-Pochhammer symbols, we refer readers to [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. In particular, Samraiz et al.

[40], studied the generalized fractional kinetic equation using the (k, s)-Hilfer-Prabhakar derivative. Saxena and Kalla [41], in 2008 studied the following Kinetic equation in sense of the Riemann-Liouville fractional integral

$$N(t) - N_0\varphi(t) = -\delta^{\gamma}\ {}_{RL}I_t^{\gamma}N(t), \tag{1.10}$$

such that ${}_{RL}I_t^{\gamma}$ is the Riemann-Liouville fractional integral operator.

In 2017, Nisar et al. [42], established the solutions of the following fractional Kinetic equations relating to the generalized k-Bessel function

$$N(t) - N_0 W_{l,c}^{k}(t) = -\delta^{\gamma}\ {}_{RL}I_t^{\gamma}N(t). \tag{1.11}$$

Very recently, the solutions of the following fractional Kinetic equations involving product of generalized k-wright function investigated by Ahmed et al. [23],

$$N(t) - N_0\varphi(t) = -\delta^{\gamma}\ {}_{H}I_t^{\gamma}N(t), \tag{1.12}$$

where ${}_{H}I_t^{\gamma}$ is the Hadamard fractional integral operator defined as

$$({}_{H}I_{0+}^{\gamma}\varphi)(t) := \frac{1}{\Gamma(\gamma)}\int_0^t \left(\log\frac{t}{s}\right)^{\gamma-1}\frac{\varphi(s)}{s}ds, \qquad (\Re(\gamma) > 0, t > 0). \tag{1.13}$$

For more details about the Hadamard fractional integral operator, can see these works [3].

In this work, we will discuss the solution of the following generalized fractional Kinetic equation involving Hadamard fractional integral operator and generalized k-Bessel function:

$$N(t) - N_0 W_{l,c}^{k}(t) = -\delta^{\gamma}\ {}_{H}I_t^{\gamma}N(t). \tag{1.14}$$

2. Solutions of Generalized Fractional Kinetic Equations

In this section, we discuss the solutions of generalized fractional Kinetic equations including the generalized k-Bessel function by applying the Mellin transform technique.

Theorem 2.1. *Consider* $l, c, t \in \mathbb{C}, \delta, \gamma > 0,$ and $k \in \mathbb{R}$. *Then, generalized fractional Kinetic equation* (1.14) *has the following solution*

$$\begin{aligned} N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + l + k)r!}\left(\frac{a}{2}\right)^{(2r+\frac{l}{k})}\log(t) \\ \times \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\left[-\left[\log(t)^{\delta}\right]^{\gamma}\right]^{\mu}\left[\log(t)^{-(2r+\frac{l}{k})}\right]^{\nu}\Gamma[1-(\mu\gamma+\nu+2)]. \end{aligned} \tag{2.1}$$

Proof. We recall the Mellin transform of Hadamard fractional integral operator

$$(\mathcal{M}\ {}_{H}I_{-}^{\gamma}\varphi)(\tau) = (\tau)^{-\gamma}\Phi(\tau),$$

where $\Phi(\tau) = \int_0^a a^{-\tau}t^{\tau-1}\varphi(t)dt$.

Now, by applying the Mellin transform on both sides of (1.14), we have

$$\mathcal{M}[N(t)] = N_0\mathcal{M}[W_{l,c}^{k}(t)] - \delta^{\gamma}\mathcal{M}[{}_{H}I_t^{\gamma}N(t)]. \tag{2.2}$$

Hence,

$$N(\tau) = N_0\mathcal{M}\Big[\int_0^a a^{-\tau}t^{\tau-1}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{t}{2}\Big)^{(2r+\frac{l}{k})}dt\Big] - \delta^{\gamma}\mathcal{M}\big[{}_HI_t^{\gamma}N(t)\big]. \quad (2.3)$$

By interchanging integration and summation order in Equ. (2.3), we get

$$\begin{aligned}N(\tau)\big[1+\delta^{\gamma}(\tau)^{-\gamma}\big] &= N_0\Big(\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{1}{2}\Big)^{(2r+\frac{l}{k})}\Big)\Big(a^{-\tau}\int_0^a t^{(\tau+2r+\frac{l}{k}-1)}dt\Big)\\ &= N_0\Big(\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{a}{2}\Big)^{(2r+\frac{l}{k})}\Big)\Big(\frac{1}{(\tau+2r+\frac{l}{k})}\Big),\end{aligned}$$

which leads to

$$N(\tau) = N_0\Big(\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{a}{2}\Big)^{(2r+\frac{l}{k})}\Big)\Big(\frac{1}{(\tau+2r+\frac{l}{k})}\Big)\sum_{\mu=0}^{\infty}(-1)^{\mu}\Big(\frac{\delta}{\tau}\Big)^{\mu\gamma}. \quad (2.4)$$

