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On Some Generalized Fractional Integral Bullen Type Inequalities With Applications

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Abstract

The theory of fractional integral inequalities plays a pivotal role in approximation theory. It is very useful in establishing the uniqueness of solutions for some fractional differential equations. Here, a generalized fractional integral identity is established to deduce new estimates for Bullen type functional and of some related inequalities to provide some applications in probability and information theory for $(s,p)-$convex function by use of basic techniques of analysis.

Keywords: Bullen's inequality, $(s,p)-$convex function, Power mean inequality, Holder's inequality.
2010 MSC: 26D10, 26A51, 34A40, 26B25 .

1. Introduction

Over few years, the fractional calculus has attained a lot of attention due to its vast nature along with its advance applications in various fields such as: Chaotic systems, chronicles of fractional order, fluid flow, rheology, geotechincal engineering, electrical networks, electromagnetic theory, probability [1, 2, 3, 4] etc. Like ordinary calculus, fractional calculus has not unified representations. Fractional integrals and derivatives are defined by many ways. For instance, fractional integral and derivative by: Riemann-Liouville [5], Hadamard [6], Katugampola [7], Caputo fractional integrals and derivatives [8] etc. Over a decade, these were further generalized by many authors, for example, the Bullen's inequality [9], Ostrowski's inequality [10, 5]. Chen [11] in 2016 proved the following Riemann–Liouville fractional integral Hermite–Hadamard type inequalities. Among mathematical inequalities, one inequality has a well-known area along with the inequality theory. It is the well-known Hermite–Hadamard inequality. The first propose of this inequality is ascribed to Hermite (1881), but until 1893, this result was not taken into a consideration in the literature and was not vastly known as Hermite's inequality.

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Theorem 1.1. *[12] Let* $f : J \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function defined on the interval* I *of real numbers and* $a, b \in I$ *with* $a < b$*, then the following inequality*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds, known in the literature as Hermite-Hadamard integral inequality for convex functions. It has several generalizations and extensions for univariable, bivariable, and multivariable convex functions see [13, 14, 15, 16, 17, 18]. Moreover, it is known that some of the classical inequalities for means can be derived from (1.1) for appropriate particular choices of the function f.

Theorem 1.2. *[12] Let* $f : [a, b] \to \mathbb{R}$ *be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \leqslant \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \leqslant \frac{f(a)+f(b)}{2}$$

The inequality (2) is known in the literature as Bullen's inequality.
Bullen type inequalities have attracted much attention since they are very remarkable in the area of applications. Due to its significance in various fields, Bullen inequality has considerable interest and concern from many scientists and mathematicians. Researchers have devoted a great deal of time and effort over the years in improving and generalizing the inequality of Bullen. Thus, for various classes of convex functions, many scientists have generalized the classical version of the popular Bullen's inequality.

2. Preliminaries

Throughout the whole discussion we consider $p \in R/\{0\}$ and I an interval in $(0, \infty)$.

Definition 1. *[19] Let* $f : I \to \mathbb{R}$ *be real−valued function, then* f *is said to be* $p-$*convex if*

$$f\left(\sqrt[p]{\tau x^p + (1-\tau)y^p}\right) \leqslant \tau f(x) + (1-\tau)f(y) \tag{2.1}$$

holds for all $x, y \in I$ *and* $\tau \in [0, 1]$*. For* f *to be* $p-$*concave, (2.1) holds in reverse order. Moreover, for* $s \in (0, 1]$*,* f *is said to be* $(s, p)-$*convex if*

$$f\left(\sqrt[p]{\tau x^p + (1-\tau)y^p}\right) \leqslant \tau^s f(x) + (1-\tau)^s f(y) \tag{2.2}$$

holds for all $x, y \in I$ *and* $\tau \in [0, 1]$*. In case of* $(s, p)-$*concave function, (2.2) holds in reverse order.*
For example, let $p > 0$*,* $s \in (0, 1]$ *and* $f : (0, \infty) \to (0, \infty)$ *be defined by* $f(x) = x^{sp}$*. Then* f *is* $(s, p)-$*convex function.*

Definition 2. *[20] The gamma function* Γ*, beta function* $\mathbb{B}$ *and the hypergeometric function* ${}_2F_1$ *are respectively, defined by:*

$$\Gamma(x) := \int_0^\infty e^{-\tau}\tau^x dt \quad (x > 0),$$

$$\mathbb{B}(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}d\tau \quad (x,y > 0)$$

$${}_2F_1(a,b;c,z) := \frac{1}{\beta(b,c-b)}\int_0^1 \tau^{b-1}(1-\tau)^{c-b-1}(1-z\tau)^{-a}d\tau \quad (|z| < 1)$$

Definition 3. *[21] The incomplete beta function* $\mathbb{B}(z;x,y)$ *is defined by:*

$$\mathbb{B}(z;x,y) := \int_0^z \tau^{x-1}(1-\tau)^{y-1}d\tau \quad (\mathrm{Re}(x) > \mathrm{Re}(y) > 0, 0 \leqslant z < 1).$$

Raina [15] introduced a class of functions as follows:

$$\mathfrak{J}^{\sigma}_{\rho,\lambda}(x) := \mathfrak{J}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k \quad (\rho,\lambda \in \mathbb{R}^+, x \in \mathbb{R})$$

where the coefficients $\sigma(k) \in \mathbb{R}^+$, $k \in \mathbb{N}_0$ *form a bounded sequence.*

Definition 4. *[22] The left-sided and right-sided fractional integral operators are defined as:*

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,a+;w}\phi\right)(x) := \int_a^x (x-\tau)^{\lambda-1}\mathfrak{J}^{\sigma(k)}_{\rho,\lambda}[w(x-\tau)^{\rho}]\phi(\tau)d\tau \quad (x > a) \tag{2.3}$$

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,b-;w}\phi\right)(x) := \int_x^b (\tau-x)^{\lambda-1}\mathfrak{J}^{\sigma(k)}_{\rho,\lambda}[w(\tau-x)^{\rho}]\phi(\tau)d\tau \quad (x < b), \tag{2.4}$$

provided that, $w \in \mathbb{R}, \lambda > 0$ *and* ϕ *is a function such that the integrals on right hand side exist. It is easy to verify that* $\mathfrak{J}^{\sigma}_{\rho,\lambda,a+;w}\phi(x)$ *and* $\mathfrak{J}^{\sigma}_{\rho,\lambda,b-;w}\phi(x)$ *are bounded integral operators on* $L(a,b)$ *for* $\mathfrak{M} := \mathfrak{J}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] < \infty$.
In fact, for $\phi \in L(a,b)$,

$$\left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,a^+;w}\phi(x)\right\|_1 \leqslant \mathfrak{M}(b-a)^{\lambda}\|\phi\|_1; \left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,b^-;w}\phi(x)\right\|_1 \leqslant \mathfrak{M}(b-a)^{\lambda}\|\phi\|_1 \tag{2.5}$$

