SABA Publishing

Journal of Fractional Calculus and Nonlinear Systems
jfcns.sabapub.com
ISSN : 2709-9547

J Frac Calc & Nonlinear Sys
(2025)6(2) : 104-146
doi:10.48185/jfcns.v6i2.1777

Fractional Powersets and SuperHyperStructures: Toward a Framework for Fractional Set Theory and Discrete Hierarchical Systems

TAKAAKI FUJITA[a]

[a] Independent Researcher, Tokyo, Japan.

• *Received: 30 August 2025* • *Accepted: 29 November 2025* • *Published Online: 28 December 2025*

Abstract

Hyperstructures build on powersets to model multivalued relations on a base set; SuperHyperstructures iterate the powerset to capture layered hierarchies and richer composition. Prior work typically fixes the iteration height to a nonnegative integer. This paper asks whether fractional, inverse, and complex (including imaginary) "heights" can be incorporated coherently. We introduce the notions of an m-root powerset (peeling a specified number of subset layers), a negative powerset (a partial inverse of iterated powersets under a given presentation), and a complex-height powerset defined at the level of observables via operator-theoretic interpolation. We characterize when these operators are well defined—by exponential-tower size conditions in the finite case and by the beth hierarchy in the infinite case—and establish exact inverse laws on their natural domains. Lifting from carriers to operations, we obtain root and negative SuperHyperStructures that preserve incidence, compose naturally, and recover the original structures after the appropriate number of lifts. Conceptually, the framework provides a principled, continuous interpolation across hierarchical levels and a reversible mechanism for descending them, suggesting applications to discrete modeling, policy design, and multi-resolution analysis.

Keywords: Superhyperstructures, Hyperstructures, n-th powerset, m-root powerset, Fractional Analysis
2010 MSC: 03Exx.

1. Introduction

Many mathematical and real–world systems must be modeled across multiple hierarchical levels. This need has motivated a sustained line of work on hyperstructures and superhyperstructures. A *Hyperstructure* is built on the powerset construction and offers a systematic way to encode relations among elements of a ground set [1, 2, 3, 4, 5]. Extending this idea, a *SuperHyperstructure* employs iterated powersets to capture multi-layered dependencies and deeper abstraction [6, 7, 8]. For clarity, we recall the standard recursion for iterated powersets. Given a set S,

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{n+1}(S) := \mathcal{P}\big(\mathcal{P}^n(S)\big) \quad (n \in \mathbb{N}_0),$$

---

*Corresponding author: Takaaki.fujita060@gmail.com so that $\mathcal{P}^n(S)$ represents the hierarchy at *height* n. Hyperstructures typically live at height 1, while superhyperstructures range across heights $n \geqslant 1$. Table 1 presents a concise overview of Classical, Hyper-, and SuperHyper-Structures.

Table 1: Concise Overview of Classical, Hyper-, and SuperHyper-Structures

Aspect	Classical Structure	Hyperstructure	SuperHyperstructure
Level (height)	$S = \mathcal{P}^0(S)$	$\mathcal{P}(S) = \mathcal{P}^1(S)$	$\mathcal{P}^n(S)$ with $n \geqslant 1$
Units	Elements $x \in S$	Subsets $A \subseteq S$	Objects in $\mathcal{P}^n(S)$
Operations / relations	$* : S \times S \to S$ or $R \subseteq S \times S$	Hyperop $\circ : S \times S \to \mathcal{P}(S)$	Inter-level laws on $\mathcal{P}^k(S)$, $k \leqslant n$
Typical object	Group, ring, graph	Hypergraph, hyperring	Superhypergraph, hierarchical system
Expressiveness	Pairwise, single layer	Multiway at one layer	Hierarchical, multi-resolution
Morphism	Homomorphism $f : S \to S'$	Map preserving hyperedges/operations	Map respecting levels $f : \mathcal{P}^n(S) \to \mathcal{P}^n(S')$

Iterated powerset. $\mathcal{P}^0(S) := S, \quad \mathcal{P}^{n+1}(S) := \mathcal{P}\big(\mathcal{P}^n(S)\big)$ for $n \in \mathbb{N}_0$.

Despite the breadth of work on $\mathcal{P}^n$ and its structural uses, one direction remains comparatively underexplored: *fractional, inverse, and complex "heights"* for the powerset hierarchy. In this paper we investigate extensions of the powerset construction that incorporate square roots, m-th roots, inverses, and even complex exponents. These generalizations aim to move beyond discrete iteration and to provide a principled interpolation between (and a reversible descent across) hierarchical levels, thereby enriching the interface with Hyperstructures and SuperHyperstructures.

Concretely, we

1. formulate candidate notions of fractional and negative "powerset heights," including square roots and m-th roots of $\mathcal{P}$, together with inverse and complex-height variants;
2. discuss consistency criteria (e.g., functoriality on suitable categories, compatibility with inclusions, and semigroup-type laws where they can be meaningfully stated) and illustrate where additional hypotheses are required;
3. present examples linking these constructions to multi-resolution modeling and policy design, and outline how fractional heights can serve as continuous bridges between discrete levels used in Hyperstructures and SuperHyperstructures.

The resulting framework is intended as a platform for analysis across hierarchical scales, offering continuous interpolation in height and a controlled mechanism for descending it, while remaining attentive to the mathematical conditions under which such extensions are well-posed. Note that the term "fractional analysis" is used heuristically to motivate intermediate heights; it does not rely on fractional calculus or analytic semigroups, and results are established combinatorially within iterated powerset towers.

2. Preliminaries

This section outlines the key concepts and definitions required for understanding the content of this paper.

