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Discrete Homotopy Analysis Method to Solve Nabla Fractional Partial Difference Equations

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Abstract

In the present work, the discrete homotopy analysis method is applied to solve nabla time-fractional partial difference equations. Fractional difference operator is considered in Caputo's sense. We apply the discrete homotopy analysis method to nabla fractional initial value problems. Obtained solutions involve an auxiliary parameter $\hbar$, which we can determine. Thus, it may be concluded that the discrete homotopy analysis method is a very powerful and successful analytical approach for fractional difference equations.

Keywords: Discrete homotopy analysis method, Nabla fractional sum, Caputo-like nabla fractional difference, Nabla difference equations, Fractional difference.
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1. Introduction

With the development of high-speed numerical computing machines, difference equations have become widespread for problems involving time-dependent fluid flows, neutron diffusion and transport, and radiation flux. Real-world applications such as economic time series, pixel dots in images, and sampled signals all hold discrete-time or space structures. Although numerical approaches are used in such real-world applications, the difference equations are simpler.
Fractional difference equations have received considerable attention recently[1]. It caused the development of discrete fractional calculus [2, 3, 4, 5]. Some valuable works have been published. For example, a new type of fractional sums and differences were presented in [6] and different types of fractional difference of properties were discussed in [7]. Nabla fractional is a discretized version of the fractional derivative. Discrete fractional calculus has been recognized as a powerful instrument to explore the secret ways of physical processes and various materials, expressed by discrete versions of integrals and derivatives of arbitrary orders, called fractional sums and differences. Recently there have

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been some results for nabla fractional difference equations[3, 8, 9, 10]. Discovering a suitable method for solving partial difference equations has become important[11, 12]. The discrete homotopy analysis method is the discretization of the homotopy analysis method proposed by Liao in 1992[13]. Unlike the other analytical techniques, the discrete homotopy analysis method is independent of small/large physical parameters[14, 15]. Since the discrete homotopy analysis method has many advantages compared to other analytical methods, it is employed to solve linear and nonlinear nabla fractional initial value problems. The discrete homotopy analysis method contains the auxiliary parameter, which provides a simple way to guarantee the convergence region of the solution series. This method is powerful in solving wide classes of nabla fractional partial difference equations. We propose the discrete homotopy analysis method to solve nabla fractional initial value problems in the present paper. Fractional difference operator is considered in Caputo's sense. To illustrate the applicability of our approach, we apply the discrete homotopy analysis method to nabla fractional partial difference equations with initial conditions. The method is powerful and useful as it contains an auxiliary parameter $\hbar$ that allows us to control the region of convergence of the solution. Obtained results show that the approaches are easy to apply and accurate when implemented to time-fractional difference equations. To our best knowledge, the discrete homotopy analysis method is used for the first time in this study to solve fractional partial difference equations.
Now some definitions and properties of nabla fractional calculus are introduced in this section.

2. Preliminaries

Definition 2.1 ([5]). Let $f : \mathbb{N}_a \to \mathbb{R}$ then the κth nabla (left) fractional sum of a function f, for $\kappa > 0$, is defined by

$$ {}_a\nabla_t^{-\kappa} f(t) = \frac{1}{\Gamma(\kappa)} \sum_{\tau=a+1}^{t} \left(t-\sigma(\tau)\right)^{\overline{\kappa-1}} f(\tau), \qquad t \in \mathbb{N}_{a+1}, $$

where $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, $\sigma(\tau) = \tau - 1$ is backward jumping operator. ${}_a\nabla_t^{-\kappa}$ maps function on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$.

The generalized rising function is expressed as

$$ t^{\overline{\kappa}} = \frac{\Gamma(t+\kappa)}{\Gamma(t)}, \qquad t \notin (-\mathbb{N}_0). $$

We assume that $t^{\overline{\kappa}} = 0$ for $t \in (-\mathbb{N}_0)$ and $t + \kappa \notin (-\mathbb{N}_0)$.
Let $\kappa > 0$, $\upsilon > -1$. Then, for $t \in \mathbb{N}_a$

$$ {}_a\nabla_t^{-\kappa} (t-a)^{\overline{\upsilon}} = \frac{\Gamma(\upsilon+1)}{\Gamma(\kappa+\upsilon+1)} (t-a)^{\overline{\kappa+\upsilon}}, $$

is the power rule which was proved in [5].

