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Opial-Jensen and functional inequalities for convex functions

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Abstract

The main of this article are presenting generalized Opial type inequalities which will be defined as the Opial-Jensen inequality for convex function. Further, new Opial type inequalities will be given for functionals defined with the help of the Opial inequalities.

Keywords: Opial inequality, Jensen inequality, Hölder's inequality, convex function.
2010 MSC: 26D15, 26A51, 26A42, 34A40.

1. Introduction

In the year 1960, in [1], Opial established an inequality involving integral of a function and its derivative as follows:

Theorem 1.1. *Let* $\psi(\tau) \in C^{(1)}[0,h]$ *be such that* $\psi(0) = \psi(h) = 0$, *and* $\psi(\tau) > 0$ *in* $(0,h)$. *Then, the following inequality holds*

$$\int_0^h |\psi(\tau)\psi'(\tau)|\, d\tau \leqslant \frac{h}{4}\int_0^h (\psi'(\tau))^2\, d\tau. \tag{1.1}$$

The constant $h/4$ *is best possible.*

In 1960, Olech [2] gave a new proof by extending the (1.1) inequality as;

Theorem 1.2. *Let* $\psi(\tau) \in C^{(1)}[0,h]$ *be such that* $\psi(0) = 0$, *and* $\psi(\tau) > 0$ *in* $(0,h)$. *Then, the following inequality holds*

$$\int_0^h |\psi(\tau)\psi'(\tau)|\, d\tau \leqslant \frac{h}{2}\int_0^h (\psi'(\tau))^2\, d\tau. \tag{1.2}$$

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Later, in 1965, Hua [3] proved Opial's inequality (1.2) by generalizing it as follows;

$$\int_0^h |\psi'(\tau)| \left|\psi^{\ell}(\tau)\right| d\tau \leqslant \frac{h^{\ell}}{(\ell+1)} \int_0^h (\psi'(\tau))^2 d\tau$$

where ℓ is a positive integer.

In 1967, Godunova and Levin [4] proved the following theorem for convex functions, a generalization of the well-known Opial inequality.

Theorem 1.3. *Let ψ be real-valued absolutely continuous function defined on $[a,b]$ with $\psi(a) = 0$. Let ψ be real-valued convex, increasing function on $[0,\infty)$ with $\psi(0) = 0$. Then, the following integral inequality holds*

$$\int_0^h |\psi'(\tau)| \psi'(|\psi(\tau)|) d\tau \leqslant \psi\left(\int_0^h |\psi'(\tau)| d\tau\right). \tag{1.3}$$

Opial's inequality and its generalizations play an important role in determining the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Recently, a large number of articles have appeared in the literature dealing with different types of evidence and various generalizations of Opial inequalities, see [5]-[9], [10]-[23].

Opial's inequality, due to its significance, experienced a lot of extensions and generalizations over time in both classical and fractional calculus. Motivated with Opial-type inequalities, together with Jensen's inequality, we improve some known results and obtain new, interesting inequalities.

Jensen's inequality is of great interest in the theory of differential and difference equations, as well as other areas of mathematics. The Jensen's inequality can be stated as follows.

Theorem 1.4 (Jensen Inequality)**.** *[24] Let $a, b, c, d \in [0,\infty)$. Let $w : [a,b] \to \mathbb{R}$ and $g : [a,b] \to (c,d)$ are nonnegative, continuous functions with $\int_a^b w(\tau)d\tau > 0$, and $\psi : (c,d) \to \mathbb{R}$ is continuous and convex function. Then, we have*

$$\psi\left(\frac{\int_a^b w(\tau)g(\tau)d\tau}{\int_a^b w(\tau)d\tau}\right) \leqslant \frac{\int_a^b w(\tau)\psi(g(\tau))\, d\tau}{\int_a^b w(\tau)d\tau}. \tag{1.4}$$

The purpose of this paper is to establish some generalizations of Opial type inequalities defined Opial-Jensen inequality for convex functions. Further, new Opial type inequalities will be given for new functionals defined with the help of Opial inequalities. Thus our results are new results for Opial type inequalities. Now, we present the main results:

2. Opial-Jensen inequalities

In this section,we will state our main results and give their proofs as follows.