Next, applying the inverse Mellin transform of the equation (2.4), we obtain

$$N(t) = N_0\Big(\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{a}{2}\Big)^{(2r+\frac{l}{k})}\Big)\sum_{\mu=0}^{\infty}(-1)^{\mu}(\delta)^{\mu\gamma}\mathcal{M}^{-1}\Big(\frac{\tau^{-\mu\gamma}}{(\tau+2r+\frac{l}{k})}\Big). \quad (2.5)$$

Since,

$$\begin{aligned}\mathcal{M}^{-1}\Big(\frac{\tau^{-\mu\gamma}}{(\tau+2r+\frac{l}{k})}\Big) &= \int_0^{\infty}\frac{t^{-\tau}}{\tau}\frac{\tau^{-\mu\gamma}}{(\tau+2r+\frac{l}{k})}d\tau\\ &= \sum_{\nu=0}^{\infty}\Big(-(2r+\frac{l}{k})\Big)^{\nu}\int_0^{\infty}t^{-\tau}\tau^{-(\mu\gamma+\nu+2)}d\tau\\ &= \sum_{\nu=0}^{\infty}\Big(-(2r+\frac{l}{k})\Big)^{\nu}\big(\log t\big)^{\mu\gamma+\nu+1}\Gamma\big[1-(\mu\gamma+\nu+2)\big].\end{aligned}$$

Then,

$$\begin{aligned}N(t) = N_0\Big(\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{a}{2}\Big)^{(2r+\frac{l}{k})}\Big)\sum_{\mu=0}^{\infty}(-1)^{\mu}(\delta)^{\mu\gamma}\\ \times\sum_{\nu=0}^{\infty}\Big(-(2r+\frac{l}{k})\Big)^{\nu}\big(\log t\big)^{\mu\gamma+\nu+1}\Gamma\big[1-(\mu\gamma+\nu+2)\big]. \quad (2.6)\end{aligned}$$

So,

$$\begin{aligned}N(t) = N_0\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{a}{2}\Big)^{(2r+\frac{l}{k})}\log(t)\\ \times\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\big[-\big[\log(t)^{\delta}\big]^{\gamma}\big]^{\mu}\big[\log(t)^{-(2r+\frac{l}{k})}\big]^{\nu}\Gamma[1-(\mu\gamma+\nu+2)]. \quad (2.7)\end{aligned}$$

□

Corollary 2.2. *Consider* $c, l, t \in \mathbb{C}$ and $\delta, \gamma > 0$. *Then, the following equation*

$$N(t) - N_0 W^1_{l,c}(t) = -\delta^\gamma \ {}_H I^\gamma_t N(t), \tag{2.8}$$

has the following solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(r+l+1)r!} \left(\frac{a}{2}\right)^{(2r+l)} \log(t) \\ \times \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \left[-\left[\log(t)^\delta\right]^\gamma\right]^\mu \left[\log(t)^{-(2r+l)}\right]^\nu \Gamma[1-(\mu\gamma+\nu+2)].$$

Proof. By putting $k = 1$ in Equ. (1.14), the proof is finished. □

Theorem 2.3. *Consider* $l, c, t \in \mathbb{C}, \delta, \gamma > 0,$ and $k \in \mathbb{R}$. *Then, the following equation*

$$N(t) - N_0 W^k_{l,c}(\delta^\gamma t^\gamma) = -\delta^\gamma \ {}_H I^\gamma_t N(t), \tag{2.9}$$

has the following solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk+l+k)r!} \left(\frac{\delta^\gamma a^\gamma}{2}\right)^{(2r+\frac{l}{k})} \log(t) \\ \times \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \left[-\left[\log(t)^\delta\right]^\gamma\right]^\mu \left[\log(t)^{-(2r\gamma+\frac{l\gamma}{k})}\right]^\nu \Gamma[1-(\mu\gamma+\nu+2)]. \tag{2.10}$$