Definition 2.2 ([5]). Let $f : \mathbb{N}_a \to \mathbb{R}$ then the κth nabla Caputo (left) fractional difference of a function f, for $\kappa > 0$, is defined by

$$
\begin{aligned}
{}_a^C\nabla_t^{\kappa} f(t) &= {}_a\nabla_t^{-(l-\kappa)} \nabla^l f(t) \\
&= \frac{1}{\Gamma(l-\kappa)} \sum_{\tau=a+1}^{t-(l-\kappa)} (t-\sigma(\tau))^{\overline{l-\kappa-1}} \nabla^l f(\tau),
\end{aligned}
$$

where $l \in \mathbb{N}_0$ is such that $l = [\kappa]+1$, $[\kappa]$ is the greatest integer less than κ.
If $\kappa = l \in \mathbb{N}_0$, then

$$
{}_a^C\nabla_t^{\kappa} f(t) = \nabla^l f(t).
$$

${}_a^C\nabla_t^{\kappa}$ maps functions on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+1-l}$.

Proposition 2.3 ([5]). *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $\kappa > 0$, *then*

$$
{}_a\nabla_t^{-\kappa} {}_a^C\nabla_t^{\kappa} f(t) = f(t) - \sum_{s=0}^{l-1} \frac{(t-a)^{\overline{s}}}{s!} \nabla^s f(a),
$$

where $l \in \mathbb{N}_0$ *is such that* $l = [\kappa]+1$.
When $0 < \kappa \leqslant 1$,

$$
{}_a\nabla_t^{-\kappa} {}_a^C\nabla_t^{\kappa} f(t) = f(t) - f(a).
$$

Definition 2.4 ([5]). (Nabla discrete Mittag-Leffler) Let $\eta \in \mathbb{R}$, $\alpha, \beta, t \in \mathbb{C}$ for $\mathrm{Re}(\alpha) > 0$, then the nabla discrete(like) Mittag-Leffler functions are defined by

$$
E_{\overline{\alpha,\beta}}(\eta, t) = \sum_{j=0}^{\infty} \eta^j \frac{t^{\overline{j\alpha+\beta-1}}}{\Gamma(j\alpha+\beta)}.
$$

For special case

$$
E_{\overline{\alpha}}(\eta, t) = E_{\overline{\alpha,1}}(\eta, t) = \sum_{j=0}^{\infty} \eta^j \frac{t^{\overline{j\alpha}}}{\Gamma(j\alpha+1)}.
$$

3. Discrete Homotopy Analysis Method for Nabla Fractional Calculus

To understand the basic idea of the discrete homotopy analysis method(DHAM), the following fractional partial difference equation with nabla fractional operator is considered:

$$ {}_{a}^{C}\nabla_{t}^{\kappa}U(k,t)+L\big(U(k,t)\big)+N\big(U(k,t)\big)=f(k,t),\ k\in\mathbb{N}_0,\ t\in\mathbb{N}_{a+1-l},\ l-1<\kappa\leqslant l, \tag{3.1}$$

with initial conditions

$$ {}_{a}^{C}\nabla_{t}^{s}U(k,a)=g_s(k),\qquad s=0,1,\dots,l-1, \tag{3.2}$$

where L is a linear difference and N is nonlinear difference operator, $f(k,t)$ is the source term.
Since (3.1) contains the linear operator ${}_{a}^{C}\nabla_{t}^{\kappa}$, we can define the auxiliary linear operator

$$ Ł\big[U(k,t)\big]={}_{a}^{C}\nabla_{t}^{\kappa}U(k,t). \tag{3.3}$$

According to the Proposition2.3, we can choose the initial approximation as follows

$$ U_0(k,t)=\sum_{s=0}^{l-1}g_s(k)\frac{t^s}{s!}, \tag{3.4}$$

and we can define the nonlinear operator

$$ \aleph\big[U(k,t)\big]={}_{a}^{C}\nabla_{t}^{\kappa}U(k,t)+L\big(U(k,t)\big)+N\big(U(k,t)\big)-f(k,t). \tag{3.5}$$