Before discussing the main results, the following assumptions are necessary to discuss:

$$\Omega(f,a_1,a_2;u) \equiv \Omega := \sum_{i=0}^{n-1}\left[\frac{-p}{n^{\beta}}\left\{\left(\mathfrak{F}^{\sigma_1}_{\rho,\beta+1}\left[w\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho}\right]-u\right)f\left(\sqrt[p]{\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right.\right.$$
$$\left.+uf\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)\right\}+\frac{p}{(a_2^p-a_1^p)^{\beta}}\times\left(\mathfrak{J}^{\sigma}_{\rho,\beta,\frac{(n-i)a_1^p+ia_2^p}{n};w}fog\right)$$
$$\left.\left(\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}\right)\right]$$

$$\delta(f,a_1,a_2) \equiv \delta := \sum_{i=0}^{n-1}\left[\frac{p}{n^{\beta}\Gamma(\beta+1)}\left\{(1-u\Gamma(\beta+1))f\left(\sqrt[p]{\frac{(n-i)a_1^p+ia_2^p}{n}}\right)+u\Gamma(\beta+1)\right.\right.$$

$$f\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)\Bigg\}-\frac{p}{(a_2^p-a_1^p)^\beta}\times\left(J^{\beta}_{\frac{(n-i)a_1^p+ia_2^p}{n}+}f\circ g\right)$$

$$\left(\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}\right)\Bigg]$$

$$H(a_1,a_2;\varpi,s+2):={}_2F_1\left(\frac{p-1}{p},\varpi;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$\alpha(k,m,s;x):={}_2F_1\left(0,k+1;m+s,\frac{|\psi(x)-\chi(x)|}{\chi(x)}\right)$$

$$\sigma_1(k):=\sigma(k)\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}\left\{\left(\left(\frac{n-i-1}{n}\right)^s|f'(a_1)|\right.\right.$$

$$\left.+\left(\frac{i+1}{n}\right)^s|f'(a_2)|\right)\mathbb{B}(\beta+\rho k+1,s+1)$$

$${}_2F_1\left(\frac{p-1}{p},\beta+\rho k+1;s+\rho k+\beta+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$+{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+s+1;\beta+\rho k+s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$\left.\left[\left(\frac{n-i}{n}\right)^s|f'(a_1)|+\left(\frac{i}{n}\right)^s|f'(a_2)|\right]\mathbb{B}\left(\beta+\rho k+s+1;1\right)\right\}$$

$$\sigma_2(k):=\sigma(k)\left\{\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{(1-p)}[\mathbb{B}(r(\beta+\rho k)+1,1)]^{\frac{1}{r}}\right.$$

$$\left.\times{}_2F_1\left(\frac{r(p-1)}{p},r(\beta+\rho k)+1;r(\beta+\rho k)+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}^{\frac{1}{r}}$$

$$\sigma_3(k):=\sigma(k)\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}\{\mathbb{B}(\beta,\rho+k+1,1)\times$$

$$\left.{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+1;\beta+\rho k+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}^{\frac{q-1}{q}}$$

$$\left\{\left\{\left(\frac{n-i-1}{n}\right)^s\left|f'(a_1)\right|^q+\left(\frac{i+1}{n}\right)^s\left|f'(a_2)\right|^q\mathbb{B}(\beta+\rho k+1,s+1)\right.\right.$$

$${}_2F_1\left(\frac{p-1}{p},1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)+\left\{\left(\frac{n-i}{n}\right)^s\left|f'(a_1)\right|^q+\right.$$

$$\left.\left(\frac{i}{n}\right)^s\left|f'(a_2)\right|^q\right\}\times{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+s+1;\beta+\rho ks+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$\left.\times\mathbb{B}(\beta+\rho k+s+1,1)\}^{\frac{1}{q}}\right\}$$

3. Main Results

Lemma 3.1. *[23] Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°*, interior of* I*,* $a_1, a_2 \in I^\circ$ *with* $a_1 < a_2$*;* $p, \rho, \beta > 0$ *; let* $g(x) = \sqrt[p]{x}$*,* $x > 0$*;* $u, w \in \mathbb{R}$ *and* $n \in \mathbb{N}$*, then*

$$\Omega := \frac{a_2^p - a_1^p}{n^{1+\beta}} \sum_{i=0}^{n-1} \int_0^1 \Bigg[\left\{ \tau^\beta \mathfrak{F}^\sigma_{\rho,\beta+1} \left[|w| \left(\frac{a_2^p - a_1^p}{n} \right)^\rho \tau^p \right] - u \right\} \times \left[(1-\tau)\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n} + \tau \frac{(n-i)a_1^p + ia_2^p}{n} \right]^{\frac{1-p}{p}} \times f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n} + \tau \frac{(n-i)a_1^p + ia_2^p}{n}} \right) \Bigg] d\tau$$

$$= \sum_{i=0}^{n-1} \Bigg[\frac{-p}{n^\beta} \left\{ \left(\mathfrak{F}^\sigma_{\rho,\beta+1} \left[w \left(\frac{a_2^p - a_1^p}{n} \right)^\rho \right] - u \right) f\left(\sqrt[p]{\frac{(n-i)a_1^p + ia_2^p}{n}} \right) + uf\left(\sqrt[p]{\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n}} \right) \right\} + \frac{p}{(a_2^p - a_1^p)^\beta} \times \left(\mathfrak{J}^\sigma_{\rho,\beta,\frac{(n-i)a_1^p + ia_2^p}{n};w} fog \right) \left(\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n} \right) \Bigg]$$

Remark 3.2.
- Lemma 3.1 is a generalization of [9, Lemma 2.1]. For $2u$, p, β, $\sigma(0) \to 1$; $w \to 0$, Lemma 3.1 reduces to [9, Lemma 2.1].
- Lemma 3.1 is the generalization of [9, Lemma 1.2]. For $n \to 2$; $2u$, p, β, $\sigma(0) \to 1$; $w \to 0$, Lemma 3.1 reduces to [9, Lemma 1.2].