Proof. we have $(\mathcal{M} \ {}_H I^\gamma_- \varphi)(\tau) = (\tau)^{-\gamma}\Phi(\tau)$, such that $\Phi(\tau) = \int_0^a a^{-\tau} t^{\tau-1} \varphi(t) dt$. Now, by taking the Mellin transform on both sides of Equ. (2.9), we get

$$\mathcal{M}[N(t)] = N_0 \mathcal{M}[W^k_{l,c}(\delta^\gamma t^\gamma)] - \delta^\gamma \mathcal{M}[{}_H I^\gamma_t N(t)]. \tag{2.11}$$

Thus,

$$N(\tau) = N_0 \mathcal{M}\Big[\int_0^a a^{-\tau} t^{\tau-1} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk+l+k)r!} \left(\frac{\delta^\gamma t^\gamma}{2}\right)^{(2r+\frac{l}{k})} dt\Big] - \delta^\gamma \mathcal{M}\big[{}_H I^\gamma_t N(t)\big]. \tag{2.12}$$

Similarly, as in Theorem 2.1, we conclude that

$$N(\tau) = N_0 \Big(\sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk+l+k)r!} \left(\frac{\delta^\gamma a^\gamma}{2}\right)^{(2r+\frac{l}{k})}\Big) \Big(\frac{1}{(\tau + 2r\gamma + \frac{l\gamma}{k})}\Big) \sum_{\mu=0}^{\infty} (-1)^\mu \left(\frac{\delta}{\tau}\right)^{\mu\gamma}. \tag{2.13}$$

Applying Mellin inverse, we have

$$N(t) = N_0 \Big(\sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk+l+k)r!} \left(\frac{\delta^\gamma a^\gamma}{2}\right)^{(2r+\frac{l}{k})}\Big) \sum_{\mu=0}^{\infty} (-1)^\mu (\delta)^{\mu\gamma} \mathcal{M}^{-1}\Big(\frac{\tau^{-\mu\gamma}}{(\tau + 2r\gamma + \frac{l\gamma}{k})}\Big). \tag{2.14}$$

Since,

$$\begin{aligned}\mathcal{M}^{-1}\left(\frac{\tau^{-\mu\gamma}}{(\tau+2r\gamma+\frac{l\gamma}{k})}\right) &= \int_0^{\infty}\frac{t^{-\tau}}{\tau}\frac{\tau^{-\mu\gamma}}{(\tau+2r\gamma+\frac{l\gamma}{k})}d\tau \\ &= \sum_{\nu=0}^{\infty}\big(-(2r\gamma+\frac{l\gamma}{k})\big)^{\nu}\int_0^{\infty}t^{-\tau}\tau^{-(\mu\gamma+\nu+2)}d\tau \\ &= \sum_{\nu=0}^{\infty}\big(-(2r\gamma+\frac{l\gamma}{k})\big)^{\nu}\big(\log t\big)^{\mu\gamma+\nu+1}\Gamma\big[1-(\mu\gamma+\nu+2)\big].\end{aligned}$$

Then,

$$\begin{aligned}N(t) = N_0\Big(\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{\delta^{\gamma}a^{\gamma}}{2}\Big)^{(2r+\frac{l}{k})}\Big)\sum_{\mu=0}^{\infty}(-1)^{\mu}(\delta)^{\mu\gamma} \\ \times\sum_{\nu=0}^{\infty}\big(-(2r\gamma+\frac{l\gamma}{k})\big)^{\nu}\big(\log t\big)^{\mu\gamma+\nu+1}\Gamma\big[1-(\mu\gamma+\nu+2)\big],\end{aligned}$$

hence, the proof is completed. □

Corollary 2.4. *Consider* $c, l, t \in \mathbb{C}$ and $\delta, \gamma > 0$. *Then, the following equation*

$$N(t) - N_0 W^{1}_{l,c}(\delta^{\gamma}t^{\gamma}) = -\delta^{\gamma}\ {}_{H}I^{\gamma}_{t}N(t), \tag{2.15}$$

has the following solution

$$\begin{aligned}N(t) = N_0\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma(r+l+1)r!}\big(\frac{\delta^{\gamma}a^{\gamma}}{2}\big)^{(2r+l)}\log(t) \\ \times\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\big[-\big[\log(t)^{\delta}\big]^{\gamma}\big]^{\mu}\big[\log(t)^{-(2r\gamma+l\gamma)}\big]^{\nu}\Gamma[1-(\mu\gamma+\nu+2)].\end{aligned}$$