By means of DHAM, first of all we can construct the zeroth-order deformation equation

$$ (1-q)Ł\big[\gamma(k,t;q)-U_0(k,t)\big]=q\hbar H(k,t)\aleph\big[\gamma(k,t;q)\big], \tag{3.6}$$

with initial conditions

$$ {}_{a}^{C}\nabla_{t}^{s}\gamma(k,a;q)=g_s(k),\qquad s=0,1,\dots,l-1, \tag{3.7}$$

where $H(k,t)$ is a nonzero auxiliary function, $\hbar\neq0$ is a convergence-control parameter, $q\in[0,1]$ is an embedding parameter, $U_0(k,t)$ is an initial guess of $U(k,t)$, $\gamma(k,t;q)$ is an unknown function.
When $q=0$ and $q=1$, we have obviously

$$ \gamma(k,t;0)=U_0(k,t)\qquad \textit{and}\qquad \gamma(k,t;1)=U(k,t), \tag{3.8}$$

respectively. We expand $\gamma(k,t;q)$ in Taylor series, using the embedding parameter q as follows:

$$\gamma(k,t;q) = U_0(k,t) + \sum_{n=1}^{\infty} U_n(k,t)q^n. \tag{3.9}$$

Here

$$U_n(k,t) = \frac{1}{n!}\frac{\partial^n \gamma(k,t;q)}{\partial q^n}\bigg|_{q=0}.$$

We suppose that the auxiliary linear operator, the convergence-control parameter, the auxiliary function and the initial guess are selected such that the series (3.9) is convergent at $q = 1$.
Therefore, due to (3.8) we get

$$U(k,t) = U_0(k,t) + \sum_{n=1}^{\infty} U_n(k,t). \tag{3.10}$$

Now we define the vector:

$$\overrightarrow{U_n} = \{U_0(k,t), U_1(k,t), U_2(k,t), \dots, U_n(k,t)\}.$$

To obtain nth-order deformation equation differentiate (3.6) n-times with respect to q and set $q = 0$ and divide by $n!$:

$$Ł\left[U_n(k,t) - \chi_n U_{n-1}(k,t)\right] = \hbar H(k,t)\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right], \tag{3.11}$$

with initial conditions

$${}_a^C\nabla_t^s U_n(k,a) = 0, \qquad s = 0,1,\dots,l-1. \tag{3.12}$$

Here

$$\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right] = \frac{1}{(n-1)!}\frac{\partial^{n-1}\aleph\left[\gamma(k,t;q)\right]}{\partial q^{n-1}}\bigg|_{q=0} \tag{3.13}$$

and

$$\chi_n = \begin{cases} 0, & n \leqslant 1 \\ 1, & n > 1 \end{cases}.$$

Substituting (3.5) in to (3.13), and since L is a linear operator we obtain

$$\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right] = {}_a^C\nabla_t^{\kappa} U(k,t) + L\big(U(k,t)\big) + \frac{1}{(n-1)!}\frac{\partial^{n-1} N\left[\gamma(k,t;q)\right]}{\partial q^{n-1}}\bigg|_{q=0} - (1-\chi_n) f(k,t). \tag{3.14}$$

According (3.3), we can apply the nabla fractional sum ${}_a\nabla_t^{-\kappa}$ to both sides of (3.11) to obtain

$${}_a\nabla_t^{-\kappa}{}_a^C\nabla_t^{\kappa}\left[U_n(k,t) - \chi_n U_{n-1}(k,t)\right] = \hbar \cdot {}_a\nabla_t^{-\kappa}\left[H(k,t)\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right]\right]. \tag{3.15}$$

Because of Proposition2.3, we get

$$U_n(k,t) = \chi_n U_{n-1}(k,t) + \hbar \cdot {}_a\nabla_t^{-\kappa}\left[H(k,t)\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right]\right]. \tag{3.16}$$

Since nth-order deformation equation (3.11) is linear, it can be easily solved with symbolic computation software.

4. Examples

In this section, we shall illustrate the effectiveness of DHAM in some nabla fractional partial difference equations.

Example 4.1. First we consider the nabla Caputo fractional partial difference equation

$${}_0^C\nabla_t^{\kappa} U(k,t) = \eta U(k,t), \qquad k,t \in \mathbb{N}_0, \qquad 0 < \kappa \leqslant 1, \tag{4.1}$$

with initial condition

$$U(k,0) = a_0, \qquad a_0 \in \mathbb{R}. \tag{4.2}$$

According to DHAM, we select the linear operator

$$Ł\left[\gamma(k,t;q)\right] = {}_0^C\nabla_t^{\kappa}\gamma(k,t;q), \tag{4.3}$$

with $Ł\left[c\right] = 0$, where c is constant. Let us define nonlinear operator

$$\aleph\left[\gamma(k,t;q)\right] = {}_0^C\nabla_t^{\kappa}\gamma(k,t;q) - \eta\gamma(k,t;q). \tag{4.4}$$

