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Positive solutions of a class of nonlinear boundary value fractional differential equations

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Abstract

In this paper, the existence of positive solutions of a class of nonlinear fractional boundary value problems is considered. Two fixed point theorems are used, namely: Banach Contraction mapping principle and Leggett-Williams fixed point theorems. The former is used to prove the existence of a unique solution, whereas the latter is used to prove the existence of at least three positive solutions to the problem. Some examples are provided to illustrate the two results.

Keywords: Fractional calculus, Caputo fractional derivative, Existence of positive solutions, Fixed point theory.
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1. Introduction

In recent years, fractional calculus has evolved as an interesting and an important area of research. This is as a result of its numerous applications in models of several phenomena in various fields of science and engineering. Indeed, a number of applications in areas such as biology, engineering, earthquake prediction, signal processing, dynamical systems and etc., abound in the literature. For more on the theories and applications of fractional calculus, see [1, 2, 3, 4, 19] and the references therein.

Owing to the fact, that positive solutions are useful in several applications, various papers dealing on the existence of positive solutions of differential equations have appeared in the literature. Most often, such results are obtained using some fixed point theorems as can be seen in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and some references therein.

In [16], the existence and uniqueness of positive solutions for the fractional differential equation,

$$\begin{cases} D_c^{\alpha}x(t) - D_c^{\beta}x(t) = f(t, x(t)), \ t \in [0, T), \ 0 < \beta < \alpha < 1 \\ x(0) = x_0, \end{cases},$$

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were studied using some new integral inequalities of Henry- Gronwall type. More recent study on existence of solutions and positive solutions of fractional differential equations can be seen in [17, 18, 19, 20].
In the case of this research work, the existence of positive solutions for the nonlinear boundary value fractional differential equation shall be considered,

$$ {}^{c}D_{0_+}^{\alpha}x(t) + k\,{}^{c}D_{0_+}^{\beta}x(t) + g(t, x(t)) = h(t),\ t \in [0,1], \tag{1.1} $$

$$ x(0) = 0, \tag{1.2} $$

$$ x'(0) = \frac{\alpha}{2}x(1), \tag{1.3} $$

where $0 < \beta < 1 < \alpha < 2$, are real constants, $x \in C^2[0,1]$ and ${}^{c}D_{0_+}^{\gamma}x(t)$ is the Caputo fractional derivative of a function x of order γ, with $\gamma = \alpha, \beta$; k is a negative constant, $g : [0,1] \times [0,\infty) \longrightarrow \mathbb{R}$ is an $L_\infty-$ Caratheódory function, $h \in L^{\frac{1}{\beta}}[0,1]$.

The rest of the paper is organized as follows: In section 2, some basic terms of fractional calculus and some useful lemmas related to the work will be presented. In section three, by using the Contraction Mapping Principle, an existence of a unique positive solution to the problem is established. Equally, existence of at least three positive solutions of equations (1)-(3) will be proved using the Leggett-Williams fixed point theorem. Lastly, in section 4, some examples are to be presented to illustrate the obtained results.

2. Preliminaries

For $n \in \mathbb{N}$, let

$$ C^n[a,b] = \left\{ f : [a,b] \longrightarrow \mathbb{C},\ \frac{d^n f(x)}{dx^n} \in C^0[a,b] \right\}, \tag{2.1} $$

where $C^0[a,b]$ is the space of continuous functions on $[a,b]$.

Also, let,

$$ E = \left\{ x \in C^2[0,1] : x(t) \geqslant 0,\ t \in [0,1] \right\}, \tag{2.2} $$

be a cone in a Banach space $C^2[0,1]$ endowed with the norm

$$ \|x\|_{C^2} = \|x\|_\infty + \|x'\|_\infty + \|x''\|_\infty. $$

Finally, let the nonnegative continuous concave functional θ on E be defined by

$$ \theta(x) = \min_{0 \leqslant t \leqslant 1} x(t). $$

Definition 2.1[1] Let x be n-th times continuously differentiable function. The Caputo fractional derivative of a function x, of order α, with lower limit 0 is defined as,

$$ {}^{c}D_{0_+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds, \tag{2.3} $$

with $n-1<\alpha<n,\ n=[\alpha]+1$; while the Riemann-Liouville fractional integral of a function x, of order $\alpha>0$, and denoted by $I^{\alpha}_{0_+}x(t)$ is defined by,

$$I^{\alpha}_{0_+}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)ds. \tag{2.4}$$