Proof. By putting $k = 1$ in Equ. (2.10), the proof is finished. □

Theorem 2.5. *Consider* $l, c, t \in \mathbb{C}, \delta, \gamma > 0,$ and $\delta \neq \alpha$. *Then, the following equation*

$$N(t) - N_0 W^{k}_{l,c}(\delta^{\gamma}t^{\gamma}) = -\alpha^{\gamma}\ {}_{H}I^{\gamma}_{t}N(t), \tag{2.16}$$

has the following solution

$$\begin{aligned}N(t) = N_0\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+l+k)r!}\Big(\frac{\delta^{\gamma}a^{\gamma}}{2}\Big)^{(2r+\frac{l}{k})}\log(t) \\ \times\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\big[-\big[\log(t)^{\alpha}\big]^{\gamma}\big]^{\mu}\big[\log(t)^{-(2r\gamma+\frac{l\gamma}{k})}\big]^{\nu}\Gamma[1-(\mu\gamma+\nu+2)].\end{aligned} \tag{2.17}$$

Proof. We can prove this theorem by same technique used in Theorem 2.3. □

Corollary 2.6. *Consider* $c, l, t \in \mathbb{C}, \delta, \gamma > 0,$ and $\delta \neq \alpha$. *Then, the following equation*

$$N(t) - N_0 W^1_{l,c}(\delta^\gamma t^\gamma) = -\alpha^\gamma \ {}_H I^\gamma_t N(t), \tag{2.18}$$

has the following solution

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(r+l+1)r!} (\frac{\delta^\gamma a^\gamma}{2})^{(2r+l)} \log(t) \times \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \big[- \big[\log(t)^\alpha\big]^\gamma\big]^\mu \big[\log(t)^{-(2r\gamma+l\gamma)}\big]^\nu \Gamma[1-(\mu\gamma+\nu+2)].$$

Proof. By putting $k = 1$ in Equ. (2.17), the proof is finished. □

3. Graphical interpretation

During this part, we will plot graphs of the solutions of Equ. (2.1). So, there are four particular solutions due to dedicating different values of the parameters have been given in each graph. In Figure(1), we will take $k = 1$ and $\gamma = 2.1, 2.2, 2.3, 2.4$. Similarly, in Figures$(2-7)$, we will take $k = 2, 3, 4, 5, 10, 20$ and value 1 for all other parameters. Thus, the effect of k is illustrated.

We observe that, initially when $t \to 0$, then $N(t)$ approaches to zero. the value of $N(t)$ increases with time and at end tends to infinity as $t \to \infty$, for all chosen parameters. So, we notice that $N(t)$ is an increasing function for $0 < t < \infty$.

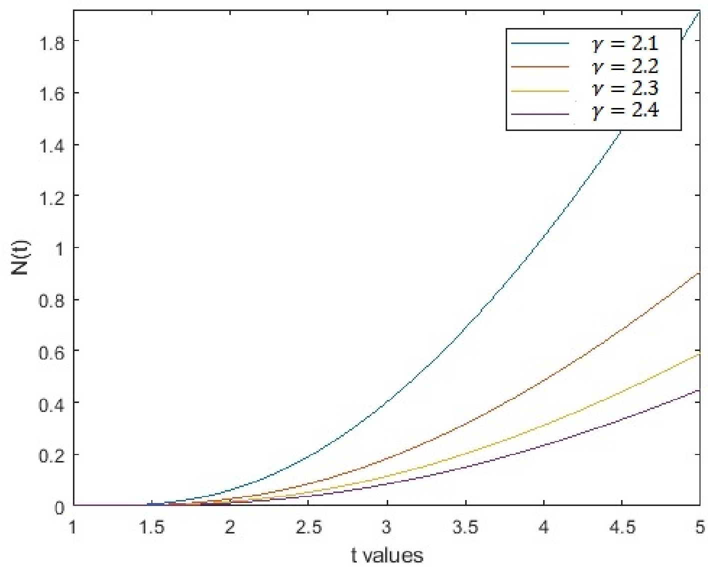

Figure 1: Solution of Equ. (2.1) for $k = 1$ and various values of γ.