Theorem 3.3. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°*, interior of* I*, such that* $a_1, a_2 \in I^\circ$*,* $\rho, \beta > 0$*,* $u, w \in \mathbb{R}$ *for* $s \in (0, 1]$*; let* $g(x) = \sqrt[p]{x}$ *for* $x > 0$ *and* $|f'|$ *is* $(s, p)-$ *convex, then*

$$|\, \Omega \,| \leqslant \sum_{i=0}^{n-1} \frac{a_2^p - a_1^p}{n^{1+\beta}} \Bigg\{ \mathfrak{F}^{\sigma_1}_{\rho,\beta+1} \left[|w| \left(\frac{a_2^p - a_1^p}{n} \right)^\rho \right] + |u| \left(\sqrt[p]{\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n}} \right)^{1-p} \mathbb{B}(1, s+1) \Bigg\{ \left| f'(a_1) \right| \Bigg(\left(\frac{n-i-1}{n} \right)^s H(a_1, a_2; 1, s+2) + \left(\frac{n-i}{n} \right)^s H(a_1, a_2; s+1, s+2) \Bigg) + \left| f'(a_2) \right| \Bigg(\left(\frac{i+1}{n} \right)^s H(a_1, a_2; 1, s+2) + \left(\frac{i}{n} \right)^s H(a_1, a_2; s+1, s+2) \Bigg) \Bigg\} \Bigg\}$$

Proof. By the properties of modulus and repeated application of $(s, p)-$convexity to the

Lemma 3.1, yield the following:

$$
\begin{aligned}
|\Omega| \leqslant & \sum_{i=0}^{n-1}\left|\frac{a_2^p-a_1^p}{n^{1+\beta}}\right|\left|\int_0^1\left[\left\{\tau^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[|w|\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho}\tau^p\right]+|u|\right\}\right.\right.\\
&\times\left[(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}\right]^{\frac{1-p}{p}}\\
&\times\left.\left.\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|\right]d\tau\right|\\
\leqslant & \sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\int_0^1\left[\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}\tau^{\rho k+\beta}}{\Gamma(\beta+\rho k+1)}+|u|\right\}\right.\\
&\times\left[(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}\right]^{\frac{1-p}{p}}\\
&\times\left.\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|\right]d\tau\\
= & \sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\int_0^1\left[\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}\tau^{\rho k+\beta}}{\Gamma(\beta+\rho k+1)}+|u|\right\}\right.\\
&\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\times\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}\\
&\left.\left\{(1-\tau)^s\left|f'\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)\right|+\tau^s\left|f'\left(\sqrt[p]{\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|\right\}\right]d\tau\\
= & \sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\int_0^1\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}\tau^{\rho k+\beta}}{\Gamma(\beta+\rho k+1)}+|u|\right\}\\
&\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}\{(1-\tau)^s\\
&\left.\left[\left(\frac{n-i-1}{n}\right)^s|f'(a_1)|+\left(\frac{i+1}{n}\right)^s|f'(a_2)|\right]+\tau^s\left[\left(\frac{n-i}{n}\right)^s|f'(a_1)|+\left(\frac{i}{n}\right)^s|f'(a_2)|\right]\right\}d\tau\\
= & \sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\right\}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\\
&\left\{\left(\frac{n-i-1}{n}\right)^s f'(a_1)|+\left(\frac{i+1}{n}\right)^s|f'(a_2)|\int_0^1\tau^{\beta+\rho k}(1-\tau)^s\right.\\
&\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}+\left[\left(\frac{n-i}{n}\right)^s|f'(a_1)|+\left(\frac{i}{n}\right)^s|f'(a_2)|\right]d\tau+
\end{aligned}
$$

$$\int_0^1 \tau^{\beta+\rho k+s}\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}} d\tau\Bigg\}$$

$$+|u|\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\left\{\left(\left(\frac{n-i-1}{n}\right)^s|f'(a_1)|+\left(\frac{i+1}{n}\right)^s|f'(a_2)|\right)\right.$$

$$\int_0^1\left(1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)^{\frac{1-p}{p}}(1-\tau)^s d\tau+\left[\left(\frac{n-i}{n}\right)^s\left|f'(a_1)\right|.+\left(\frac{i}{n}\right)^s\right.$$

$$\left.\left|f'(a_2)\right|\right]\int_0^1\tau^s\left(1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)^{\frac{1-p}{p}} d\tau\Bigg\}$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{p-1}}\left\{\left(\left(\frac{n-i-1}{n}\right)^s\left|f'(a_1)\right|\right.\right.\right.$$

$$\left.+\left(\frac{i+1}{n}\right)^s\left|f'(a_2)\right|\right)\mathbb{B}(\beta+\rho k+1,s+1)$$

$$\times\,{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+1;s+\rho k+\beta+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$+{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+s+1;\beta+\rho k+s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$\times\left[\left(\frac{n-i}{n}\right)^s\left|f'(a_1)\right|+\left(\frac{i}{n}\right)^s\left|f'(a_2)\right|\right]\mathbb{B}\left(\beta+\rho k+s+1;1\right)\Bigg\}$$

$$+|u|\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}\left\{\left[\left(\frac{n-i-1}{n}\right)^s\left|f'(a_1)\right|+\left(\frac{i+1}{n}\right)^s\right.\right.$$

$$\left.\left|f'(a_2)\right|\right]\mathbb{B}\left(1,s+1\right)\times{}_2F_1\left(\frac{p-1}{p},1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)+\left[\left(\frac{n-i}{n}\right)^s\right.$$

$$\left.\left.\left.\left|f'(a_1)\right|+\left(\frac{i}{n}\right)^s\left|f'(a_2)\right|\right]\mathbb{B}\left(1,s+1\right){}_2F_1\left(\frac{p-1}{p},s+1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}\right]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\mathfrak{F}^{\sigma_1}_{\rho,\beta+1}\left[|w|\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho}\right]+\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}|u|\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}$$

$$\left\{\left[\left(\frac{n-i-1}{n}\right)^s\left|f'(a_1)\right|+\left(\frac{i+1}{n}\right)^s\left|f'(a_2)\right|\right]\mathbb{B}\left(1,s+1\right)\right.$$

$$\times{}_2F_1\left(\frac{p-1}{p},1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)+\left[\left(\frac{n-i}{n}\right)^s\left|f'(a_1)\right|\right.$$

$$\left.+\left(\frac{i}{n}\right)^s\left|f'(a_2)\right|\right]\mathbb{B}\left(1,s+1\right){}_2F_1\left(\frac{p-1}{p},s+1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\Bigg\}$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\mathfrak{F}^{\sigma_1}_{\rho,\beta+1}\left[|w|\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho}\right]+\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}|u|\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}$$

$$\mathbb{B}(1,s+1)\left\{\left|f'(a_1)\right|\left(\left(\frac{n-i-1}{n}\right)^s\times{}_2F_1\left(\frac{p-1}{p},1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right.\right.$$

$$+\left(\frac{n-i}{n}\right)^{s} \times {}_2F_1\left(\frac{p-1}{p}, s+1; s+2, \frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\Bigg) + |f'(a_2)| \left(\left(\frac{i+1}{n}\right)^{s}\right.$$
$$\times {}_2F_1\left(\frac{p-1}{p}, 1; s+2, \frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right) + \left(\frac{i}{n}\right)^{s} \times$$
$$\left.{}_2F_1\left(\frac{p-1}{p}, s+1; s+2, \frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right)\Bigg\}$$