Observe from equations (6) and (7) that,

$$^{c}D^{\alpha}_{0_+}x(t)=I^{n-\alpha}_{0_+}x^{(n)}(t). \tag{2.5}$$

Definition 2.2.[1] A map γ is said to be a nonnegative continuous concave functional on a cone E of real Banach space S, if $\gamma:E\longrightarrow[0,\infty)$ is continuous and

$$\gamma(tx+(1-t)y)\geqslant t\gamma(x)+(1-t)\gamma(y),$$

for all $x,y\in E$ and $0\leqslant t\leqslant 1$.
Similarly, we say that the map ψ is a nonnegative continuous convex functional on a cone E of a real Banach space S, if $\psi:E\longrightarrow[0,\infty)$ is continuous and

$$\psi(tx+(1-t)y)\leqslant t\psi(x)+(1-t)\psi(y),$$

for all $x,y\in E$ and $0\leqslant t\leqslant 1$.

Definition 2.3. [21] A function $f:[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is said to be a Carathéodory function if it satisfies the following conditions

- $f(t,x)$ is Lebesgue measurable with respect to t in $[a,b]$,
- $f(t,x)$ is continuous with respect to x on $\mathbb{R}$

A function $f(t,x)$ defined on $[a,b]\times\mathbb{R}$ is said to be an L^p- Carathéodory function, $p\geqslant 1$, if it is a Carathéodory function and $\forall r>0$, there exists $h_r\in L^p(a,b)$, such that $\forall x\in[-r,r]$ and $\forall t\in[a,b]$, then $f(t,x)\leqslant h_r(t)$.

Lemma 2.1. *[22] The space $AC^n[a,b]$ consists of those and only functions f, which can be represented in the form*

$$x(t)=I^{n}_{a_+}\varphi(t)+\sum_{k=0}^{n-1}c_k(t-a)^k, \tag{2.6}$$

where $\varphi\in L_1(a,b),\ c_k(k=0,1,2,\cdots,n-1)$ are arbitrary constants.

Lemma 2.2. *[21] If $x\in AC^n[a,b]$ or $x\in C^n[a,b]$, then the equality*

$$I^{\alpha}_{a_+}(^{c}D^{\alpha}_{a_+}x(t))=x(t)-\sum_{j=0}^{n-1}\frac{x^j(a)}{j!}(t-a)^j. \tag{2.7}$$

holds.

A particular case, where $0 < \beta < 1 < \alpha < 2$, then equation (10) becomes

$$I^{\alpha}_{a_+} {}^{c}D^{\beta}_{a_+} x(t) = I^{\alpha-\beta}_{a_+} x(t) - \frac{(t-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} x(a). \tag{2.8}$$

Lemma 2.3. *If $x \in C^2[0,1]$, then x satisfies the equations (1) - (3) if, and only if x satisfies the Volterra integral equation*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 G_1(t,s)[h(s) - g(s,x(s))]ds - \frac{k}{\Gamma(\alpha-\beta)} \int_0^1 G_2(t,s)x(s)ds,$$

where

$$G_1(t,s) = \begin{cases} \frac{\alpha t(1-s)^{\alpha-1}}{2-\alpha} + (t-s)^{\alpha-1} & \text{if, } 0 \leqslant s \leqslant t \\ \frac{\alpha t(1-s)^{\alpha-1}}{2-\alpha} & \text{if, } t \leqslant s \leqslant 1 \end{cases},$$

$$G_2(t,s) = \begin{cases} \frac{\alpha t(1-s)^{\alpha-\beta-1}}{2-\alpha} + (t-s)^{\alpha-\beta-1} & \text{if, } 0 \leqslant s \leqslant t \\ \frac{\alpha t(1-s)^{\alpha-\beta-1}}{2-\alpha} & \text{if, } t \leqslant s \leqslant 1 \end{cases}.$$

Proof. Suppose that $x \in C^2[a,b]$ satisfies equations (1) - (3), then we show that x satisfies the Volterra integral equation above. By lemma 2.2, we can apply $I^{\alpha}_{a_+}$ to both sides of (1), to obtain

$$I^{\alpha}_{0_+}\left({}^{c}D^{\alpha}_{0_+} x(t) + k\,{}^{c}D^{\beta}_{0_+} x(t) + g(t,x(t)) = h(t)\right)$$

$$\implies I^{\alpha}_{0_+}({}^{c}D^{\alpha}_{0_+} x(t)) + I^{\alpha}_{0_+}(k\,{}^{c}D^{\beta}_{0_+} x(t)) = I^{\alpha}_{0_+}[h(t) - g(t,x(t))]$$

$$\implies x(t) - x(0) - tx'(0) + kI^{\alpha-\beta}_{0_+} x'(t) = I^{\alpha}_{0_+}[h(t) - g(t,x(t))]$$

$$\implies x(t) = tx'(0) + \frac{kx(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - kI^{\alpha-\beta}_{0_+} x(t) + I^{\alpha}_{0_+}[h(t) - g(t,x(t))] \tag{2.9}$$