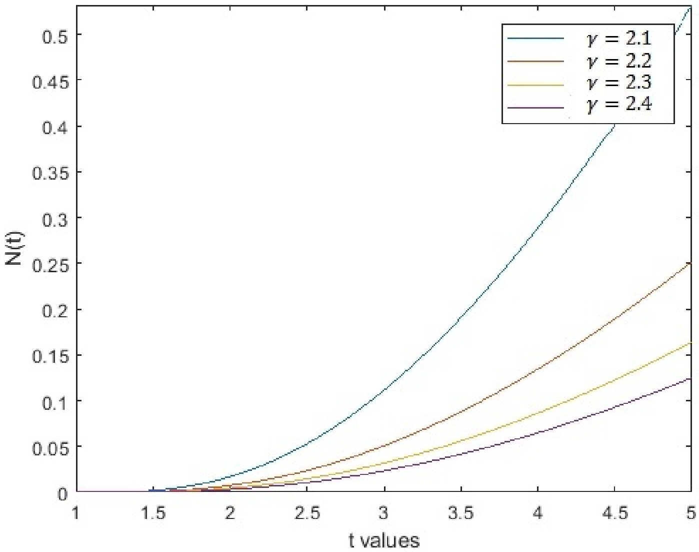

Figure 2: Solution of Equ. (2.1) for $k = 2$ and various values of γ.

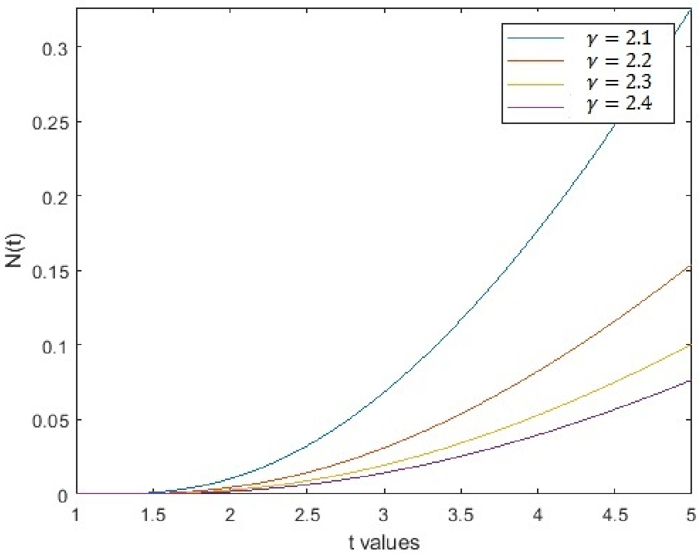

Figure 3: Solution of Equ. (2.1) for $k = 3$ and various values of γ.

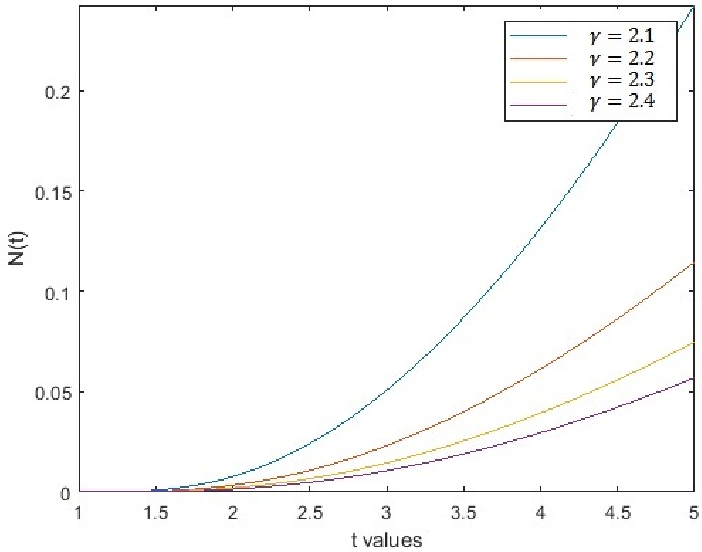

Figure 4: Solution of Equ. (2.1) for $k = 4$ and various values of γ.