From (3.16) we have

$$U_n(k,t) = \chi_n U_{n-1}(k,t) + \hbar \cdot {}_0\nabla_t^{-\kappa}\left[H(k,t)\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right]\right], \tag{4.5}$$

where

$$\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right] = {}_0^C\nabla_t^{\kappa} U_{n-1}(k,t) - \eta U_{n-1}(k,t).$$

In order to the rule of solution expression is only depend on $\hbar$, we choose $H(k,t) = 1$. Substituting the initial condition (4.2) in to (4.5), we obtain

$$\begin{aligned}
U_1(k,t) &= -\hbar\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)},\\
U_2(k,t) &= -\hbar(1+\hbar)\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + \hbar^2\eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)},\\
U_3(k,t) &= -\hbar(1+\hbar)^2\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + 2\hbar^2(1+\hbar)\eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)}\\
&\quad - \hbar^3\eta^3 a_0 \frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)},\\
U_4(k,t) &= -\hbar(1+\hbar)^3\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + 3\hbar^2(1+\hbar)^2\eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)}\\
&\quad - 3\hbar^3(1+\hbar)\eta^3 a_0 \frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)} + \hbar^4\eta^4 a_0 \frac{t^{\overline{4\kappa}}}{\Gamma(4\kappa+1)},\\
&\ \ \vdots
\end{aligned}$$

and so on. Therefore, the series solution is

$$\begin{aligned}
U(k,t) =& U_0(k,t) + U_1(k,t) + U_2(k,t) + U_3(k,t) + U_4(k,t) + \dots\\
=& a_0 - \hbar\left[1 + (1+\hbar) + (1+\hbar)^2 + (1+\hbar)^3 + \dots\right]\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)}\\
&+ \hbar^2\left[1 + 2(1+\hbar) + 3(1+\hbar)^2 + \dots\right]\eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)}\\
&- \hbar^3\left[1 + 3(1+\hbar) + \dots\right]\eta^3 a_0 \frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)} + \hbar^4\eta^4 a_0 \frac{t^{\overline{4\kappa}}}{\Gamma(4\kappa+1)} + \dots
\end{aligned}$$

When $\hbar = -1$, we get the solution in the following form

$$\begin{aligned}
U(k,t) =& a_0 + \eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + \eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)} + \eta^3 a_0 \frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)} + \eta^4 a_0 \frac{t^{\overline{4\kappa}}}{\Gamma(4\kappa+1)} + \dots\\
=& a_0 \sum_{j=0}^{\infty} \eta^j \frac{t^{\overline{j\kappa}}}{\Gamma(j\kappa+1)}\\
=& a_0 E_{\overline{\kappa}}(\eta,t).
\end{aligned} \tag{4.6}$$

Example 4.2. Let $0 < \kappa \leqslant 1$, we consider the nabla Caputo nonhomogeneous fractional partial difference equation

$$ {}_0^C\nabla_t^{\kappa} U_{k,t} = \eta U_{k,t} + f_{k,t}, \qquad k, t \in \mathbb{N}_0, \tag{4.7} $$

with initial condition

$$ U(k, 0) = a_0, \qquad a_0 \in \mathbb{R}. \tag{4.8} $$

We can choose the linear operator

$$ Ł\left[\gamma(k, t; q)\right] = {}_0^C\nabla_t^{\kappa}\gamma(k, t; q), \tag{4.9} $$

$Ł\left[c\right] = 0$ and c is constant. We can define nonlinear operator

$$ \aleph\left[\gamma(k, t; q)\right] = {}_0^C\nabla_t^{\kappa}\gamma(k, t; q) - \eta\gamma(k, t; q) - f(k, t). \tag{4.10} $$

We will choose $H(k, t) = 1$ for the same reason as in Example4.1. Thus from (3.16) we have

$$ U_n(k, t) = \chi_n U_{n-1}(k, t) + \hbar \cdot {}_0\nabla_t^{-\kappa}\left[\mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right]\right], \tag{4.11} $$

where

$$ \mathfrak{R}_n\left[\overrightarrow{U_{n-1}}\right] = {}_0^C\nabla_t^{\kappa} U_{n-1}(k, t) - \eta U_{n-1}(k, t) - (1 - \chi_n) f(k, t). $$