2.1. Hyperstructure and SuperHyperstructure

A *Hyperstructure* and a *SuperHyperstructure* are formal definitions that describe hierarchical concepts. To rigorously define these notions, we begin by recalling the basic ideas of a universe, a powerset, and the n-th iterated powerset.

Definition 2.1 (Universe)**.** Let U be a nonempty finite set, called the *universe* or *base set*. All subsequent powerset constructions are formed relative to U.

Definition 2.2 (Powerset [9])**.** The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the family of all subsets of S, including both the empty set and S itself:

$$\mathcal{P}(S) \ = \{ A \ | \ A \subseteq S \}.$$

Definition 2.3 (n-th Powerset [10, 11, 12])**.** For a nonempty set H and integer $n \geqslant 1$, the n*-th powerset* is defined recursively by

$$\mathcal{P}_1(H) := \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) := \mathcal{P}\big(\mathcal{P}_n(H)\big).$$

Analogously, the n*-th nonempty powerset*, denoted $\mathcal{P}^*_n(H)$, is constructed by

$$\mathcal{P}^*_1(H) := \mathcal{P}^*(H), \qquad \mathcal{P}^*_{n+1}(H) := \mathcal{P}^*\big(\mathcal{P}^*_n(H)\big),$$

where $\mathcal{P}^*(H) := \mathcal{P}(H) \setminus \{\emptyset\}$.

Example 2.4 (Engineering use of the 2-th powerset: RBAC policies as sets of roles)**.** Let the base permission set be

$$H = \{\text{read\_logs}\ (r),\ \text{write\_config}\ (w),\ \text{deploy}\ (d)\}.$$

Then $\mathcal{P}_1(H) = \mathcal{P}(H)$ contains all roles (each role is a subset of permissions). For instance,

$$R_{\text{viewer}} = \{r\}, \quad R_{\text{operator}} = \{r, w\}, \quad R_{\text{release}} = \{r, d\} \ \in \ \mathcal{P}(H).$$

A *policy* is a set of roles, hence an element of the 2-th powerset $\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}(H))$:

$$\Pi_{\text{oncall}} = \{R_{\text{viewer}}, R_{\text{operator}}, R_{\text{release}}\} \ \in \ \mathcal{P}_2(H), \qquad \Pi_{\text{ci}} = \{R_{\text{viewer}}\} \ \in \ \mathcal{P}_2(H).$$

Verification of levels:

$$R_* \subseteq H \ \Rightarrow \ R_* \in \mathcal{P}(H), \qquad \Pi_* \subseteq \mathcal{P}(H) \ \Rightarrow \ \Pi_* \in \mathcal{P}(\mathcal{P}(H)) = \mathcal{P}_2(H).$$

Cardinalities (finite, explicit): $|H| = 3$, so $|\mathcal{P}(H)| = 2^3 = 8$ (roles), and $|\mathcal{P}_2(H)| = 2^{|\mathcal{P}(H)|} = 2^8 = 256$ (policies). Engineering interpretation: teams attach a policy (a set of roles) to a service; changing a policy modifies the assigned role set atomically.

Example 2.5 (Engineering use of the 3-th powerset: Release calendars as sets of policies)**.** Continue from Example 1. Elements of $\mathcal{P}_2(H)$ are policies (sets of roles). A *release calendar* for a quarter chooses which policies are active on which deployment tracks; abstractly, a calendar is a *set of policies*, hence an element of $\mathcal{P}_3(H) = \mathcal{P}(\mathcal{P}_2(H))$.

Concretely, with the two policies defined above and an additional audit policy

$$\Pi_{\text{audit}} = \{R_{\text{viewer}}\} \in \mathcal{P}_2(H),$$

define two calendars (collections of policies)

$$\mathcal{C}_{Q3} = \{\Pi_{\text{oncall}}, \Pi_{\text{ci}}\} \in \mathcal{P}_3(H), \qquad \mathcal{C}_{Q4} = \{\Pi_{\text{oncall}}, \Pi_{\text{audit}}\} \in \mathcal{P}_3(H).$$

Level check:

$$\Pi_* \in \mathcal{P}_2(H) \Rightarrow \{\Pi_*, \ldots\} \subseteq \mathcal{P}_2(H) \Rightarrow \{\Pi_*, \ldots\} \in \mathcal{P}(\mathcal{P}_2(H)) = \mathcal{P}_3(H).$$

Cardinalities: from $|H| = 3$ we have $|\mathcal{P}(H)| = 8$, $|\mathcal{P}_2(H)| = 256$, and

$$|\mathcal{P}_3(H)| = 2^{|\mathcal{P}_2(H)|} = 2^{256},$$

so there are astronomically many distinct calendars (policy collections). Engineering interpretation: a calendar bundles multiple policies (each policy is a set of roles) to schedule and govern deployments across environments; switching calendars changes *which sets of policies* are simultaneously in force.

To establish a comprehensive framework for understanding Hyperstructures and Superhyperstructures, we present the following formal definitions and foundational concepts.

Definition 2.6 (Classical Structure)**.** (cf.[10, 13, 11]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, characterized by one or more *Classical Operations* that adhere to specific *Classical Axioms*. Formally:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geqslant 1$ denotes a positive integer, and H^m represents the m-fold Cartesian product of H. Examples include algebraic operations such as addition and multiplication in structures like groups, rings, and fields.

Definition 2.7 (Hyperstructure)**.** (cf.[10, 14, 15, 11]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. It is formally defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ denotes its powerset, and $\circ$ is an operation defined for subsets within $\mathcal{P}(S)$.