Making use of the boundary condition (2) in (3), we have

$$\begin{aligned} x(1) &= x'(0) - \frac{k}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} x(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}[h(s) - g(s,x(s))]ds \\ &\implies \left(\frac{2}{\alpha} - 1\right) x'(0) = -\frac{k}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}[h(s) - g(s,x(s))]ds \\ &\implies x'(0) = \frac{\alpha}{2-\alpha}\left(\frac{-k}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} x(s)ds\right) \\ &+ \frac{\alpha}{2-\alpha}\left(\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}[h(s) - g(s,x(s))]ds\right) \end{aligned}$$

Substituting the above into (3), we have

$$\begin{aligned} x(t) &= \frac{\alpha t}{2-\alpha}\left\{-\frac{k}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[h(s)-g(s,x(s))]ds\right\} \\ & -\frac{k}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[h(s)-g(s,x(s))]ds \\ & \Longrightarrow x(t) = \frac{-k}{\Gamma(\alpha-\beta)}\int_0^t [\frac{(\alpha t(1-s)^{\alpha-\beta-1}}{2-\alpha} + (t-s)^{\alpha-\beta-1}]x(s)ds \\ & + \frac{-k}{\Gamma(\alpha-\beta)}\int_t^1 \frac{\alpha t(1-s)^{\alpha-\beta-1}}{2-\alpha}x(s)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t [\frac{\alpha t(1-s)^{\alpha-1}}{2-\alpha} + (t-s)^{\alpha-1}](h(s)-g(s,x(s)))ds \\ & + \frac{1}{\Gamma(\alpha)}\int_t^1 \frac{\alpha t(1-s)^{\alpha-1}}{2-\alpha}(h(s)-g(s,x(s)))ds \end{aligned}$$

Therefore,

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 G_1(t,s)[h(s)-g(s,x(s))]ds - \frac{k}{\Gamma(\alpha-\beta)}\int_0^1 G_2(t,s)x(s)ds,$$

Thus, the volterra integral equation is satisfied.
On the other hand, suppose that x satisfies the volterra integral equation, then operating ${}^cD_{0+}^{\alpha}$ to both sides of the volterra integral equation, we obtain (1). Equally, it can easily be seen that the boundary conditions (2) and (3) are satisfied. Hence, the lemma is proved. □

Lemma 2.4. $\forall t,s \in [0,1]$, *and* $\alpha-\beta-1>0$, *the Green's function defined in lemma 2.3 satisfies the following :*

(i) $G_i(t,s) \geqslant 0, i=1,2.$

(ii) $G_i(t,s) \leqslant G_i(1,s), i=1,2.$

(iii) $\frac{t(1-s)^{\alpha-\beta-1}}{3} \leqslant G_i(t,s) \leqslant \frac{3t(1-s)^{\alpha-1}}{2-\alpha},\ i=1,2.$

Proof. (i) From the definition of $G_i, i=1,2$, and $\forall t[0,1]$, we have that

$$G_i(t,s) \geqslant \frac{\alpha t(1-s)^{\alpha-\beta-1}}{2-\alpha} \geqslant 0$$

(ii) For $s \in [0,t],\ t \in [0,1]$,

$$G_1(t,s) = \frac{\alpha t(1-s)^{\alpha-1}}{2-\alpha} + (t-s)^{\alpha-1}$$

$$\leqslant \frac{\alpha(1-s)^{\alpha-1}}{2-\alpha} + (1-s)^{\alpha-1} = G_1(1,s)$$

Also, for $s \in [t,1],\ t \in [0,1]$,

$$G_1(t,s) = \frac{\alpha t(1-s)^{\alpha-1}}{2-\alpha} \leqslant \frac{\alpha(1-s)^{\alpha-1}}{2-\alpha} = G_1(1,s)$$

Similarly, we can show that $G_2(t,s) \leqslant G_2(1,s), \forall t,s \in [0,1]$.

Therefore, $G_i(t,s) \leqslant G_i(1,s), i = 1,2$.