This completes the proof. □

Corollary 3.4. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°*, interior of* I*, such that* $a_1, a_2 \in I^\circ$ *with* $a_1 < a_2$*,* $\beta > 0$*,* $u, w \in \mathbb{R}$*,* $s \in (0, 1]$*; let* $|f'|$ *be* $(s, p)-$*convex, then*

$$\left|\frac{f(a_1)+f(a_2)}{2} - \frac{1}{a_2^p-a_1^p}\int_{a_1}^{a_2} f(x)dx\right| \leqslant \frac{a_2^p-a_1^p}{p a_2^{p-1}}\left[\left\{\frac{{}_2F_1\left(\frac{p-1}{p}, 2; s+3, \frac{a_2^p-a_1^p}{a_2^p}\right)|f'(a_2)|}{(s+1)(s+2)}\right.\right.$$
$$\left.+\frac{{}_2F_1\left(\frac{p-1}{p}, s+2; s+3, \frac{a_2^p-a_1^p}{a_2^p}\right)|f'(a_1)|}{(s+2)}\right\} + \frac{1}{2(s+1)}\left\{{}_2F_1\left(\frac{p-1}{p}, 1; s+2, \frac{a_2^p-a_1^p}{a_2^p}\right)\right.$$
$$\left.\left.|f'(a_2)| + {}_2F_1\left(\frac{p-1}{p}, s+1; s+2, \frac{a_2^p-a_1^p}{a_2^p}\right)|f'(a_1)|\right\}\right]$$

Proof. The proof directly follows from Theorem 3.3 for $2u$, β, $\sigma(0)$, $n \to 1$ and $w \to 0$. □

Corollary 3.5. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°*, interior of* I*, such that* $a_1, a_2 \in I^\circ$ *with* $a_1 < a_2$*,* $u \in \mathbb{R}$*,* $\beta, s \in (0, 1]$*; let* $|f'|$ *be* $(s, p)-$*convex, then*

$$|\delta| \leqslant \sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}|u|\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}\left(\left(\frac{n-i-1}{n}\right)^{s}|f'(a_1)|\right.$$
$$\left.+\left(\frac{i+1}{n}\right)^{s}|f'(a_2)|\right)\Bigg[\mathbb{B}(1, s+1)\left\{2 \times {}_2F_1\left(\frac{p-1}{p}, \left[1, s+2; \sqrt[\beta]{u\Gamma(\beta+1)}\right],\right.\right.$$
$$\left.\left.\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right) - {}_2F_1\left(\frac{p-1}{p}, 1; s+2, \frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}$$
$$-\mathbb{B}(\beta+1, s+1)\left\{2 \times {}_2F_1\left(\frac{p-1}{p}, \left[\beta+1, s+\beta+2; \sqrt[\beta]{u\Gamma(\beta+1)}\right],\right.\right.$$
$$\left.\left.\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right) - {}_2F_1\left(\frac{p-1}{p}, \beta+1; s+\beta+2, \frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}\Bigg]$$
$$+\left(\left(\frac{n-i}{n}\right)^{s}|f'(a_1)| + \left(\frac{i}{n}\right)^{s}|f'(a_2)|\right)[\mathbb{B}(s+1, 1)$$

$$\left\{2{}_2F_1\left(\frac{p-1}{p},\left[s+1,s+2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right.$$
$$\left.-{}_2F_1\left(\frac{p-1}{p},s+1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}-\mathbb{B}\left(\beta+s+1,1\right)$$
$$\left\{2\times{}_2F_1\left(\frac{p-1}{p},\left[\beta+s+1,\beta+s+2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right.$$
$$\left.\left.-{}_2F_1\left(\frac{p-1}{p},\beta+s+1;\beta+s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}\right]$$

Remark 3.6. For p, β, $2u$, $s\to 1$, $n\to 1,2$, Corollary 3.5, respectively, reduces to [24, Theorem 2.2] and [26, Corollary 3.4].

Theorem 3.7. *Let $f: I\to\mathbb{R}$ be a differentiable function on I°, interior of I, such that $a_1,a_2\in I^\circ$, ρ, $\beta>0$, $s\in(0,1]$, u, $w\in\mathbb{R}$, $q>1$, $r=\frac{q}{q-1}$; let $g(x)=\sqrt[p]{x}$ for $x>0$ and $|f'|^q$ is $(s,p)-$convex, then*

$$|\Omega|\leqslant\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left\{\mathfrak{F}^{\sigma_2}_{\rho,\beta+1}\left[|w|\left(\frac{a_2^p-a_1^p}{n}\right)^\rho\right]+|u|\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}\right.$$
$$\sqrt[r]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{a_2^p-a_1^p}}\mathbb{B}\left(\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p};1,\frac{r(p-1)+p}{p}\right)$$
$$\left.\sqrt[q]{\frac{(n-i-1)^s+(n-i)^s\left|f(a_1)'\right|^q+(i+1)^s+i^s\left|f(a_2)'\right|^q}{n^s(s+1)}}\right\}$$

Proof. By the properties of modulus, Hölder's inequality and repeated application of $(s,p)-$convexity to the Lemma 3.1, yield the following:

$$|\Omega|=\left|\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\int_0^1\left[\left\{\tau^\beta\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[|w|\left(\frac{a_2^p-a_1^p}{n}\right)^\rho\tau^\rho\right]-u\right\}\times\right.\right.$$
$$\left[(1-\tau)\left(\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}\right)\right]^{\frac{1-p}{p}}\times$$
$$\left.\left.f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right]\right|d\tau$$
$$\leqslant\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\int_0^1\left[\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}\tau^{\rho k+\beta}}{\Gamma(\beta+\rho k+1)}+|u|\right\}\times\right.$$
$$\left[(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}\right]^{\frac{1-p}{p}}\times$$

$$\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|\Bigg] d\tau$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left\{\int_0^1\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.$$

$$\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}$$

$$\left.\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|d\tau\right\}$$

$$+|u|\left\{\int_0^1\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}\times\right.$$

$$\left.\left.\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|d\tau\right\}\right]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left\{\int_0^1\left(\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.\right.$$