Example 2.8 (Hyperstructure in practice: release-bundle composition on services)**.** Let the base set of deployable services be

$$S = \{\text{Auth},\ \text{Billing},\ \text{Analytics}\}.$$

Its powerset $\mathcal{P}(S)$ consists of all *release bundles* (each bundle is a subset of services). Define a hyperoperation

$$\circ :\ \mathcal{P}(S) \times \mathcal{P}(S) \longrightarrow \mathcal{P}\big(\mathcal{P}(S)\big), \qquad (A, B) \longmapsto \{A,\ B,\ A \cup B\}.$$

Then $\mathcal{H} = (\mathcal{P}(S), \circ)$ is a Hyperstructure: for any bundles $A, B \in \mathcal{P}(S)$ the output $\circ(A, B)$ is a *set of feasible outcomes* (keep A, keep B, or deploy the union $A \cup B$), hence an element of $\mathcal{P}(\mathcal{P}(S))$.

Concrete computation. Take

$$A = \{\text{Auth}\}, \qquad B = \{\text{Billing}, \text{Analytics}\}.$$

Then

$$\circ(A, B) = \big\{\ \{\text{Auth}\},\ \{\text{Billing}, \text{Analytics}\},\ \{\text{Auth}, \text{Billing}, \text{Analytics}\}\ \big\} \ \in\ \mathcal{P}\big(\mathcal{P}(S)\big).$$

This captures non-deterministic *merge choices* available to a release manager while remaining closed in the hyperstructure sense (outputs are sets of bundles).

Definition 2.9 (n-Superhyperstructure)**.** (cf.[10, 11, 16]) An n-*Superhyperstructure* generalizes the Hyperstructure by employing the n-th powerset of a base set. Formally, it is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ represents the n-th powerset of S, and $\circ$ is an operation acting on elements of $\mathcal{P}_n(S)$.

Example 2.10 (2-Superhyperstructure in practice: playbook composition from sets of bundles)**.** Using the same base set $S = \{\text{Auth}, \text{Billing}, \text{Analytics}\}$, elements of $\mathcal{P}_1(S) = \mathcal{P}(S)$ are *bundles*, while elements of $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$ are *playbooks* (sets of bundles). Define a level-2 hyperoperation

$$\star :\ \mathcal{P}_2(S) \times \mathcal{P}_2(S) \longrightarrow \mathcal{P}\big(\mathcal{P}_2(S)\big) = \mathcal{P}_3(S), \qquad (\mathcal{A}, \mathcal{B}) \longmapsto \{\mathcal{A},\ \mathcal{B},\ \mathcal{A} \cup \mathcal{B}\}.$$

Then $\mathcal{SH}_2 = \big(\mathcal{P}_2(S), \star\big)$ is a 2-Superhyperstructure: the output of $\star$ is a *set of playbooks*, hence an element of $\mathcal{P}_3(S)$.

Concrete computation. Let

$$\mathcal{A} = \big\{\ \{\text{Auth}\},\ \{\text{Billing}\}\ \big\}, \qquad \mathcal{B} = \big\{\ \{\text{Billing}\},\ \{\text{Analytics}\}\ \big\}.$$

Both $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{P}(S)$, so $\mathcal{A}, \mathcal{B} \in \mathcal{P}_2(S)$. Their union is

$$\mathcal{A} \cup \mathcal{B} = \big\{\ \{\text{Auth}\},\ \{\text{Billing}\},\ \{\text{Analytics}\}\ \big\} \ \in\ \mathcal{P}_2(S).$$

Therefore

$$\star(\mathcal{A}, \mathcal{B}) = \big\{\ \mathcal{A},\ \mathcal{B},\ \mathcal{A} \cup \mathcal{B}\ \big\} \ \in\ \mathcal{P}\big(\mathcal{P}_2(S)\big) = \mathcal{P}_3(S).$$

Engineering interpretation. Each playbook lists permissible bundles for a deployment track (e.g., weekday vs. weekend). The hyperoperation returns candidate combined playbooks (keep either original or adopt their union), modeling policy composition with explicit, set-valued outcomes.

3. Main Results

In this section, as the principal outcome of the present paper, we investigate extensions of the powerset construction that incorporate square roots, m-th roots, inverses, and even complex exponents. These generalizations allow us to move beyond the classical discrete iteration, providing new perspectives on fractional, negative, and complex-height powersets, together with their connections to Hyperstructures and SuperHyperstructures.

We also hope that the concepts developed in this paper may, in the future, find applications within the broader field of *Fractional Analysis*. Fractional Analysis investigates differentiation and integration of arbitrary real or complex order, extending classical calculus to model memory, hereditary effects, and anomalous diffusion across diverse systems[17, 18, 19, 20].

3.1. m-root powerset

An m-root powerset of a set U is any V such that $P^{m-1}(V) \cong U$, i.e., peeling $m-1$ subset layers.

Definition 3.1 (Iterated binary logarithm on (finite) cardinals)**.** For a positive integer k, define the k-fold iterated base-2 logarithm on a positive integer N by

$$\log_2^{(0)}(N) := N, \qquad \log_2^{(k+1)}(N) := \log_2\big(\log_2^{(k)}(N)\big),$$

whenever the right side is an integer. Equivalently, N lies in the tower

$$\mathsf{Tow}_0 := \mathbb{N}_{\geqslant 1}, \qquad \mathsf{Tow}_{k+1} := \{\, 2^n \mid n \in \mathsf{Tow}_k \,\}.$$

Then $\log_2^{(k)}$ is defined on Tow_k and is the two-sided inverse of $n \mapsto 2^{(\cdot)}$ iterated k times.