(iii)

$$G_i(t,s) \geqslant \frac{\alpha t(1-s)^{\alpha-\beta-1}}{2-\alpha} \geqslant \frac{t(1-s)^{\alpha-\beta-1}}{2-\alpha} \geqslant \frac{t(1-s)^{\alpha-\beta-1}}{3(2-\alpha)} \geqslant \frac{t(1-s)^{\alpha-\beta-1}}{3}$$

Equally,

$$G_i(t,s) \leqslant \frac{\alpha t(1-s)^{\alpha-1}}{2-\alpha} + (t-s)^{\alpha-1} \leqslant \frac{2(1-s)^{\alpha-1}}{2-\alpha} + (1-s)^{\alpha-1} \leqslant \frac{2(1-s)^{\alpha-1}}{2-\alpha} + \frac{(1-s)^{\alpha-1}}{2-\alpha} = \frac{3(1-s)^{\alpha-1}}{2-\alpha}$$

□

Lemma 2.5. [18] *Let E be a cone in a real Banach space* $C^2[0,1]$ $E_e = \{x \in E : \|x\|_{C^2} \leqslant e\}$, θ *is a nonnegative continuous concave functional on* E *such that* $\theta(x) \leqslant \|x\|_{C^2}$, $\forall x \in \bar{E}_e$, *and* $E(\theta, d, f) = \{x \in E : d \leqslant \theta(x),\ \|x\|_{C^2} \leqslant f\}$. *Suppose that* $T : \bar{E}_e \longrightarrow \bar{E}_e$ *is completely continuous and there exist positive constants* $0 < c < d < e \leqslant f$ *such that*

(i) $\{x \in E(\theta, d, f) : \theta(x) > d\} \neq \emptyset$ *and* $\theta(T(x)) > d\ \forall x \in E(\theta, d, f)$

(ii) $\|T\|_{C^2} \leqslant c$ *for* $x \leqslant c$

(iii) $\theta(Tx) > d$ *for some* $x \in E(\theta, d, e)$ *with* $|T(x)\|_{C^2} > f$.

Then T *has at least three fixed points* x_1, x_2, x_3 *with* $\|x_1\|_{C^2} < c,\ d < \theta(x_2),\ c < \|x_3\|_{C^2}$ *with* $\theta(x_3) < d$.

Lemma 2.6. *[22] ((Banach's contraction principle) Let (X,d) be a complete metric space, and consider a mapping $J : X \longrightarrow X$, which is strictly contractive, i.e.,*

$$d(Jx, Jy) \leqslant Ld(x, y), \ \forall x, y \in X,$$

for some (Lipschitz constant) $L < 1$. Then,

1. *The mapping J has one, and only one, fixed point $x^\star = J(x^\star)$;*
2. *the fixed point $x^\star$ is globally attractive, i.e.,*

$$\lim_{n \longrightarrow \infty} J^n x = x^\star;$$

3. $$d(J^n x, x^\star) \leqslant L^n d(x, x^\star), \ \forall n \geqslant 0, \ \forall x \in X;$$

4. $$d(J^n x^n, x^\star) \leqslant \frac{1}{1-L} d(J^n x, J^{n+1} x), \ \forall n \geqslant 0, \ \forall x \in X;$$

5. $$d(x, x^\star) \leqslant \frac{1}{1-L} d(x, Tx), \ \forall x \in X$$

Lemma 2.7. *(Hölder inequality)*[20] *Assume that $p, q \geqslant 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(J, X)$, $g \in L^q(J, X)$, then for $1 \leqslant p \leqslant \infty$, $fg \in L^1(J, X)$ and*

$$\|fg\| \leqslant \|f\|_{L^p} \|g\|_{L^q},$$

where $J = [0, 1]$.

3. Existence results

First, By applying the standard contraction mapping principle, the existence of a unique positive solution to the class of equations (1)-(3)is proved.

Theorem 3.1. *Let E be defined as in (5) and suppose that,*

$$|g(t, x(t)) - g(t, y(t)))| \leqslant |k||(x(t) - y(t)|,$$

and $h - g$ is a nonnegative function, where k is a negative constant as defined in (1), with

$$| \, k \, | < \frac{(2-\alpha)\Gamma(\alpha-\beta)}{6}.$$

Then, the problem (1)-(3) has a unique solution on E.