$$\left.\left.\times\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}\right)^r d\tau\right\}^{\frac{1}{r}}$$

$$\times\sqrt[q]{\int_0^1\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|^q d\tau}+|u|$$

$$\sqrt[r]{\int\left[\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}\right]^r d\tau}$$

$$\left.\sqrt[q]{\int_0^1\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|^q d\tau}\right]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left\{\int_0^1\left(\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.\right.$$

$$\left.\left.\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{\frac{1-p}{p}}\right)^r d\tau\right\}^{\frac{1}{r}}\left[\int_0^1(1-\tau)^s\left[\left(\frac{n-i-1}{n}\right)^s|f'(a_1)|^q\right.\right.$$

$$+\left(\frac{i+1}{n}\right)^{s}|f'(a_2)|^q\Bigg]+\tau^s\left[\left(\frac{n-i}{n}\right)^{s}|f'(a_1)|^q+\left(\frac{i}{n}\right)^{s}|f'(a_2)|^q\right]d\tau\Bigg]^{\frac{1}{q}}+$$

$$|u|\left(\left[\int_0^1\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\left(1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)^{\frac{1-p}{p}}\right]^r d\tau\right)^{\frac{1}{r}}$$

$$\Bigg[\int_0^1(1-\tau)^s\left[\left(\frac{n-i-1}{n}\right)^{s}|f'(a_1)|^q+\left(\frac{i+1}{n}\right)^{s}\right.$$

$$\left.|f'(a_2)|^q\right]+\tau^s\left[\left(\frac{n-i}{n}\right)^{s}|f'(a_1)|^q+\left(\frac{i}{n}\right)^{s}|f'(a_2)|^q\right]d\tau\Bigg]^{\frac{1}{q}}\Bigg]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\times$$

$$\sqrt[r]{\int_0^1\tau^{r(\rho k+\beta)}\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{r\frac{1-p}{p}}d\tau}$$

$$+|u|\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\sqrt[r]{\int_0^1\left(1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)^{p}d\tau}$$

$$\Bigg[\int_0^1(1-\tau)^s\left[\left(\frac{n-i-1}{n}\right)^{s}|f'(a_1)|^q+\left(\frac{i+1}{n}\right)^{s}|f'(a_2)|^q\right]$$

$$+\tau^s\left[\left(\frac{n-i}{n}\right)^{s}|f'(a_1)|^q+\left(\frac{i}{n}\right)^{s}|f'(a_2)|^q\right]d\tau\Bigg]^{\frac{1}{q}}\Bigg]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\times$$

$$\sqrt[r]{\int_0^1\tau^{r(\rho k+\beta)}\left[1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right]^{r\frac{1-p}{p}}d\tau}+|u|\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}$$

$$\sqrt[r]{\int_0^{\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}(1-u)^{r\frac{1-p}{p}}\frac{(n-i-1)a_1^p+(i+1)a_2^p}{a_2^p-a_1^p}du}\Bigg[\int_0^1(1-\tau)^s\Bigg[\left(\frac{n-i-1}{n}\right)^{s}$$

$$+\left(\frac{i+1}{n}\right)^{s}|f'(a_2)|^q\Bigg]+t^s\left[\left(\frac{n-i}{n}\right)^{s}|f'(a_1)|^q+\left(\frac{i}{n}\right)^{s}|f'(a_2)|^q\right]d\tau\Bigg]^{\frac{1}{q}}\Bigg]|f'(a_1)|^q\Bigg]$$

$$= \sum_{i=0}^{n-1} \frac{a_2^p - a_1^p}{n^{1+\beta}} \left[\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{a_2^p - a_1^p}{}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)} \left(\sqrt[p]{\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n}} \right)^{1-p} \times \right.$$

$$\sqrt[r]{\mathbb{B}(r(\beta+\rho k)+1,1)_2F_1\left(r(\beta+\rho k)+1; r(\beta+\rho k)+2, \frac{a_2^p - a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)}$$

$$+|u| \left[\sqrt[p]{\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n}} \right]^{1-p} \left[\frac{(n-i-1)a_1^p + (i+1)a_2^p}{a_2^p - a_1^p} \times \right.$$

$$\left. \mathbb{B}\left(\frac{a_2^p - a_1^p}{(n-i-1)a_1^p + (i+1)a_2^p}; 1, \frac{r(1-p)}{p} + 1 \right) \right]^{\frac{1}{r}} \times$$

$$\left. \sqrt[q]{\frac{(n-i-1)^s + (n-i)^s \left|f(a_1)'\right|^q + (i+1)^s + i^s \left|f(a_2)'\right|^q}{n^s(s+1)}} \right]$$

$$\leqslant \sum_{i=0}^{n-1} \frac{a_2^p - a_1^p}{n^{1+\beta}} \left[\mathfrak{F}^{\sigma_2}_{\rho,\beta+1} \left[|w| \left(\frac{a_2^p - a_1^p}{n} \right)^{\rho} \right] + |u| \left(\sqrt[p]{\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n}} \right)^{1-p} \right.$$

$$\sqrt[r]{\frac{(n-i-1)a_1^p + (i+1)a_2^p}{a_2^p - a_1^p} \mathbb{B}\left(\frac{a_2^p - a_1^p}{(n-i-1)a_1^p + (i+1)a_2^p}; 1, \frac{r(1-p)+p}{p} \right)}$$

$$\left. \sqrt[q]{\frac{(n-i-1)^s + (n-i)^s \left|f(a_1)'\right|^q + (i+1)^s + i^s \left|f(a_2)'\right|^q}{n^s(s+1)}} \right]$$

This completes the proof. □

Corollary 3.8. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°*, interior of* I*, such that* $a_1, a_2 \in I^\circ$ *with* $a_1 < a_2$, $p > 0$, $u, w > 0$, $s \in (0,1]$ *and* $q = \frac{r}{r-1} > 1$*; let* $|f'|^q$ *is* $(s,p)-$ *convex, then*

$$\left| \frac{f(a_1)+f(a_2)}{2} - \frac{1}{a_2^p - a_1^p} \int_{a_1}^{a_2} f(x)dx \right| \leqslant \frac{a_2^p - a_1^p}{p a_2^{p-1}} \left[\sqrt[r]{\mathbb{B}(r+1,1)_2F_1\left(r+1; r+2, \frac{a_2^p - a_1^p}{a_2^p}\right)} \right.$$

$$\left. +\frac{1}{2} \sqrt[r]{\frac{a_2^p}{a_2^p - a_1^p} \mathbb{B}\left(\frac{a_2^p - a_1^p}{a_2^p}; 1, \frac{r(1-p)+p}{p} \right)} \times \frac{\left\{|f'(a_1)|^q + |f'(a_2)|^q\right\}^{\frac{1}{q}}}{s+1} \right]$$