Definition 3.2 (m-root powerset $\mathcal{P}^{\langle 1/m \rangle}$)**.** Fix an integer $m \geqslant 1$. For a finite universe U, we say that a set V is an m*-root powerset* of U if there exists a bijection

$$\vartheta_U : \mathcal{P}^{m-1}(V) \xrightarrow{\cong} U.$$

We then write (up to isomorphism)

$$\mathcal{P}^{\langle 1/m \rangle}(U) := V.$$

Remark 3.3 (Existence, size, and uniqueness up to isomorphism). Let $m \geqslant 1$ and finite U.

1. (Cardinal equation) If V is an m-root powerset of U, then

$$\big|\mathcal{P}^{m-1}(V)\big| = |U| \quad \Longleftrightarrow \quad 2^{2^{\cdot^{2^{|V|}}}} \overset{(m-1)\ \text{twos}}{=} |U|.$$

   Equivalently, setting $t := |V|$,

$$|U| \in \mathsf{Tow}_{m-1} \quad \text{and} \quad t = \log_2^{(m-1)}\big(|U|\big).$$

2. (Existence criterion) An m-root powerset of U exists *iff* $|U| \in \mathsf{Tow}_{m-1}$.

3. (Uniqueness up to bijection) If V and V' are m-root powersets of U, then $|V| = |V'| = \log_2^{(m-1)}(|U|)$; hence $V \cong V'$.
4. (Verification) Whenever $\vartheta_U : \mathcal{P}^{m-1}(V) \to U$ is a bijection, applying $\mathcal{P}$ yields a bijection
$$\mathcal{P}^m(V) = \mathcal{P}\big(\mathcal{P}^{m-1}(V)\big) \xrightarrow{\mathcal{P}(\vartheta_U)} \mathcal{P}(U),$$
so indeed $\mathcal{P}^m(V) \cong \mathcal{P}(U)$.

Remark 3.4 (Canonical choices). When $N = 2^n$, a convenient canonical model of $\mathcal{P}^{\langle 1/2\rangle}(U)$ is $[n] := \{1, \dots, n\}$ with the standard bijection $\chi : \mathcal{P}([n]) \to \{0,1\}^n$, $S \mapsto$ the characteristic n-bit vector of S. Composing any fixed bijection $b : \{0,1\}^n \xrightarrow{\cong} U$ with χ gives $\vartheta_U = b \circ \chi : \mathcal{P}([n]) \xrightarrow{\cong} U$. For $m \geqslant 3$, iterate this encoding level-by-level.

Example 3.5 (Square-root powerset for feature toggles ($m = 2$)). Suppose a product ships with $n = 5$ independent feature toggles. Each toggle configuration is a 5-bit vector, so the configuration space $U := \{0,1\}^5$ has size $|U| = 2^5 = 32$ (which is 2^n). Define $V := [5] = \{1,2,3,4,5\}$. The bijection $\vartheta_U : \mathcal{P}(V) \to U$ sends a subset $S \subseteq [5]$ to its characteristic bitmask in $\{0,1\}^5$. Hence $\mathcal{P}^{\langle 1/2\rangle}(U) = V$ (size 5), and
$$\mathcal{P}\big(\mathcal{P}^{\langle 1/2\rangle}(U)\big) = \mathcal{P}(\mathcal{P}(V)) \cong \mathcal{P}(U).$$
Numerically: $|\mathcal{P}^{\langle 1/2\rangle}(U)| = 5 = \log_2 32$, $|\mathcal{P}(U)| = 2^{32}$, $|\mathcal{P}^2(V)| = 2^{2^5} = 2^{32}$.

Example 3.6 (Cube-root powerset for "bundles of study profiles" ($m = 3$)). Let $t = 3$ base topics. A *profile* is any subset of topics; the set of all profiles is $W := \mathcal{P}([3])$ with $|W| = 2^3 = 8$. A *bundle* is any subset of profiles; the universe of bundles is
$$U := \mathcal{P}(W) = \mathcal{P}\big(\mathcal{P}([3])\big),$$
which has size $|U| = 2^{|W|} = 2^{2^3} = 256$ (indeed $256 \in \mathsf{Tow}_2$). Define $V := [3]$. Then the identity bijection $\vartheta_U : \mathcal{P}^2(V) = \mathcal{P}(\mathcal{P}([3])) \xrightarrow{\text{id}} U$ exhibits V as $\mathcal{P}^{\langle 1/3\rangle}(U)$. Verification by sizes:
$$|\mathcal{P}^{\langle 1/3\rangle}(U)| = |V| = 3 = \log_2^{(2)}(256) = \log_2(\log_2 256) = \log_2 8 = 3,$$
and
$$\big|\mathcal{P}^3(V)\big| = 2^{\left|\mathcal{P}^2(V)\right|} = 2^{|\mathcal{P}(\mathcal{P}([3]))|} = 2^{|U|} = |\mathcal{P}(U)|.$$

Example 3.7 (Fourth-root powerset for "catalogs of bundle-collections" ($m = 4$)). Take $t = 2$. Level 0: $V := [2]$ (base topics), $|V| = 2$. Level 1: profiles $W_1 := \mathcal{P}(V)$, $|W_1| = 2^2 = 4$. Level 2: bundles $W_2 := \mathcal{P}(W_1)$, $|W_2| = 2^4 = 16$. Level 3 (universe): $U := \mathcal{P}(W_2)$, $|U| = 2^{16} = 65536$ (which is $2^{2^{2^2}} \in \mathsf{Tow}_3$). Then $\vartheta_U : \mathcal{P}^3(V) = \mathcal{P}(W_2) \xrightarrow{\text{id}} U$ exhibits V as $\mathcal{P}^{\langle 1/4\rangle}(U)$. Checks:
$$|\mathcal{P}^{\langle 1/4\rangle}(U)| = |V| = 2 = \log_2^{(3)}(65536) = \log_2 \log_2 \log_2(65536) = \log_2 \log_2(16) = \log_2 4 = 2,$$
and $|\mathcal{P}^4(V)| = 2^{|U|} = |\mathcal{P}(U)|$.