Proof: Define $T : E \longrightarrow E$ by

$$Tx(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 G_1(t, s)[h(s) - g(s, x(s))]ds - \frac{k}{\Gamma(\alpha-\beta)} \int_0^1 G_2(t, s)x(s)ds.$$

First, observe that for each $x \in E$,

$$Tx(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 G_1(t,s)[h(s) - g(s,x(s))]ds - \frac{k}{\Gamma(\alpha-\beta)}\int_0^1 G_2(t,s)x(s)ds$$

$$\geqslant \frac{1}{\Gamma(\alpha)}\int_0^1 \frac{t(1-s)^{\alpha-\beta-1}}{3}(h(s) - g(s,x(s)))ds + \frac{|k|}{\Gamma(\alpha-\beta}\int_0^1 \frac{t(1-s)^{\alpha-\beta-1}}{3}x(s)ds$$

$$\geqslant 0, \ 0 \leqslant t,s \leqslant 1.$$

Thus, T is well-defined.

Next,

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 G_1(t,s)|g(s,x(s)) - g(s,y(s))|ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_2(t,s)|x(s) - y(s)|ds \\
&\leqslant \frac{|k|}{\Gamma(\alpha)}\int_0^1 \frac{3(1-s)^{\alpha-1}}{2-\alpha}|x(s)-y(s)|ds + \frac{|k|}{\Gamma(\alpha-\beta)}\int_0^1 \frac{3(1-s)^{\alpha-1}}{2-\alpha}|x(s)-y(s)|ds \\
&\leqslant \frac{2|k|}{\Gamma(\alpha-\beta)}\|x-y\|\int_0^1 \frac{3(1-s)^{\alpha-1}}{2-\alpha} \\
&\leqslant \frac{2|k|}{\Gamma(\alpha-\beta)}\|x-y\|\frac{3}{\alpha(2-\alpha)} \\
&\leqslant \frac{6|k|}{(2-\alpha)\Gamma(\alpha-\beta)}\|x-y\|.
\end{aligned}$$

This implies that,

$$\|Tx - Ty\|_{C^2} \leqslant \frac{6|k|}{(2-\alpha)\Gamma(\alpha-\beta)}\|x-y\|_{C^2},$$

with $\frac{6|k|}{(2-\alpha)\Gamma(\alpha-\beta)} < 1$.

Therefore, by the Banach Contraction principle of theorem 2.6, it has been proved that T, has a unique fixed point on E, which is a solution of the problem.

Next, applying the famous Leggett-Williams fixed point theorem of lemma 2.5, the existence of at least three positive solutions to the problem is obtained. Before proceeding, here is a theorem that is necessary for the proof of the next result.

Theorem 3.2. *Let* E *be a cone in a Banach space as defined by (5). Then, the operator* $T : E \longrightarrow E$ *is completely continuous.*

Proof. First, define a map $T : E \longrightarrow E$ by

$$Tx(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 G_1(t,s)[h(s) - g(s,x(s))]ds - \frac{k}{\Gamma(\alpha-\beta)}\int_0^1 G_2(t,s)x(s)ds.$$

Then, from theorem 3.1, the map T is well-defined.

Next, it suffices to show that the operator $T : E \longrightarrow E$ is continuous. Pick any sequence

$x_n \in E$ such that $x_n \longrightarrow x \in E$, we show that $\|Tx_n - Tx\|_{C^2} \longrightarrow 0$.
Now,

$$|Tx_n(t) - Tx(t)| \leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 G_1(t,s)|g(s,x_n(s)) - g(s,x(s))ds$$
$$+ \frac{|k|}{\Gamma(\alpha-\beta)}\int_0^1 G_2(t,s)|x_n(s) - x(s)|ds, \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This implies that,

$$\|Tx_n - Tx\|_{C^2} \longrightarrow 0,$$

as $n \longrightarrow \infty$. Therefore, T is continuous on E. Next, it is required to show that T maps bounded sets of E into bounded sets in E.
Now, let

$$V = \{x \in E : \|x\| < M\},$$

then we show that $\|Tx\|_{C^2} < L$, for some positive constants M, L.
Thus,

$$\begin{aligned}
|Tx(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 G_1(t,s)|h(s) - g(s,x(s))|ds + \frac{|k|}{\Gamma(\alpha-\beta)}\int_0^1 G_2(t,s)|x(s)|ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 \frac{3(1-s)^{\alpha-1}}{2-\alpha}|h(s) - g(s,x(s))|ds + \frac{|k|}{\Gamma(\alpha-\beta)}\int_0^1 \frac{3(1-s)^{\alpha-1}}{2-\alpha}|x(s)|ds \\
&\leqslant \frac{3}{(2-\alpha)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|h(s) - g(s,x(s))|ds + \frac{3M|k|}{(2-\alpha)\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-1}ds \\
&\leqslant \frac{3}{(2-\alpha)\Gamma(\alpha)}\left(\int_0^1 |h(s) - g(s,x(s))|^{\frac{1}{\beta}}ds\right)^{\beta}\left(\int_0^1 (1-s)^{\frac{\alpha-1}{1-\beta}}ds\right)^{1-\beta} \\
&+ \frac{3M|k|}{(2-\alpha)\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-1}ds \\
&\leqslant \frac{3}{(2-\alpha)\Gamma(\alpha)(\gamma+1)^{1-\beta}}\|h-g\|_{L^{\frac{1}{\beta}}} + \frac{3M|k|}{\alpha(2-\alpha)\Gamma(\alpha-\beta)}
\end{aligned}$$