Theorem 2.1. *Let ψ be real-valued absolutely continuous function defined on $[a,b]$ with $\psi(a)=0$. Let ψ be a differentiable mapping defined on $[0,\infty)$ with $\psi(0)=0$ and ψ' be real-valued convex, increasing function on $[0,\infty)$. Then, the following integral inequalities hold*

$$\psi'\left(\frac{\int_a^b |\psi'(\tau)|\,|\psi(\tau)|\,d\tau}{\int_a^b |\psi'(\tau)|\,d\tau}\right) \leqslant \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\psi\left(\int_a^b \left|\psi'(\tau)\right| d\tau\right) \tag{2.1}$$

and

$$\psi'\left(\frac{\int_a^b |\psi'(\tau)|\,|\psi(\tau)|\,d\tau}{\int_a^b |\psi'(\tau)|\,d\tau}\right) \tag{2.2}$$
$$\leqslant \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\left[\frac{\psi'(b)-\psi'(a)}{2}\int_a^b \left|\psi'(\tau)\right|^2 d\tau + \frac{b\psi'(a)-a\psi'(b)}{b-a}\int_a^b \left|\psi'(\tau)\right| d\tau\right].$$

If ψ' is a concave and decreasing fuctions, then inequalities (2.1) and (2.2) are reversed.

Proof. We consider $y(\tau)=\int_a^\tau |\psi'(s)|\,ds$ such that $y'(\tau)=|\psi'(\tau)|$ and $y(\tau)\geqslant|\psi(\tau)|$. Since ψ' is convex and increasing, by using Jensen inequality, we get

$$\begin{aligned}\psi'\left(\frac{\int_a^b |\psi'(\tau)|\,|\psi(\tau)|\,d\tau}{\int_a^b |\psi'(\tau)|\,d\tau}\right) &\leqslant \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\int_a^b \left|\psi'(\tau)\right| \psi'\left(|\psi(\tau)|\right) d\tau \\ &\leqslant \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\int_a^b y'(\tau)\psi'\left(|y(\tau)|\right) d\tau \\ &= \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\psi\left(\int_a^b \left|\psi'(\tau)\right| d\tau\right)\end{aligned} \tag{2.3}$$

which completes the proof the inequality (2.1).

Since ψ' is a convex function on $[a,b]\subset[0,\infty)$, from the first inequality of (2.3) and with the help of the inequality (1.2), we have

$$\psi'\left(\frac{\int_a^b |\psi'(\tau)|\,|\psi(\tau)|\,d\tau}{\int_a^b |\psi'(\tau)|\,d\tau}\right) \leqslant \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\int_a^b \left|\psi'(\tau)\right| \psi'\left(|\psi(\tau)|\right) d\tau$$

$$= \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\int_a^b \left|\psi'(\tau)\right| \psi'\left(\frac{|\psi(\tau)|-a}{b-a}b+\frac{b-|\psi(\tau)|}{b-a}a\right) d\tau$$

$$\leqslant \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\left[\frac{\psi'(b)-\psi'(a)}{b-a}\int_a^b \left|\psi'(\tau)\right| |\psi(\tau)|\,d\tau+\frac{b\psi'(a)-\psi'(b)}{b-a}\int_a^b \left|\psi'(\tau)\right| d\tau\right]$$

$$\leqslant \frac{1}{\int_a^b |\psi'(\tau)|\,d\tau}\left[\frac{\psi'(b)-\psi'(a)}{2}\int_a^b \left|\psi'(\tau)\right|^2 d\tau+\frac{b\psi'(a)-a\psi'(b)}{b-a}\int_a^b \left|\psi'(\tau)\right| d\tau\right]$$

which completes the proof the inequality (2.2). □

Remark 2.2. In Theorem 2.1, if we take $\psi(s)=\frac{s^2}{2}$ on $[0,h]$, $\psi'(s)=s$ is a convex function on $[0,h]$,

i) the inequality (2.1) reduces to the inequality

$$\int_0^h \left|\psi'(\tau)\right| |\psi(\tau)|\,d\tau \leqslant \frac{1}{2}\left(\int_0^h \left|\psi'(\tau)\right| d\tau\right)^2.$$

By using the Cauchy-Schwarz inequality, it holds the inequality (1.2).

ii) the inequality (2.2) reduces to the inequality (1.2).