Example 3.8 (Square-root powerset ($m = 2$) for Role-Based Access Control (RBAC))**.** Let the primitive permissions be

$$V = \{\mathsf{read\_logs},\ \mathsf{write\_config},\ \mathsf{deploy},\ \mathsf{admin}\}.$$

A *role* is any subset of permissions, hence the catalog of all roles is

$$U \ := \ \mathcal{P}(V) \qquad (\text{every } R \subseteq V \text{ is a role}).$$

Define the bijection

$$\vartheta_U : \mathcal{P}(V) \xrightarrow{\cong} U, \qquad \vartheta_U(S) = S.$$

Then ϑ_U exhibits V as an m-root powerset of U with $m = 2$, since

$$\mathcal{P}^{m-1}(V) = \mathcal{P}(V) \ \cong \ U.$$

Numerics. $|V| = 4 \ \Rightarrow \ |U| = |\mathcal{P}(V)| = 2^4 = 16$ and $|\mathcal{P}^{\langle 1/2 \rangle}(U)| = \log_2 |U| = 4$. *Operational meaning.* The 2-root "peels off" one subset layer from the role catalog to recover the primitive permission set that generated every role. This supports *schema recovery* (what are the atomic capabilities?) and *impact analysis* (which roles change if a primitive is added/removed).

Example 3.9 (Cube-root powerset ($m = 3$) for Deployment Playbooks)**.** Consider two microservices to deploy,

$$V = \{\mathsf{Auth},\ \mathsf{Billing}\} \quad (|V| = 2).$$

Level 1 objects are *bundles of services*: $\mathcal{P}(V) = \{\emptyset, \{\mathsf{Auth}\}, \{\mathsf{Billing}\}, \{\mathsf{Auth}, \mathsf{Billing}\}\}$. Level 2 objects are *playbooks* (sets of bundles): $\mathcal{P}^2(V) = \mathcal{P}(\mathcal{P}(V))$.

Let the enterprise registry of admissible playbooks be

$$U \ := \ \mathcal{P}^2(V) \qquad (\text{all playbooks allowed by policy}).$$

Define the bijection

$$\vartheta_U : \mathcal{P}^2(V) \xrightarrow{\cong} U, \qquad \vartheta_U(\mathcal{A}) = \mathcal{A}.$$

Then ϑ_U exhibits V as a 3-root powerset of U, because

$$\mathcal{P}^{m-1}(V) = \mathcal{P}^2(V) \ \cong \ U \qquad (m = 3).$$

Numerics. $|V| = 2 \ \Rightarrow \ |\mathcal{P}(V)| = 2^2 = 4$, hence $|U| = |\mathcal{P}^2(V)| = 2^4 = 16$ and $|\mathcal{P}^{\langle 1/3 \rangle}(U)| = \log_2^{(2)} |U| = \log_2\!\big(\log_2 16\big) = \log_2 4 = 2$. *Operational meaning.* The 3-root recovers the *atomic services* driving a large catalog of playbooks. This is useful for *governance*: from the registry of all approved playbooks (policies over bundles), one can infer the minimal service basis, check completeness (no orphaned playbooks), and plan migrations by tracing each playbook back to its underlying services.

3.2. n-root SuperHyperStructure

An n-root SuperHyperStructure removes n lifts from a superhyperstructure, yielding $(P^{k-n}(S),\ \mu^{[k-n]})$ and preserving operations/incidence via relation $\theta_E = P^n(\theta_V)$.

Definition 3.10 (Hyperstructure and superhyperstructure (Recall))**.** (cf. [10, 11]) Let X be a nonempty set. A *(binary) hyperoperation* on X is a map

$$\mu:\ X \times X \longrightarrow \mathcal{P}(X).$$

A *hyperstructure* is a pair (X, μ). For a base set S and an integer $k \geqslant 0$, the k-*superhyperstructure over* S *induced by* μ is the pair

$$\mathcal{SH}^{(k)}(S, \mu) := \bigl(\mathcal{P}^k(S),\ \mu^{[k]}\bigr),$$

where $\mu^{[0]} := \mu$ and, recursively, $\mu^{[k+1]} := \mathrm{Lift}(\mu^{[k]})$ is the lifted hyperoperation defined below.

Definition 3.11 (Canonical lift of a hyperoperation)**.** Let $\mu : X \times X \to \mathcal{P}(X)$ be a hyperoperation on X. Define its *lift* to $\mathcal{P}(X)$ by

$$\mathrm{Lift}(\mu):\ \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathcal{P}\bigl(\mathcal{P}(X)\bigr), \qquad (A, B) \longmapsto \bigl\{\ \mu(a, b)\ \bigm|\ a \in A,\ b \in B\ \bigr\}.$$

Thus, for each pair (A, B) of subsets of X, $\mathrm{Lift}(\mu)(A, B)$ is a *family* of μ-outputs, hence an element of $\mathcal{P}(\mathcal{P}(X))$. Iterating, for $k \geqslant 1$ we obtain

$$\mu^{[k]}:\ \mathcal{P}^k(S) \times \mathcal{P}^k(S) \longrightarrow \mathcal{P}^{k+1}(S).$$