Therefore,

$$\|Tx\|_{C^2} \leqslant L,$$

with $L = \frac{3}{(2-\alpha)\Gamma(\alpha)(\gamma+1)^{1-\beta}}\|h-g\|_{L^{\frac{1}{\beta}}} + \frac{3M|k|}{\alpha(2-\alpha)\Gamma(\alpha-\beta)}$

Next, for any $t_1, t_2 \in [0,1]$, with $t_1 < t_2$ and $x \in E$, we have that

$$|Tx(t_2) - Tx(t_1)| \leqslant \frac{1}{\Gamma(\alpha}\int_0^1 G_1(t_2,s) - G_1(t_1,s)|h(s) - g(s,x(s)ds$$
$$+ \frac{|k|}{\Gamma(\alpha-\beta}\int_0^1 G_2(t_2,s) - G_2(t_1,s)|x(s)|ds \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2.$$

This shows that T is equicontinuous on E.
Therefore, The operator $T : E \longrightarrow E$ is a completely continuous operator. □

Theorem 3.3. *Suppose that* $\alpha - \beta - 1 > 0$ *and the assumptions of Theorem 3.2 hold. If there exist constants* $0 < c < d < e$ *such that,*

1.

$$c > \frac{3\alpha\|h-g\|_{L^{\frac{1}{\beta}}}}{(\gamma+1)^{1-\beta}\left(\alpha(2-\alpha)\Gamma(\alpha-\beta)-3|k|\right)},$$

provided that $\alpha(2-\alpha)\Gamma(\alpha-\beta)-3|k| > 0$.

2. for some $\xi \in (0,1)$,

$$|k| > \frac{3\Gamma(\alpha-\beta+1)}{\xi}$$

Then our problem has at least three positive solutions x_1, x_2, x_3, *with* $\|x_1\|_{C^2} < c$, $d < \theta(x_2)$, $c < \|x_3\|_{C^2}$, *with* $\theta(x_3) < d$,

Proof. It suffices to show that the conditions of lemma 2.5 are satisfied. Supposing that $x \in \bar{E_c}$, then $\|x\| \leqslant c$.

$$\begin{aligned}
|Tx(t)| &\leqslant |\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(t,s)(h(s)-g(s,x(s))ds + \frac{|k|}{\Gamma(\alpha-\beta)}\int_0^1 G_2(t,s)x(s)ds| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 \frac{3(1-s)^{\alpha-1}}{2-\alpha}|h(s)-g(s,x(s)|ds + \frac{|k|}{\Gamma(\alpha-\beta)}\int_0^1 \frac{3(1-s)^{\alpha-1}}{2-\alpha}|x(s)|ds \\
&\leqslant \frac{3}{(2-\alpha)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|h(s)-g(s,x(s)|ds + \frac{3c|k|}{(2-\alpha)\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-1}ds \\
&\frac{3}{(2-\alpha)\Gamma(\alpha)}\left(\int_0^1 |h(s)-g(s,x(s))|^{\frac{1}{\beta}}ds\right)^{\beta}\left(\int_0^1 (1-s)^{\frac{\alpha-1}{1-\beta}}ds\right)^{1-\beta} + \frac{3c|k|}{\alpha(2-\alpha)\Gamma(\alpha-\beta)} \\
&\leqslant x(0)(1+\frac{|k|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}) + x'(0)t + \frac{|k|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x(s)|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s)-g(s,x(s))|ds
\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}
|Tx(t)| \leqslant x(0)\left(1+\frac{|k|}{\Gamma(\alpha-\beta+1)}\right) + x'(0) + \frac{|k|c}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}ds + \\
\frac{1}{\Gamma(\alpha)}\left(\int_0^t |h(s)-g(s,x(s))|^{\frac{1}{\beta}}ds\right)^{\beta}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\beta}}ds\right)^{1-\beta} \\
\leqslant x(0)\left(1+\frac{|k|}{\Gamma(\alpha-\beta+1)}\right) + x'(0) + \frac{c|k|}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha)}\|h-g\|_{L^{\frac{1}{\beta}}}\frac{1}{(\gamma+1)^{1-\beta}} \\
\leqslant c,
\end{aligned}$$

by condition (1) of theorem 3.3 This implies that,

$$\|Tx\|_{C^2} \leqslant c.$$

Then by theorem 3.2, $T : \bar{E_c} \longrightarrow \bar{E_c}$ is completely continuous. Equally, employing the same argument, it follows from condition (1) that if $x \in \bar{E_d}$ then $\|Tx\|_{C^2} \leqslant d$.