Corollary 2.3. *Under the hypotheses of Theorem 2.1, we have*

$$\int_0^h \left|\psi'(\tau)\right| |\psi(\tau)|\,d\tau \leqslant \frac{h}{(\ell+1)^{\frac{1}{\ell}}}\int_0^h \left|\psi'(\tau)\right|^2 d\tau \tag{2.4}$$

for $\ell \geqslant 1$.

Proof. In Theorem 2.1, if we choose $\psi(s)=\frac{s^{\ell+1}}{\ell+1}$ on $[0,h]$, $\psi'(s)=s^{\ell}$ is a convex function on $[0,h]$ for $\ell \geqslant 1$, the inequality (2.1) reduces to

$$\int_0^h \left|\psi'(\tau)\right| |\psi(\tau)|\,d\tau \leqslant \frac{1}{(\ell+1)^{\frac{1}{\ell}}}\left(\int_0^h \left|\psi'(\tau)\right| d\tau\right)^2.$$

By using the Cauchy-Schwarz inequality, it follows that

$$\int_0^h |\psi'(\tau)| \, |\psi(\tau)| \, d\tau \leqslant \frac{h}{(\ell+1)^{\frac{1}{\ell}}} \int_0^h |\psi'(\tau)|^2 \, d\tau$$

holds the inequality (2.4). □

Remark 2.4. In Corollary 2.3, if we take $\ell = 1$, the inequality (2.4) reduces to the inequality (1.2).

Corollary 2.5. *Under the hypotheses of Theorem 2.1, for $\ell \geqslant 1$, we have*

$$\int_0^h |\psi'(\tau)| \, |\psi(\tau)| \, d\tau \leqslant \frac{h^{\frac{3\ell-1}{2\ell}}}{2^{\frac{1}{\ell}}} \left(\int_0^h |\psi'(\tau)|^2 \, d\tau \right)^{\frac{\ell+1}{2\ell}}. \tag{2.5}$$

Proof. In Theorem 2.1, if we choose $\psi'(s) = s^{\ell}$ on $[0, h]$ for $\ell > 1$, the inequality (2.2) reduces to

$$\int_0^h |\psi'(\tau)| \, |\psi(\tau)| \, d\tau \leqslant \frac{h}{2^{\frac{1}{\ell}}} \left(\int_0^h |\psi'(\tau)|^2 \, d\tau \right)^{\frac{1}{\ell}} \left(\int_0^h |\psi'(\tau)| \, d\tau \right)^{\frac{\ell-1}{\ell}}.$$

By applying the Cauchy-Schwarz inequality, we have

$$\int_0^h |\psi'(\tau)| \, |\psi(\tau)| \, d\tau \leqslant \frac{h^{\frac{3\ell-1}{2\ell}}}{2^{\frac{1}{\ell}}} \left(\int_0^h |\psi'(\tau)|^2 \, d\tau \right)^{\frac{1}{\ell}} \left(\int_0^h |\psi'(\tau)|^2 \, d\tau \right)^{\frac{\ell-1}{2\ell}}$$

which completes the proof. □

Remark 2.6. In Corollary 2.5, if we take $\ell = 1$, the inequality (2.5) reduces to the inequality (1.2).

3. Functional inequalities via Opial type inequalities

By the inequalities (1.2) and (1.3), we define the following functionals

$$\Phi_1(s) = \frac{s-a}{2} \int_a^s |\psi'(\tau)|^2 \, d\tau - \int_a^s |\psi'(\tau)| \, |\psi(\tau)| \, d\tau, \tag{3.1}$$

$$\Phi_{\psi}(s) = \psi \left(\int_a^s (|\psi'(\tau)|) \, d\tau \right) - \int_a^s |\psi'(\tau)| \, \psi'(|\psi(\tau)|) \, d\tau \tag{3.2}$$

where the function ψ is as in Theorem 1.2 and 1.3. By the change of the variable $\tau = \lambda s + (1-\lambda)a$, we rewrite the (3.2) as follows

$$\Phi_\psi(s) = \psi\left((s-a)\int_0^1 \left(\left|\psi'(\lambda s+(1-\lambda)a)\right|\right)d\lambda\right) - (s-a)\int_0^1 \left|\psi'(\lambda s+(1-\lambda)a)\right|\psi'\left(\left|\psi(\lambda s+(1-\lambda)a)\right|\right)d\lambda \tag{3.3}$$

for $s \in (a,b]$.