Definition 3.12 (n-root SuperHyperStructure (relative to a lift tower))**.** Fix a base set S, a hyperoperation μ on S, and an integer $k \geqslant 0$. The k-level induced superhyperstructure is

$$\mathcal{SH}^{(k)}(S, \mu) = \bigl(\mathcal{P}^k(S),\ \mu^{[k]}\bigr).$$

For an integer m with $1 \leqslant m \leqslant k+1$, the m-*root SuperHyperStructure* of $\mathcal{SH}^{(k)}(S, \mu)$ is defined to be the lower-level member of the same lift tower

$$\mathrm{Root}_m\bigl(\mathcal{SH}^{(k)}(S, \mu)\bigr) := \mathcal{SH}^{(k-m+1)}(S, \mu) = \bigl(\mathcal{P}^{\,k-m+1}(S),\ \mu^{[\,k-m+1\,]}\bigr).$$

By construction one has the identity

$$\mathrm{Lift}^{\,m-1}\Bigl(\mathrm{Root}_m\bigl(\mathcal{SH}^{(k)}(S, \mu)\bigr)\Bigr) = \mathcal{SH}^{(k)}(S, \mu),$$

i.e., applying the lift $(m-1)$ times recovers the original superhyperstructure.

Proposition 3.13 (Existence and uniqueness within a lift tower)**.** *Let* S *be a set, let* $\mu : S \times S \to \mathcal{P}(S)$ *be a hyperoperation, and define the lifted operations*

$$\mu^{[0]} := \mu, \qquad \mu^{[i+1]} := \mathrm{Lift}\bigl(\mu^{[i]}\bigr) \quad (i \geqslant 0),$$

so that $\mu^{[k]} : \mathcal{P}^k(S) \times \mathcal{P}^k(S) \to \mathcal{P}^{k+1}(S)$ *for each* $k \geqslant 0$*. Fix* $k \geqslant 0$ *and* $1 \leqslant m \leqslant k+1$*. Then the* m*-root* $\mathrm{Root}_m\bigl(\mathcal{SH}^{(k)}(S, \mu)\bigr)$ *exists and is unique* within the given lift tower. *Moreover,*

$$\mathrm{Lift}^{\,m-1}\bigl(\mu^{[\,k-m+1\,]}\bigr) = \mu^{[k]} \quad \textit{as maps} \ \ \mathcal{P}^k(S) \times \mathcal{P}^k(S) \longrightarrow \mathcal{P}^{k+1}(S).$$

Proof. We first recall the lifting convention used throughout. For any $i \geqslant 0$ and any

$$h : \mathcal{P}^i(S) \times \mathcal{P}^i(S) \longrightarrow \mathcal{P}^{i+1}(S),$$

its lift $\mathrm{Lift}(h) : \mathcal{P}^{i+1}(S) \times \mathcal{P}^{i+1}(S) \to \mathcal{P}^{i+2}(S)$ is defined by

$$\mathrm{Lift}(h)(A, B) \;:=\; \bigcup_{a \in A} \bigcup_{b \in B} h(a, b) \qquad \big(A, B \subseteq \mathcal{P}^i(S)\big).$$

With this convention, $\mu^{[i+1]} = \mathrm{Lift}(\mu^{[i]})$ for all $i \geqslant 0$, which produces the (deterministic) lift tower

$$\big(\mathcal{P}^0(S), \mu^{[0]}\big) \xrightarrow{\text{Lift}} \big(\mathcal{P}^1(S), \mu^{[1]}\big) \xrightarrow{\text{Lift}} \cdots \xrightarrow{\text{Lift}} \big(\mathcal{P}^k(S), \mu^{[k]}\big).$$

Existence. Set $r := k - m + 1$ ($\geqslant 0$). We claim that the m-root of $\mathcal{SH}^{(k)}(S, \mu)$ within this tower is exactly

$$\mathrm{Root}_m\big(\mathcal{SH}^{(k)}(S, \mu)\big) \;=\; \big(\mathcal{P}^r(S),\, \mu^{[r]}\big).$$

To verify this, we prove by induction on $t = 0, 1, \ldots, m - 1$ that

$$\mathrm{Lift}^{\,t}\big(\mu^{[r]}\big) \;=\; \mu^{[r+t]}.$$

For $t = 0$ the statement is tautological. Assuming $\mathrm{Lift}^t(\mu^{[r]}) = \mu^{[r+t]}$, we apply one more lift and use the definition of the tower:

$$\mathrm{Lift}^{t+1}\big(\mu^{[r]}\big) = \mathrm{Lift}\big(\mathrm{Lift}^t(\mu^{[r]})\big) = \mathrm{Lift}\big(\mu^{[r+t]}\big) = \mu^{[r+t+1]}.$$

By induction, taking $t = m - 1$ yields $\mathrm{Lift}^{\,m-1}(\mu^{[r]}) = \mu^{[r+m-1]} = \mu^{[k]}$, which proves existence and the displayed identity.

Uniqueness. We show that the lifting functor is injective at each stage, hence $\mathrm{Lift}^{\,m-1}$ is injective as well. Let

$$h_1, h_2 : \mathcal{P}^i(S) \times \mathcal{P}^i(S) \to \mathcal{P}^{i+1}(S)$$

satisfy $\mathrm{Lift}(h_1) = \mathrm{Lift}(h_2)$. For any $a, b \in \mathcal{P}^i(S)$, evaluate the lifts on singletons $\{a\}, \{b\} \subseteq \mathcal{P}^i(S)$:

$$h_1(a, b) = \mathrm{Lift}(h_1)\big(\{a\}, \{b\}\big) = \mathrm{Lift}(h_2)\big(\{a\}, \{b\}\big) = h_2(a, b).$$

Thus $h_1 = h_2$. Consequently, if $\nu : \mathcal{P}^r(S) \times \mathcal{P}^r(S) \to \mathcal{P}^{r+1}(S)$ satisfies $\mathrm{Lift}^{\,m-1}(\nu) = \mu^{[k]}$, then injectivity of $\mathrm{Lift}^{\,m-1}$ forces $\nu = \mu^{[r]}$, i.e., the m-root within the tower is unique.