Define,

$$G(\theta, d, e) = \{x \in E : d \leqslant \theta(x), \|x\|_{C^2} \leqslant e\},$$

and let

$$x(t) = \frac{d+e}{2}, \ t \in [0,1],$$

then we show that

$$x(t) \in G(\theta, d, e).$$

We observe that $x(t)$ as defined is nonnegative and so is contained in E.
Therefore,

$$\begin{aligned} |x(t)| &= \frac{(d+e)}{2} \\ &= \frac{e+e}{2} \\ &= e. \end{aligned}$$

$$\implies \|x\|_{C^2} \leqslant e$$

Taking $\theta(x) = \frac{d+e}{\alpha}$, we have that

$$\theta(x) \geqslant d$$

Thus,

$$\{x \in G(\theta, d, e) : \theta(x) > d\} \neq \Phi$$

.
Next,

$$\begin{aligned} \theta(Tx) &= \min_{\xi \leqslant t \leqslant b} |Tx(t)| \\ &\geqslant \min_{\xi \leqslant t \leqslant b} Tx(t) \\ &\geqslant \frac{1}{\Gamma(\alpha)} \int_0^1 G_1(t,s)(h(s) - g(s, x(s)))ds + \frac{k}{\Gamma(\alpha-\beta)} \int_0^1 G_2(t,s)x(s)ds \\ &\geqslant \frac{-ek}{3\Gamma(\alpha-\beta)} \int_0^1 t(1-s)^{\alpha-\beta-1} ds \\ &\geqslant \frac{\xi ek}{3\Gamma(\alpha-\beta+1)} (1-s)^{\alpha-\beta} \Big|_0^1 \\ &= \frac{e\xi|k|}{3\Gamma(\alpha-\beta+1)} \\ &> e \end{aligned}$$

for some $\xi \in (0,1)$

$$\implies \theta(Tx) > e$$

for $x \in G(\theta, d, e)$.
Therefore, by lemma 2.5, our problem has at least three positive solutions x_1, x_2, x_3, with

$$\|x_1\|_{C^2} < c \text{ and } d < \theta(x_2), c < \|x_3\|_{C^2}, \text{ with } \theta(x_3) < d. \qquad \square$$

4. Some Examples

Example 1. Any well-posed boundary value problem formed by the fractional differential equation

$$^{c}D^{\frac{\pi}{2}}_{0_+}x(t) - \frac{e}{15\pi}\,^{c}D^{\frac{e}{4}}_{0_+}x(t) + (t - \frac{e}{4})(x^2(t) + 6)^{\frac{-1}{2}} = t \tag{4.1}$$

Proof. To see this, it suffices to show that example (4.1) satisfies the conditions of theorem 3.1. First, we observe that $g(t, x(t)) = (t - \frac{e}{4})(x^2(t) + 6)^{\frac{-1}{2}}$ is an L^∞- Carathéodory function and equally Lipschitz continuous, since it is continuously differentiable with respect to x.
Also,

$$\begin{aligned} |k| = \frac{e}{15} &\leqslant \frac{(2-\alpha)\Gamma(\alpha-\beta)}{6} \\ &= 0.07695697757 \end{aligned}$$

Equally,

$$h(t) - g(t, x(t)) = t - (t - \frac{e}{4})(x^2(t) + 6)^{\frac{-1}{2}} \geqslant 0$$

Thus the conditions of theorem 3.1 is verified, hence the proof. $\square$

Example 2.
The equation

$$^{c}D^{\frac{3}{2}}_{0_+}x(t) - \frac{1}{5}\,^{c}D^{\frac{1}{3}}_{0_+}x(t) + 1 = \sin^{\frac{4}{3}} x(t) \tag{4.2}$$

with the conditions

$$x(0) = 0, \; x'(0) = \frac{3}{4}x(1) \tag{4.3}$$

has at least three positive solutions x_1, x_2, x_3, with

$$\|x_1\|_{C^2} < c \text{ and } d < \theta(x_2), c < \|x_3\|_{C^2}, \text{ with } \theta(x_3) < d.$$