Proof. The proof directly follows from Theorem 3.7 for $2u$, β, $\sigma(0)$, $n \to 1$ and $w \to 0$. □

Theorem 3.9. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°*, interior of* I, $a_1, a_2 \in I^\circ$ ρ, β, $p > 0$, $q \geqslant 1$, $u, w \in \mathbb{R}$, $s \in (0,1]$*; let* $g(x) = \sqrt[p]{x}$ *for* $x > 0$ *and* $|f'|$ *is* $(s,p)-$*convex, then*

$$|\Omega(f, a_1, a_2; u)| \leqslant \sum_{i=0}^{n-1} \frac{a_2^p - a_1^p}{n^{1+\beta}} \left[\mathfrak{F}^{\sigma_3}_{\rho,\beta+1} \left[|w| \left(\frac{a_2^p - a_1^p}{n} \right)^{\rho} \right] + |u| \left(\sqrt[p]{\frac{(n-i-1)a_1^p + (i+1)a_2^p}{n}} \right)^{1-p} \right.$$

$$\left(\sqrt[q]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{a_2^p-a_1^p}\mathbb{B}\left(\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p};1,\frac{1}{p}\right)}\right)^{q-1}\left\{\left\{\left(\left(\frac{n-i-1}{n}\right)^s\right.\right.\right.$$
$$\left.\left.|f'(a_1)|^q+\left(\frac{i+1}{n}\right)^s|f'(a_2)|^q\right)\right\}\mathbb{B}(1,s+1)H(a_1,a_2;1,s+2)+\left[\left(\left(\frac{n-i}{n}\right)^s|f'(a_1)|^q+\right.\right.$$
$$\left.\left.\left.\left(\frac{i}{n}\right)^s|f'(a_2)|^q\right)\right]\mathbb{B}(s+1,1)H(a_1,a_2;s+1,s+2)\right\}^{\frac{1}{q}}\Bigg]$$

Proof. By the properties of modulus, power mean inequality and repeated applications of $(s,p)-$convexity to the Lemma 3.1, yield the following:

$$|\Omega|=\left|\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\int_0^1\left[\left\{\tau^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{a_2^p-a_1^p}{n}\right)^\rho\right]-u\right\}\right.\right.$$
$$\left[(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}\right]^{\frac{1-p}{p}}$$
$$\left.\left.f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right]d\tau\right|$$
$$\leqslant\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left\{\int_0^1\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.$$
$$\times\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}$$
$$\left.\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|d\tau\right\}$$
$$+|u|\left\{\int_0^1\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}\times\right.$$
$$\left.\left.\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|d\tau\right\}\right]$$
$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left\{\int_0^1\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.$$
$$\left.\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}d\tau\right\}^{\frac{q-1}{q}}\left[\int_0^1\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.$$

$$\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}$$

$$\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|^q d\tau\Bigg]^{\frac{1}{q}}$$

$$+|u|\left[\left(\left(\left[\int_0^1\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}d\tau\right)^{\frac{q-1}{q}}\right.\right.$$

$$\left(\int_0^1\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}\right.$$

$$\left.\left.\left.\left|f'\left(\sqrt[p]{(1-\tau)\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}+\tau\frac{(n-i)a_1^p+ia_2^p}{n}}\right)\right|^q d\tau\right)^{\frac{1}{q}}\right]\right]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left\{\int_0^1\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.$$

$$\left.\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}d\tau\right\}^{\frac{q-1}{q}}\left[\int_0^1\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.$$

$$\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}\left\{(1-\tau)^s\left[\left(\frac{n-i-1}{n}\right)^s|f'(a_1)|^q+\left(\frac{i+1}{n}\right)^s|f'(a_2)|^q\right]\right.$$

$$\left.\left.+\tau^s\left[\left(\frac{n-i}{n}\right)^s|f'(a_1)|^q+\left(\frac{i}{n}\right)^s|f'(a_2)|^q\right]\right\}d\tau\right]^{\frac{1}{q}}+|u|\left[\left(\left(\left[\int_0^1\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.\right.$$

$$\left.\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}d\tau\right)^{\frac{q-1}{q}}\left[\int_0^1\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.$$

$$\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}\left\{(1-\tau)^s\left[\left(\frac{n-i-1}{n}\right)^s|f'(a_1)|^q+\left(\frac{i+1}{n}\right)^s|f'(a_2)|^q\right]\right.$$

$$\left.\left.\left.+t^s\left[\left(\frac{n-i}{n}\right)^s|f'(a_1)|^q+\left(\frac{i}{n}\right)^s|f'(a_2)|^q\right]\right\}d\tau\right]^{\frac{1}{q}}\right]\right]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)}\left\{\int_0^1\tau^{\rho k+\beta}\left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.$$

$$\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} d\tau\Bigg\}^{\frac{q-1}{q}} \left[\left(\frac{n-i-1}{n}\right)^s |f'(a_1)|^q + \left(\frac{i+1}{n}\right)^s |f'(a_2)|^q\right.$$

$$\int_0^1 \tau^{\rho k+\beta}(1-\tau)^s \left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p} \sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} d\tau$$

$$+\left[\left(\frac{n-i}{n}\right)^s |f'(a_1)|^q + \left(\frac{i}{n}\right)^s |f'(a_2)|^q\right] \int_0^1 \tau^{\rho k+\beta} t^s \left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}$$

$$\left.\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} d\tau\right]^{\frac{1}{q}} + |u| \left[\left(\left[\int_0^1 \sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}\right.\right.$$

$$\left.\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} d\tau\right)^{\frac{q-1}{q}} \left[\left[\left(\frac{n-i-1}{n}\right)^s |f'(a_1)|^q + \left(\frac{i+1}{n}\right)^s\right.\right.$$

$$\left.|f'(a_2)|^q\right] \int_0^1 (1-\tau)^s \left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p} \sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}$$

$$d\tau + \left[\left(\frac{n-i}{n}\right)^s |f'(a_1)|^q + \left(\frac{i}{n}\right)^s |f'(a_2)|^q\right] \int_0^1 \tau^s \left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p}$$

$$\left.\left.\left.\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} dt\right]^{\frac{1}{q}}\right]\right]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}} \left[\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{a_2^p-a_1^p}{n}\right)^{\rho k}}{\Gamma(\beta+\rho k+1)} \left[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right]^{1-p} \left\{\int_0^1 \tau^{\rho k+\beta}\times\right.\right.$$