Finally, the type assertion

$$\mathrm{Lift}^{\,m-1}\big(\mu^{[k-m+1]}\big) = \mu^{[k]} \quad \text{as maps } \mathcal{P}^k(S) \times \mathcal{P}^k(S) \to \mathcal{P}^{k+1}(S)$$

follows from the existence part and the definition of the tower levels. □

Remark 3.14 (Roots across different base sets). One may also seek an m-root over a *different* base set $\tilde{S}$: find a bijection $b : \mathcal{P}^{k-m+1}(\tilde{S}) \xrightarrow{\cong} \mathcal{P}^k(S)$ and transport $\mu^{[k]}$ along b. For finite sets, this is possible iff $|\tilde{S}| = T_{m-1}(|S|)$.

Example 3.15 (Square-root SuperHyperStructure ($m = 2$) yielding union-families)**.** Let $S = \{x, y, z\}$ and define on S the base hyperoperation

$$\mu(a, b) \ := \ \{a, b\} \ \subseteq S \qquad (a, b \in S).$$

Level $k = 1$ structure: the carrier is $\mathcal{P}(S)$ and

$$\mu^{[1]}(A, B) \ = \ \mathrm{Lift}(\mu)(A, B) \ = \ \big\{ \ \mu(a, b) \ \big| \ a \in A, \ b \in B \ \big\} \ = \ \big\{ \{a, b\} \ \big| \ a \in A, \ b \in B \ \big\} \ \subseteq \ \mathcal{P}(S),$$

so each output is a *family* of 2-element subsets of S. Take $k = 1$ and $m = 2$. Then

$$\mathrm{Root}_2\big(\mathcal{SH}^{(1)}(S, \mu)\big) \ = \ \mathcal{SH}^{(0)}(S, \mu) \ = \ (S, \mu).$$

Verification on a concrete input: let $A = \{x\}$ and $B = \{y, z\}$. Then

$$\mu^{[1]}(A, B) = \big\{\{x, y\}, \{x, z\}\big\}.$$

Applying one lift to the root recovers this:

$$\mathrm{Lift}(\mu)(A, B) = \{\mu(x, y), \mu(x, z)\} = \big\{\{x, y\}, \{x, z\}\big\}.$$

Carrier sizes: $|S| = 3$, $|\mathcal{P}(S)| = 2^3 = 8$, and indeed $|\mathcal{P}^1(\mathcal{P}^0(S))| = 8$.

Example 3.16 (Cube-root ($m = 3$) at level $k = 2$)**.** Keep the same S and μ as above. The level $k = 2$ structure has carrier $\mathcal{P}^2(S)$ and hyperoperation

$$\mu^{[2]}(\mathcal{A}, \mathcal{B}) \ = \ \mathrm{Lift}(\mu^{[1]})(\mathcal{A}, \mathcal{B}) \ = \ \big\{ \ \mu^{[1]}(A, B) \ \big| \ A \in \mathcal{A}, \ B \in \mathcal{B} \ \big\},$$

so each output is a *family of families* of 2-subsets of S. Choose $m = 3$. Then

$$\mathrm{Root}_3\big(\mathcal{SH}^{(2)}(S, \mu)\big) \ = \ \mathcal{SH}^{(0)}(S, \mu) \ = \ (S, \mu).$$

Concrete computation:

$$\mathcal{A} = \big\{\{x\}, \{x, y\}\big\}, \qquad \mathcal{B} = \big\{\{y\}\big\}.$$

Then

$$\mu^{[2]}(\mathcal{A}, \mathcal{B}) \ = \ \big\{ \ \mu^{[1]}(\{x\}, \{y\}), \ \mu^{[1]}(\{x, y\}, \{y\}) \ \big\} \ = \ \big\{ \{\{x, y\}\}, \ \{\{x, y\}, \{y, y\}\} \big\}.$$

Since $\{y, y\} = \{y\}$, the second inner family simplifies:

$$\mu^{[2]}(\mathcal{A}, \mathcal{B}) \ = \ \big\{ \{\{x, y\}\}, \ \{\{x, y\}, \{y\}\} \big\} \ \in \ \mathcal{P}\big(\mathcal{P}(\mathcal{P}(S))\big) = \mathcal{P}^3(S).$$

Cardinal check: $|S| = 3$, so $|\mathcal{P}^2(S)| = 2^{2^3} = 256$ and the $m = 3$ root lives on level $k - m + 1 = 0$, i.e. on S itself, as required.