Theorem 3.1. *Under the hypotheses of Theorem 1.2, let* $|\psi(\tau)|$ *and* $|\psi'(\tau)|$ *be convex functions on* $[a,b]$*. Then, the following integral inequality holds*

$$|\Phi_1(s)| \leqslant \frac{(s-a)^2}{6}\left(\left[\left|\psi'(s)\right|^2+\left|\psi'(a)\right|^2\right]+\left|\psi'(s)\right|\left|\psi'(a)\right|\right) + \frac{(s-a)}{6}\left[2\left|\psi'(s)\right|+\left|\psi'(a)\right|\right]|\psi(s)|$$

for $s \in (a,b]$.

Proof. From the functional (3.1), we have

$$|\Phi_1(s)| \leqslant \frac{s-a}{2}\int_a^s \left|\psi'(\tau)\right|^2 d\tau + \int_a^s \left|\psi'(\tau)\right||\psi(\tau)|\,d\tau.$$

By using the convexity of $|\psi(\tau)|$ and $|\psi'(\tau)|$, we get

$$\begin{aligned} |\Phi_1(s)| &\leqslant \frac{1}{2(s-a)}\int_a^s \left[(\tau-a)^2\left|\psi'(s)\right|^2+2(\tau-a)(s-\tau)\left|\psi'(s)\right|\left|\psi'(a)\right|+(s-\tau)^2\left|\psi'(a)\right|^2\right]d\tau \\ &+\frac{1}{(s-a)^2}\int_a^s [(\tau-a)|\psi(s)|]\left[(\tau-a)\left|\psi'(s)\right|+(s-\tau)\left|\psi'(a)\right|\right]d\tau \\ &= \frac{(s-a)^2}{6}\left(\left[\left|\psi'(s)\right|^2+\left|\psi'(a)\right|^2\right]+\left|\psi'(s)\right|\left|\psi'(a)\right|\right) \\ &+\frac{(s-a)}{6}\left[2\left|\psi'(s)\right|+\left|\psi'(a)\right|\right]|\psi(s)| \end{aligned}$$

which completes the proof. □

A similar results can be obtained for the different convex functions as follows:

Theorem 3.2. *Under the hypotheses of Theorem 1.2, let* $|\psi(\tau)|$, $|\psi'(\tau)|$ *and* $|\psi'(\tau)|^2$ *be convex functions on* $[a,b]$*. Then, the following integral inequality holds*

$$|\Phi_1(s)| \leqslant \frac{(s-a)^2}{4}\left(\left|\psi'(s)\right|^2+\left|\psi'(a)\right|^2\right)+\frac{(s-a)}{6}\left[2\left|\psi'(s)\right|+\left|\psi'(a)\right|\right]|\psi(s)|$$

for $s \in (a,b]$.

Proof. From the functional (3.1), we have

$$|\Phi_1(s)| \leqslant \frac{s-a}{2}\int_a^s |\psi'(\tau)|^2 d\tau + \int_a^s |\psi'(\tau)|\,|\psi(\tau)|\, d\tau.$$

By using the convexity of $|\psi(\tau)|$, $|\psi'(\tau)|$ and $|\psi'(\tau)|^2$, we get

$$\begin{aligned}
|\Phi_1(s)| &\leqslant \frac{1}{2}\int_a^s \left[(\tau-a)\left|\psi'(s)\right|^2 + (s-\tau)\left|\psi'(a)\right|^2\right] d\tau \\
&+ \frac{1}{(s-a)^2}\int_a^s [(\tau-a)\,|\psi(s)|]\left[(\tau-a)\left|\psi'(s)\right| + (s-\tau)\left|\psi'(a)\right|\right] d\tau \\
&= \frac{(s-a)^2}{4}\left(\left|\psi'(s)\right|^2 + \left|\psi'(a)\right|^2\right) + \frac{(s-a)}{6}\left[2\left|\psi'(s)\right| + \left|\psi'(a)\right|\right]|\psi(s)|
\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Under the hypotheses of Theorem 1.2, let* $|\psi(\tau)|^p$, $|\psi'(\tau)|^p$ *and* $|\psi'(\tau)|^q$ *be convex functions on* $[a,b]$, $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then, the following integral inequality holds*

$$|\Phi_1(s)| \leqslant (s-a)\left(\frac{|\psi'(s)|^q + |\psi'(a)|^q}{2}\right)^{\frac{1}{q}}\left[\frac{s-a}{2}\left(\frac{|\psi'(s)|^p + |\psi'(a)|^p}{2}\right)^{\frac{1}{p}} + |\psi(s)|\right]$$

for $s \in (a,b]$.