$$\left.\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} d\tau\right\}^{\frac{q-1}{q}} \left[\left[\left(\frac{n-i-1}{n}\right)^s |f'(a_1)|^q + \left(\frac{i+1}{n}\right)^s |f'(a_2)|^q\right]\right.$$

$$\int_0^1 \tau^{\rho k+\beta}(1-\tau)^s \sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} d\tau + \left[\left(\frac{n-i}{n}\right)^s |f'(a_1)|^q + \left(\frac{i}{n}\right)^s\right.$$

$$\left.\left.|f'(a_2)|^q\right] \int_0^1 \tau^{\rho k+\beta+s} \sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p} d\tau\right]^{\frac{1}{q}} + |u|$$

$$\Bigg[\Bigg(\Bigg(\Bigg[\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\Bigg]^{1-p}\int_0^1\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}d\tau\Bigg)^{\frac{q-1}{q}}$$

$$\Bigg[\Bigg[\left(\frac{n-i-1}{n}\right)^s\left|f'(a_1)\right|^q+\left(\frac{i+1}{n}\right)^s\left|f'(a_2)\right|^q\Bigg]\int_0^1(1-\tau)^s$$

$$\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}d\tau+\Bigg[\left(\frac{n-i}{n}\right)^s\left|f'(a_1)\right|^q+\left(\frac{i}{n}\right)^s\left|f'(a_2)\right|^q\Bigg]$$

$$\int_0^1\tau^s\sqrt[p]{1-\tau\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}}^{\,1-p}dt\Bigg]^{\frac{1}{q}}\Bigg]\Bigg]$$

$$=\sum_{i=0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}\Bigg\{\mathfrak{F}^{\sigma_3}_{\rho,\beta+1}\Bigg[|w|\left(\frac{a_2^p-a_1^p}{n}\right)^\rho\Bigg]+|u|\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}$$

$$\times\left(\sqrt[q]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{a_2^p-a_1^p}}\mathbb{B}\left(\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p};1,\frac{1}{p}\right)\right)^{q-1}\Bigg\{\Bigg\{\left(\frac{n-i-1}{n}\right)^s$$

$$\left|f'(a_1)\right|^q+\left(\frac{i+1}{n}\right)^s\left|f'(a_2)\right|^q\Bigg\}\mathbb{B}(1,s+1){}_2F_1\left(\frac{p-1}{p},1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$+\Bigg\{\left(\frac{n-i}{n}\right)^s\left|f'(a_1)\right|^q+\left(\frac{i}{n}\right)^s\left|f'(a_2)\right|^q\Bigg\}\mathbb{B}(s+1,1)$$

$$\times{}_2F_1\left(\frac{p-1}{p},s+1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\Bigg\}^{\frac{1}{q}}\Bigg\}$$

This completes the proof. □

Corollary 3.10. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°, *interior of* I, *such that* $a_1, a_2 \in I^\circ$ *with* $a_1 < a_2$, $p > 0$, $u, w > 0$, $s \in (0,1]$ *and* $q = \frac{r}{r-1} > 1$; *let* $|f'|^q$ *is* $(s,p)-$*convex, then*

$$\left|\frac{f(a_1)+f(a_2)}{2}-\frac{1}{a_2^p-a_1^p}\int_{a_1}^{a_2}f(x)dx\right|\leqslant\frac{a_2^p-a_1^p}{pa_2^{p-1}}\left\{\left(\sqrt[q]{\frac{{}_2F_1\left(\frac{p-1}{p},2;3,\frac{a_2^p-a_1^p}{a_2^{p-1}}\right)}{2}}\right)^{q-1}\right.$$

$$\sqrt[q]{\frac{{}_2F_1\left(\frac{p-1}{p},2;s+3,\frac{a_2^p-a_1^p}{a_2^{p-1}}\right)|f'(a_2)|^q}{(s+1)(s+2)}+\frac{{}_2F_1\left(\frac{p-1}{p},2;s+3,\frac{a_2^p-a_1^p}{a_2^{p-1}}\right)|f'(a_1)|^q}{s+2}}$$

$$+\frac{1}{2}\left(\sqrt[q]{\frac{a_2^p}{a_2^p-a_1^p}\mathbb{B}\left(\frac{a_2^p}{a_2^p-a_1^p};1,\frac{1}{p}\right)}\right)^{q-1}\left(\frac{1}{\sqrt[q]{s+1}}\right)\left\{{}_2F_1\left(\frac{p-1}{p},1;s+2,\frac{a_2^p-a_1^p}{a_2^{p-1}}\right)\right.$$
$$\left.\left.|f'(a_2)|^q+{}_2F_1\left(\frac{p-1}{p},s+1;s+2,\frac{a_2^p-a_1^p}{a_2^{p-1}}\right)|f'(a_1)|^q\right\}^{\frac{q-1}{q}}\right\}$$

Proof. The proof directly follows from Theorem 3.9 for $2u$, β, $\sigma(0)$, $n\to 1$ and $w\to 0$. □

Corollary 3.11. *Let $f: I\to\mathbb{R}$ be a differentiable function on I°, interior of I, such that $a_1,a_2\in I^\circ$ with $a_1<a_2$, $p>0$, $u>0$, β, $s\in(0,1]$ and $q=\frac{r}{r-1}>1$; let $|f'|^q$ is $(s,p)-$convex, then*

$$|\delta|\leqslant\sum_{i-0}^{n-1}\frac{a_2^p-a_1^p}{n^{1+\beta}}|u|\left(\sqrt[p]{\frac{(n-i-1)a_1^p+(i+1)a_2^p}{n}}\right)^{1-p}[\mathbb{B}(1,1)$$
$$\left[2\times_2F_1\left(\frac{p-1}{p},\left[1,2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)-_2F_1\left(\frac{p-1}{p},1;2,\right.\right.$$
$$\left.\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)-\frac{\mathbb{B}(\beta+1,1)}{u\Gamma(\beta+1)}\left[2\times_2F_1\left(\frac{p-1}{p},\left[\beta+1,\beta+2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\right.\right.$$
$$\left.\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)-_2F_1\left(\frac{p-1}{p},\beta+1;\beta+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)^{\frac{q-1}{q}}$$
$$\left[\left(\left(\frac{n-i-1}{n}\right)^s|f'(a_1)|^q+\left(\frac{i+1}{n}\right)^s|f'(a_2)|^q\right)\right.$$
$$\left[\mathbb{B}(1,s+1)\left\{2\times_2F_1\left(\frac{p-1}{p},\left[1,s+2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right.\right.$$
$$\left.-_2F_1\left(\frac{p-1}{p},1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}-\mathbb{B}\left(\beta+1,s+1\right)$$
$$\left\{2\times_2F_1\left(\frac{p-1}{p},\left[\beta+1,s+\beta+2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right.$$
$$\left.\left.\left.-_2F_1\left(\frac{p-1}{p},\beta+1;s+\beta+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}\right]+\left(\left(\frac{n-i}{n}\right)^s|f'(a_1)|^q\right.$$
$$\left.+\left(\frac{i}{n}\right)^s|f'(a_2)|^q\right)\left[\mathbb{B}\left(s+1,1\right)\left\{2\times_2F_1\left(\frac{p-1}{p},\left[s+1,s+2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\right.\right.\right.$$
$$\left.\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)-_2F_1\left(\frac{p-1}{p},s+1;s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)$$