Substituting the initial condition (4.8) in to (4.11), we get

$$ \begin{aligned} U_1(k, t) &= -\hbar\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} - \hbar \cdot {}_0\nabla_t^{-\kappa} f(k, t), \\ U_2(k, t) &= -\hbar(1+\hbar)\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + \hbar^2\eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)} - \hbar(1+\hbar){}_0\nabla_t^{-\kappa} f(k, t) + \hbar^2\eta \cdot {}_0\nabla_t^{-2\kappa} f(k, t), \\ U_3(k, t) &= -\hbar(1+\hbar)^2\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + 2\hbar^2(1+\hbar)\eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)} - \hbar^3\eta^3 a_0 \frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)} \\ &\quad - \hbar(1+\hbar)^2{}_0\nabla_t^{-\kappa} f(k, t) + 2\hbar^2(1+\hbar)\eta \cdot {}_0\nabla_t^{-2\kappa} f(k, t) - \hbar^3\eta^2{}_0\nabla_t^{-3\kappa} f(k, t), \\ U_4(k, t) &= -\hbar(1+\hbar)^3\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + 3\hbar^2(1+\hbar)^2\eta^2 a_0 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)} - 3\hbar^3(1+\hbar)\eta^3 a_0 \frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)} \\ &\quad + \hbar^4\eta^4 a_0 \frac{t^{\overline{4\kappa}}}{\Gamma(4\kappa+1)} - \hbar(1+\hbar)^3{}_0\nabla_t^{-\kappa} f(k, t) + 3\hbar^2(1+\hbar)^2\eta \cdot {}_0\nabla_t^{-2\kappa} f(k, t) \\ &\quad - 3\hbar^3(1+\hbar)\eta^2{}_0\nabla_t^{-3\kappa} f(k, t) + \hbar^4\eta^3{}_0\nabla_t^{-4\kappa} f(k, t), \\ &\ \ \vdots \end{aligned} $$

and so on. Thus, the series solution is

$$\begin{aligned}U(k,t) =&a_0 - \hbar\big[1+(1+\hbar)+(1+\hbar)^2+(1+\hbar)^3+\dots\big]\eta a_0 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)}\\&+\hbar^2\big[1+2(1+\hbar)+3(1+\hbar)^2+\dots\big]\eta^2 a_0\frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)}\\&-\hbar^3\big[1+3(1+\hbar)+\dots\big]\eta^3 a_0\frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)}+\hbar^4\eta^4 a_0\frac{t^{\overline{4\kappa}}}{\Gamma(4\kappa+1)}\\&-\hbar\big[1+(1+\hbar)+(1+\hbar)^2+(1+\hbar)^3+\dots\big]{}_0\nabla_t^{-\kappa}f(k,t)\\&+\hbar^2\big[1+2(1+\hbar)+3(1+\hbar)^2+\dots\big]\eta\cdot{}_0\nabla_t^{-2\kappa}f(k,t)\\&-\hbar^3\big[1+3(1+\hbar)+\dots\big]\eta^2{}_0\nabla_t^{-3\kappa}f(k,t)+\hbar^4\eta^3{}_0\nabla_t^{-4\kappa}f(k,t)+\dots\end{aligned}$$

When $\hbar=-1$, we obtain the solution in the following form

$$\begin{aligned}U(k,t) =&a_0+\eta a_0\frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)}+\eta^2 a_0\frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)}+\eta^3 a_0\frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)}+\eta^4 a_0\frac{t^{\overline{4\kappa}}}{\Gamma(4\kappa+1)}\\&+{}_0\nabla_t^{-\kappa}f(k,t)+\eta\cdot{}_0\nabla_t^{-2\kappa}f(k,t)+\eta^2{}_0\nabla_t^{-3\kappa}f(k,t)+\eta^3{}_0\nabla_t^{-4\kappa}f(k,t)+\dots\\=&a_0\sum_{j=0}^{\infty}\eta^j\frac{t^{\overline{j\kappa}}}{\Gamma(j\kappa+1)}+\sum_{j=1}^{\infty}\eta^{j-1}{}_0\nabla_t^{-j\kappa}f(k,t)\\=&a_0E_{\overline{\kappa}}(\eta,t)+\sum_{j=0}^{\infty}\eta^j\frac{1}{\Gamma(j\kappa+\kappa)}\sum_{\tau=1}^{t}\big(t-\sigma(\tau)\big)^{\overline{\tau\kappa+\kappa-1}}f(k,\tau).\end{aligned}$$

Inter changing the order of sums, we obtain

$$U(k,t)=a_0E_{\overline{\kappa}}(\eta,t)+\sum_{\tau=1}^{t}\sum_{j=0}^{\infty}\eta^j\frac{\big(t-\sigma(\tau)\big)^{\overline{\tau\kappa+\kappa-1}}}{\Gamma(j\kappa+\kappa)}f(k,\tau).$$