Proof. From the functional (3.1) and by using Hölder's inequality, we have

$$\begin{aligned}
|\Phi_1(s)| &\leqslant \frac{s-a}{2}\int_a^s \left|\psi'(\tau)\right|^2 d\tau + \int_a^s \left|\psi'(\tau)\right|\,|\psi(\tau)|\,d\tau \\
&\leqslant \left(\int_a^s \left|\psi'(\tau)\right|^q d\tau\right)^{\frac{1}{q}}\left[\frac{s-a}{2}\left(\int_a^s \left|\psi'(\tau)\right|^p d\tau\right)^{\frac{1}{p}} + \left(\int_a^s |\psi(\tau)|^p d\tau\right)^{\frac{1}{p}}\right].
\end{aligned}$$

By using the convexity of $|\psi(\tau)|^p$, $|\psi'(\tau)|^p$ and $|\psi'(\tau)|^q$ on $[a,b]$, we get

$$\begin{aligned}
|\Phi_1(s)| &\leqslant \left(\int_a^s \left[\frac{\tau-a}{s-a}\left|\psi'(s)\right|^q + \frac{s-\tau}{s-a}\left|\psi'(a)\right|^q\right] d\tau\right)^{\frac{1}{q}} \\
&\times\left[\frac{s-a}{2}\left(\int_a^s \left[\frac{\tau-a}{s-a}\left|\psi'(s)\right|^p + \frac{s-\tau}{s-a}\left|\psi'(a)\right|^p\right] d\tau\right)^{\frac{1}{p}} + \left(\int_a^s \frac{\tau-a}{s-a}|\psi(s)|^p d\tau\right)^{\frac{1}{p}}\right] \\
&= (s-a)^{\frac{1}{q}}\left(\frac{|\psi'(s)|^q + |\psi'(a)|^q}{2}\right)^{\frac{1}{q}}\left[\frac{s-a}{2}(s-a)^{\frac{1}{p}}\left(\frac{|\psi'(s)|^p + |\psi'(a)|^p}{2}\right)^{\frac{1}{p}} + (s-a)^{\frac{1}{p}}|\psi(s)|\right]
\end{aligned}$$

which this completes the proof. □

Theorem 3.4. *Under the hypotheses of Theorem 1.3, let* $|\psi(\tau)|$ *and* $|\psi'(\tau)|$ *be convex functions on* $[a,b]$. *Then, the following integral inequality holds*

$$\begin{aligned}
&\left|\Phi_{\psi}(s)\right| \\
\leqslant\ & \psi\left((s-a)\left(\frac{|\psi'(s)|+|\psi'(a)|}{2}\right)\right) \\
&+(s-a)\left[\frac{|\psi'(s)|}{|\psi(s)|}\psi\left((|\psi(s)|)\right)+\frac{|\psi'(a)|-|\psi'(s)|}{|\psi(s)|}\int_0^1\psi\left((\lambda\,|\psi(s)|)\right)d\lambda\right]
\end{aligned} \tag{3.4}$$

for $s\in(a,b]$.

Proof. From the functional (3.3), we have

$$\begin{aligned}
&\left|\Phi_{\psi}(s)\right| \\
\leqslant\ & \psi\left((s-a)\int_0^1\left(\left|\psi'(\lambda s+(1-\lambda)a)\right|\right)d\lambda\right) \\
&+(s-a)\int_0^1\left|\psi'(\lambda s+(1-\lambda)a)\right|\psi'\left(|\psi(\lambda s+(1-\lambda)a)|\right)d\lambda.
\end{aligned} \tag{3.5}$$

By using the convexity of $|\psi'(\tau)|$, we get

$$\begin{aligned}
&(s-a)\int_0^1\left(\left|\psi'(\lambda s+(1-\lambda)a)\right|\right)d\lambda \\
\leqslant\ & (s-a)\int_0^1\left(\lambda\left|\psi'(s)\right|+(1-\lambda)\left|\psi'(a)\right|\right)d\lambda \\
=\ & (s-a)\left(\frac{|\psi'(s)|+|\psi'(a)|}{2}\right).
\end{aligned} \tag{3.6}$$