$$-\mathbb{B}\left(\beta+s+1,1\right)\left\{2\times_2 F_1\left(\frac{p-1}{p},\left[\beta+s+1,\beta+s+2;\sqrt[\beta]{u\Gamma(\beta+1)}\right],\right.\right.$$
$$\left.\left.\left.\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)-_2F_1\left(\frac{p-1}{p},\beta+s+1;\beta+s+2,\frac{a_2^p-a_1^p}{(n-i-1)a_1^p+(i+1)a_2^p}\right)\right\}\right]^{\frac{1}{q}}$$

Remark 3.12. For p, β, $2u$, $s\to1$, Corollary 3.11 reduces to [9, Theorem 2.1] and hence for $n\to1,2$ and $q\to1$, it reduces to, respectively, [28, Theorem 2], [24, Theorem 2.2] and [26, Corollary 3.4]. Moreover for $n\to3$, it reduces to [9, Corollary 2.1].

4. Applications

In this section, we provide some applications on $f-$divergence measure and probability density function.

4.1. $f-$divergence measures:

let ϕ be a set and ν a σ finite measure. Consider

$$\Omega=\left\{\chi|\chi:\phi\to\mathbb{R},\chi(x)>0,\int_\phi\chi(x)d\nu(x)=1\right\},\tag{4.1}$$

set of all probability densities on ν and $f:I\to\mathbb{R}$ a real$-$valued function. Then the Csisźar $f-$divergence , $D_f(\chi,\psi)$, is defined by :

$$D_f(\chi,\psi)=\int_\phi\chi(x)f\left[\frac{\chi(x)}{\psi(x)}\right]d\nu(x)\quad(\chi,\psi\in\Omega)\tag{4.2}$$

For f to be convex, the Hermite$-$Hadamard (HH) divergence denoted by $D^f_{HH}(\chi,\psi)$ is defined as:

$$D^f_{HH}(\chi,\psi)=\int_\phi\chi(x)\frac{\int_\phi^{\frac{\psi(x)}{\chi(x)}}f(\tau)d\tau}{\frac{\psi(x)}{\chi(x)}-1}d\nu(x)\quad(\chi,\psi\in\Omega)\tag{4.3}$$

with $f(1)=0$. Note that $D^f_{HH}(\chi,\psi)\geqslant0$ and $D^f_{HH}(\chi,\psi)=0$ if and only if $\chi=\psi$.

Proposition 1. *Let $f:I\to\mathbb{R}$ be a differentiable function on I°, interior of I, such that $a_1,a_2\in I^\circ$ with $a_1<a_2$, $s\in(0,1]$ and $|f'|$ is $s-$convex and $f(1)=0$, then*

$$\left|\frac{D_f(\chi,\psi)}{2}-D^f_{HH}(\chi,\psi)\right|\leqslant\frac{1}{s+1}\left|f^{'}(1)\right|\int_\phi|\psi(x)-\chi(x)|\left[\frac{\alpha\left(1,3,s;x\right)}{s+2}+\frac{\alpha\left(0,2,s;x\right)}{2}\right]d\nu(x)$$
$$+\left|f'\left(\frac{\psi(x)}{\chi(x)}\right)\right|\int_\phi|\psi(x)-\chi(x)|\left[\frac{\alpha\left(s+1,3,s;x\right)}{s+2}+\frac{\alpha\left(s,2,s;x\right)}{2(s+1)}\right]d\nu(x)$$

4.2. Probability density functions:

Let $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$ and $g : [a_1, a_2] \to [0, 1]$ be the probability density function of a continuous random variable X with cummulative distribution function F given by

$$F(x) = p(X \leqslant x) = \int_{a_1}^{x} g(\tau)d\tau; \quad E(X) = \int_{a_1}^{a_2} \tau dF(\tau) = a_2 - \int_{a_1}^{a_2} F(\tau)d\tau. \tag{4.4}$$

Then from Corollary 3.4 , for $p \to 1$, we see that:

$$\left|\frac{1}{2} - \frac{a_2 - E(X)}{a_2 - a_1}\right| \leqslant (a_2 - a_1)\left[\left[\left\{\frac{|g(a_2)| \times {}_2F_1\left(0, 2; s+3, \frac{a_2-a_1}{a_1}\right)}{(s+1)(s+2)} + {}_2F_1\left(0, s+2; s+3, \frac{a_2 - a_1}{a_1}\right)\right.\right.\right.$$
$$\left.\frac{|g(a_1)|}{s+2}\right\} + \frac{1}{2(s+1)}\left\{|g(a_2)| \times {}_2F_1\left(0, 1; s+2, \frac{a_2 - a_1}{a_1}\right)\right.$$
$$\left.\left. + |g(a_1)| \times {}_2F_1\left(0, s+1; s+2, \frac{a_2 - a_1}{a_1}\right)\right\}\right]$$
$$\left|\frac{1}{2} - \frac{a_2 - E(X)}{a_2 - a_1}\right| \leqslant (a_2 - a_1)\left[|g(a_1)|\left\{\frac{{}_2F_1\left(0, s+2; s+3, \frac{a_2-a_1}{a_1}\right)}{s+2} + \frac{{}_2F_1\left(0, s+1; s+2, \frac{a_2-a_1}{a_1}\right)}{2(s+1)}\right\}\right.$$
$$\left. + |g(a_2)|\left\{\frac{{}_2F_1\left(0, 2; s+3, \frac{a_2-a_1}{a_1}\right)}{(s+1)(s+2)} + \frac{{}_2F_1\left(0, 1; s+2, \frac{a_2-a_1}{a_1}\right)}{2(s+1)}\right\}\right]$$

5. Conclusion

• A new identity regarding Bullen type has been established.
• New weighted Bullen type inequalities for (s, p) convex functions using above identity were deduced.
• Some applications in probability and information theory for (s, p) convex have been given.
• We hope that our results can be applied to obtain several new results in different areas of pure and applied sciences.

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