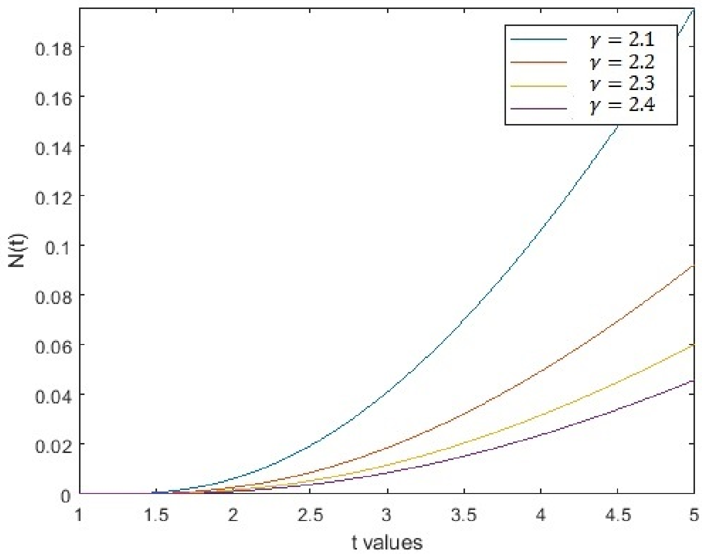

Figure 5: Solution of Equ. (2.1) for $k = 5$ and various values of γ.

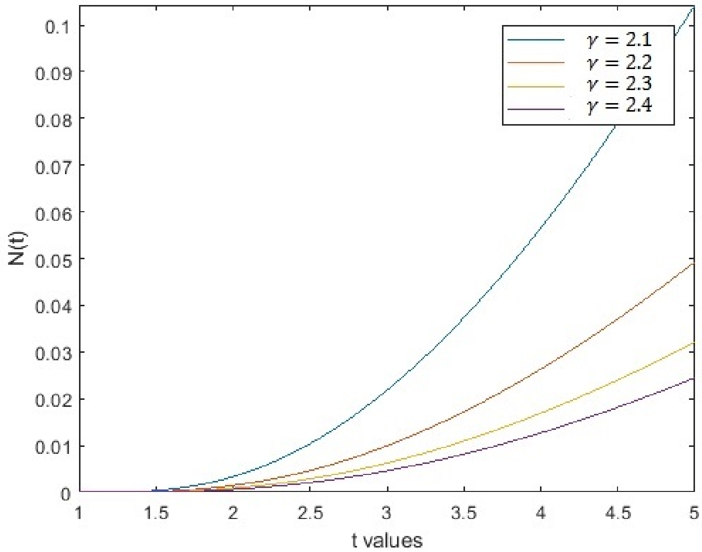

Figure 6: Solution of Equ. (2.1) for $k = 10$ and various values of γ.

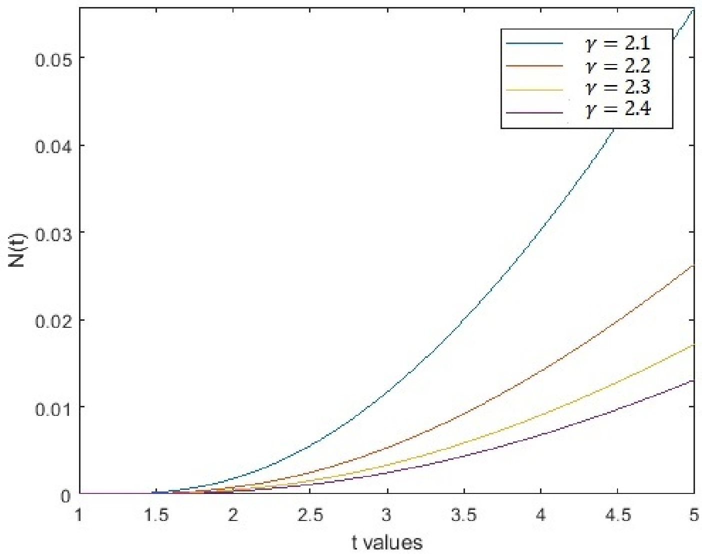

Figure 7: Solution of Equ. (2.1) for $k = 20$ and various values of γ.

4. Conclusion

In mathematical modeling, kinetic equations are basic equations of mathematical physics and the natural sciences and describe the continuity of motion of the materials. In this study, a new solutions of generalized Hadamard fractional Kinetic equation relating to generalized k-Bessel function were discussed by applying the Mellin transform technique.

The graphics interpretation of solutions were given as applications of our results. We deduce that $N(t) > 0$ for various values of the parameters during different time t. We can easily construct varying known and new fractional Kinetic equations due to closed relationship of the generalized k-Bessel function with many special functions.

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