Since ψ is a increasing function, it follows that the (3.6) can be written

$$\psi\left((s-a)\int_0^1\left(\left|\psi'(\lambda s+(1-\lambda)a)\right|\right)d\lambda\right)\leqslant\psi\left((s-a)\left(\frac{|\psi'(s)|+|\psi'(a)|}{2}\right)\right). \tag{3.7}$$

On the other hand, using the convexity of $|\psi(\tau)|$, we get

$$\begin{aligned} & (s-a)\int_0^1 \left|\psi'(\lambda s+(1-\lambda)a)\right| \psi'\left(|\psi(\lambda s+(1-\lambda)a)|\right) d\lambda \\ \leqslant\ & (s-a)\int_0^1 \left(\lambda\left|\psi'(s)\right|+(1-\lambda)\left|\psi'(a)\right|\right)\psi'\left(\lambda|\psi(s)|\right) d\lambda \\ =\ & (s-a)\left|\psi'(s)\right|\int_0^1 \lambda\psi'\left(\lambda|\psi(s)|\right) d\lambda+(s-a)\left|\psi'(a)\right|\int_0^1 (1-\lambda)\psi'\left(\lambda|\psi(s)|\right) d\lambda. \end{aligned}$$

Here, we apply integration by parts in integrals, we have

$$\begin{aligned} & (s-a)\int_0^1 \left|\psi'(\lambda s+(1-\lambda)a)\right| \psi'\left(|\psi(\lambda s+(1-\lambda)a)|\right) d\lambda \qquad (3.8) \\ \leqslant\ & (s-a)\left[\frac{|\psi'(s)|}{|\psi(s)|}\psi\left(|\psi(s)|\right)+\frac{|\psi'(a)|-|\psi'(s)|}{|\psi(s)|}\int_0^1 \psi\left(\lambda|\psi(s)|\right) d\lambda\right]. \end{aligned}$$

Substitute the inequalities (3.7) and (3.8) in the inequality (3.5), we obtain that

$$\begin{aligned} \left|\Phi_\psi(s)\right| \ \leqslant\ & \psi\left((s-a)\left(\frac{|\psi'(s)|+|\psi'(a)|}{2}\right)\right) \\ & +(s-a)\left[\frac{|\psi'(s)|}{|\psi(s)|}\psi\left(\left(|\psi(s)|\right)\right)+\frac{|\psi'(a)|-|\psi'(s)|}{|\psi(s)|}\int_0^1 \psi\left(\left(\lambda|\psi(s)|\right)\right) d\lambda\right] \end{aligned}$$

which this completes the proof of the (3.4). □

Corollary 3.5. *Under the hypotheses of Theorem 3.4, if we take $\psi(u)=u$ on $[a,b]$, we have*

$$|\Phi_u(s)| = \left|\int_a^s \left|\psi'(\tau)\right| d\tau - \int_a^s \left|\psi'(\tau)\right| \left(|\psi(\tau)|\right) d\tau\right| \leqslant (s-a)\left(\left|\psi'(s)\right|+\left|\psi'(a)\right|\right).$$

for $s\in(a,b]$.

Corollary 3.6. *Under the hypotheses of Theorem 3.4, if we take $\psi(u)=u^2$ on $[a,b]$, we have*

$$\begin{aligned} |\Phi_{u^2}(s)| \ =\ & \left|\left(\int_a^s \left(\left|\psi'(\tau)\right|\right) d\tau\right)^2 - 2\int_a^s \left|\psi'(\tau)\right| |\psi(\tau)|\, d\tau\right| \\ \leqslant\ & (s-a)^2\left(\frac{|\psi'(s)|+|\psi'(a)|}{2}\right)^2+(s-a)\,|\psi(s)|\,\frac{|\psi'(a)|+2\,|\psi'(s)|}{3} \end{aligned}$$

for $s\in(a,b]$.

4. Conclusion

In this paper, some generalizations of Opial type inequalities defined Opial-Jensen inequality for convex function were established and also presented some extensions of the analogues. Further, new Opial type inequalities will be given for new functionals defined with the help of Opial inequalities. Thus our results are new results for Opial type inequalities. In future work, we recommend researchers can be obtained new results in different types of convex functions, fractional calculus or inner product spaces with the method used in this study.

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