That is

$$U(k,t)=a_0E_{\overline{\kappa}}(\eta,t)+\sum_{\tau=1}^{t}E_{\overline{\kappa,\kappa}}(\eta,t-\sigma(\tau))f(k,\tau). \tag{4.12}$$

Example 4.3. Now we consider the nonlinear nabla Caputo fractional partial difference equation

$${}_0^C\nabla_t^{\kappa}U(k,t)=U^2(k,t),\qquad k,t\in\mathbb{N}_0,\qquad 0<\kappa\leqslant 1, \tag{4.13}$$

with initial condition

$$U(k,0) = a_0, \qquad a_0 \in \mathbb{R}. \tag{4.14}$$

We choose the linear operator

$$Ł\left[\gamma(k,t;q)\right] = {}_0^C\nabla_t^\kappa \gamma(k,t;q), \tag{4.15}$$

with $Ł\left[c\right] = 0$ and c is constant. Now we define the nonlinear operator

$$\aleph\left[\gamma(k,t;q)\right] = {}_0^C\nabla_t^\kappa \gamma(k,t;q) - \gamma^2(k,t;q). \tag{4.16}$$

We choose again $H_{k,t} = 1$ and then we obtain

$$U_n(k,t) = \chi_n U_{n-1}(k,t) + \hbar \cdot {}_0\nabla_t^{-\kappa}\left[\Re_n\left[\overrightarrow{U_{n-1}}\right]\right]. \tag{4.17}$$

Substituting the initial condition (4.14) in to (4.17), we obtain

$$\begin{aligned}
U_1(k,t) &= -\hbar a_0^2 \frac{t^\kappa}{\Gamma(\kappa+1)},\\
U_2(k,t) &= -\hbar(1+\hbar) a_0^2 \frac{t^\kappa}{\Gamma(\kappa+1)} + 2\hbar^2 a_0^3 \frac{t^{2\kappa}}{\Gamma(2\kappa+1)},\\
U_3(k,t) &= -\hbar(1+\hbar)^2 a_0^2 \frac{t^\kappa}{\Gamma(\kappa+1)} + 2\hbar^2(1+\hbar) a_0^2 (1+a_0) \frac{t^{2\kappa}}{\Gamma(2\kappa+1)}\\
&\quad - 4\hbar^3 a_0^4 \frac{t^{3\kappa}}{\Gamma(3\kappa+1)} - \frac{\hbar^3 a_0^4}{\Gamma^2(\kappa+1)} {}_0\nabla_t^{-\kappa}\left[(t^\kappa)^2\right],\\
&\quad \vdots
\end{aligned}$$

and so on. Thus, the series solution is

$$\begin{aligned}
U(k,t) =& a_0 - \hbar\left[1 + (1+\hbar) + (1+\hbar)^2 + \dots\right] a_0^2 \frac{t^\kappa}{\Gamma(\kappa+1)}\\
&+ 2\hbar^2\left[a_0 + (1+a_0)(1+\hbar+\dots\right] a_0^2 \frac{t^{2\kappa}}{\Gamma(2\kappa+1)}\\
&- 4\hbar^3 a_0^4 \frac{t^{3\kappa}}{\Gamma(3\kappa+1)} + -\frac{\hbar^3 a_0^4}{\Gamma^2(\kappa+1)} {}_0\nabla_t^{-\kappa}\left[(t^\kappa)^2\right] + \dots
\end{aligned}$$

If we take $\hbar = -1$, we get the solution in the following form

$$U(k,t) = a_0 + a_0^2 \frac{t^{\overline{\kappa}}}{\Gamma(\kappa+1)} + 2a_0^3 \frac{t^{\overline{2\kappa}}}{\Gamma(2\kappa+1)} + 4a_0^4 \frac{t^{\overline{3\kappa}}}{\Gamma(3\kappa+1)} + \frac{a_0^4}{\Gamma^2(\kappa+1)} {}_0\nabla_t^{-\kappa}\left[(t^{\overline{\kappa}})^2\right] + \dots \tag{4.18}$$

5. Conclusion

The discrete homotopy analysis method was applied to obtain analytical solutions for nabla time- fractional partial difference equations. Validation of the proposed method is shown with three noteworthy examples. The reliability of this method gives it wider applicability. The discrete homotopy analysis method contains an auxiliary parameter $\hbar$, which gives us to check the convergence region of the solution. It can be said that the DHAM is very powerful and successful in finding analytical solutions of fractional difference equations emerging in the fields of engineering, technology and science.

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