Proof. To see this, from theorem 3.2, the operator

$$Tx(t) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 G_1(t,s)[1 - \sin^{\frac{4}{3}} x(t)]ds + \frac{1}{5\Gamma(\frac{7}{6})}\int_0^1 G_2(t,s)x(s)ds, \tag{4.4}$$

where

$$G_1(t,s) = \begin{cases} \frac{3t(1-s)^1}{2} + (t-s)^{\frac{1}{2}} & \text{if, } 0 \leqslant s \leqslant t \\ \frac{3t(1-s)^1}{2} & \text{if, } t \leqslant s \leqslant 1 \end{cases},$$

$$G_2(t,s) = \begin{cases} 3t(1+s)^{\frac{1}{6}} + (t-s)^{\frac{1}{6}} & \text{if, } 0 \leqslant s \leqslant t \\ 3t(1-s)^{\frac{1}{6}} & \text{if, } t \leqslant s \leqslant 1 \end{cases}$$

is completely continuous.
Next, from equation (15), $\alpha = \frac{3}{2},\ \beta = \frac{1}{3},\ k = \frac{-1}{5}$ then, we estimate $\|h-g\|_{L^{\frac{1}{\beta}}}$.
Now,

$$\begin{aligned}\|h-g\|_{L^{\frac{1}{\beta}}} &= \left(\int_0^t (1-\sin^{\frac{4}{3}} x(s))^3 ds\right)^{\frac{1}{3}} \\ &= \left(\int_0^t (\cos^{\frac{4}{3}} x(s))^3 ds\right)^{\frac{1}{3}} \\ &= \left(\int_0^t \cos^4 x(s) ds\right)^{\frac{1}{3}}.\end{aligned}$$

We recall from trigonometric identities that

$$\cos^4 x(t) = \frac{3+4\cos(2x(t)+\cos(4x(t)}{8}$$

This implies that

$$\begin{aligned}\|h-g\|_{L^{\frac{1}{\beta}}} &= \left(\int_0^t \frac{3+4\cos(2x(s)+\cos(4x(s)}{8} ds\right)^{\frac{1}{3}} \\ &= \left(\frac{3t+2\cos(2x(t))+\frac{1}{4}\cos(4x(t))-\frac{9}{4}}{8}\right)^{\frac{1}{3}} \\ &\leqslant \sqrt[3]{\frac{3}{8}}.\end{aligned}$$

So,

$$0 \leqslant \|h-g\|_{L^{\frac{1}{\beta}}} \leqslant \sqrt[3]{\frac{3}{8}}.$$

Also,

$$\begin{aligned}\frac{3\alpha\|h-g\|_{L^{\frac{1}{\beta}}}}{(\gamma+1)^{1-\beta}\,(\alpha(2-\alpha)\Gamma(\alpha-\beta)-3|k|)} &\leqslant \frac{3\times\frac{3}{2}\times\sqrt[3]{\frac{3}{8}}}{(\frac{7}{4})^{\frac{2}{3}}\left[\frac{3}{4}\times\Gamma(\frac{7}{6})-\frac{3}{5}\right]} \\ &= \frac{\frac{9}{4}\times\sqrt[3]{3}}{(\frac{7}{4})^{\frac{2}{3}}\times 0.0957895002} \\ &= \frac{3.245061532}{0.1391051706} \\ &= 23.32811583.\end{aligned}$$

Therefore, any $c \geqslant 23.32811583 > \frac{3\alpha\|h-g\|_{L^{\frac{1}{\beta}}}}{(\gamma+1)^{1-\beta}(\alpha(2-\alpha)\Gamma(\alpha-\beta)-3|k|)}$ will satisfy the first condition of theorem 3.3.
For the second condition of theorem 3.3, we need some $\xi \in (0,1)$ such that

$$|k| > \frac{3\Gamma(\alpha-\beta+1)}{\xi}$$

$$\implies \frac{1}{5} > \frac{3\Gamma(\frac{13}{6})}{\xi}$$
$$\implies \xi < \frac{1}{16.23508834}$$

So, any $\xi < \frac{1}{16.23508834}$ will suffice. Thus, by Theorem 3.3, the problem has at least three positive solutions x_1, x_2, x_3, with

$$\|x_1\|_{C^2} < c \text{ and } d < \theta(x_2), c < \|x_3\|_{C^2}, \text{ with } \theta(x_3) < d. \qquad \square$$

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