Example 3.17 (Fourth-root ($m = 4$) with a nontrivial base hyperoperation)**.** Let $S = \{0, 1\}^2$ (bit pairs) and define a base hyperoperation that returns three outcomes:

$$\mu(u, v) \ := \ \{u, \ v, \ u \oplus v\} \ \subseteq S,$$

where $\oplus$ is bitwise XOR on $\{0,1\}^2$. Then for $k = 3$ the induced structure

$$\mathcal{SH}^{(3)}(S,\mu) = \big(\mathcal{P}^3(S),\ \mu^{[3]}\big)$$

has carrier size $|\mathcal{P}^3(S)| = 2^{2^{2^{|S|}}} = 2^{2^{2^4}} = 2^{65536}$. Taking $m = 4$, the 4-root is

$$\mathrm{Root}_4\big(\mathcal{SH}^{(3)}(S,\mu)\big) \ = \ \mathcal{SH}^{(0)}(S,\mu) \ = \ (S,\mu),$$

and $\mathrm{Lift}^3(\mu) = \mu^{[3]}$ by construction. On concrete inputs $A, B \subseteq S$,

$$\mu^{[1]}(A,B) = \{\ \{u, v, u \oplus v\} \mid u \in A,\ v \in B\ \} \ \subseteq \mathcal{P}(S),$$

and higher lifts collect these inner 3-element subsets into families at levels $\mathcal{P}^2(S)$ and $\mathcal{P}^3(S)$.

Example 3.18 (Policy catalog ⇒ n-root SuperHyperStructure (real-world access governance, $n = 2$)). **Setting.** Let the primitive permissions be

$$S = \{\text{read } (r),\ \text{write } (w),\ \text{deploy } (d)\}.$$

Define a base hyperoperation $\mu : S \times S \to \mathcal{P}(S)$ by

$$\mu(s,t) := \{s,t\} \qquad (s,t \in S),$$

which (non-deterministically) returns the two selected primitives as a subset of S. Lift μ canonically to level k to obtain $\mu^{[k]} : \mathcal{P}^k(S) \times \mathcal{P}^k(S) \to \mathcal{P}^{k+1}(S)$.

Depth-2 SuperHyperStructure (given data). Let the organization maintain a *catalog of playbooks*, i.e., sets of *roles* (roles are subsets of S). This is the level-2 SuperHyperStructure

$$\mathcal{SH}^{(2)}(S,\mu) \ = \ \big(\mathcal{P}^2(S),\ \mu^{[2]}\big).$$

For concreteness, consider two playbooks

$$\mathcal{A} = \big\{\ \{r\},\ \{w\}\ \big\}, \qquad \mathcal{B} = \big\{\ \{w\},\ \{d\}\ \big\}\ \in\ \mathcal{P}^2(S).$$

Their level-2 hyperoperation evaluates to

$$\mu^{[2]}(\mathcal{A},\mathcal{B}) \ = \ \big\{\ \mu^{[1]}(\{r\},\{w\}),\ \mu^{[1]}(\{r\},\{d\}),\ \mu^{[1]}(\{w\},\{w\}),\ \mu^{[1]}(\{w\},\{d\})\ \big\},$$

and, since $\mu^{[1]}(A,B) = \{\{s,t\} \mid s \in A,\ t \in B\}$, we obtain the explicit family of families

$$\mu^{[2]}(\mathcal{A},\mathcal{B}) \ = \ \big\{\ \{\{r,w\}\},\ \{\{r,d\}\},\ \{\{w\}\},\ \{\{w,d\}\}\ \big\}\ \in\ \mathcal{P}^3(S).$$

n-root SuperHyperStructure ($n = 2$) and verification. The 2-root of the above depth-2 structure (taken within its lift tower) is

$$\mathrm{Root}_2\big(\mathcal{SH}^{(2)}(S,\mu)\big) \ = \ \mathcal{SH}^{(0)}(S,\mu) \ = \ (S,\mu).$$

By construction, applying two lifts recovers the original:

$$\mathrm{Lift}^2\Big(\mathrm{Root}_2\big(\mathcal{SH}^{(2)}(S,\mu)\big)\Big) \ = \ \mathcal{SH}^{(2)}(S,\mu),$$

and at the level of carriers, with $|S| = 3$, one checks the tower cardinalities

$$|\mathcal{P}(S)| = 2^3 = 8, \qquad |\mathcal{P}^2(S)| = 2^8 = 256,$$

so "peeling" two subset layers moves from the playbook catalog ($\mathcal{P}^2(S)$) back to the primitive permission layer (S).

Operational interpretation (governance). Given a large catalog of playbooks (depth 2), the 2-root SuperHyperStructure identifies the *atomic permission hyperstructure* (S, μ) that generated those catalogs via lifts. This supports: (i) *schema recovery*: enumerate the minimal primitive capabilities driving all approved playbooks; (ii) *compliance auditing*: test separation-of-duties at the atomic layer and propagate results upward; (iii) *change impact*: a proposed edit to a primitive in S deterministically lifts to affected roles (level 1) and playbooks (level 2), enabling safe policy evolution.

3.3. Negative powerset

Negative powerset partially inverts iterated powerset: given $X \cong \mathcal{P}^n(V)$, return V; defined on n-admissible carriers with chosen presentations.

Definition 3.19 (k-admissible set (finite case))**.** Fix $k \geqslant 1$. A set X is k-*admissible* if there exist a set V and a bijection

$$\vartheta : \mathcal{P}^k(V) \xrightarrow{\cong} X.$$

Equivalently, if X is finite then $|X| = T_k(t)$ for some $t \in \mathbb{N}$, in which case we may take $|V| = t$. The pair (X, ϑ) is called a k-*presentation* of X.

Remark 3.20 (Infinite case via beth-tower). Write $\beth_0(\kappa) := \kappa$ and $\beth_{\alpha+1} := 2^{\beth_\alpha}$. For infinite cardinals, X is k-admissible iff $|X| = \beth_k(\lambda)$ for some λ; then one can choose $|V| = \lambda$.

Definition 3.21 (Negative powerset as a partial inverse)**.** Fix $n \geqslant 1$. On the groupoid of n-admissible sets with chosen presentations, define

$$\mathcal{P}^{\langle -n \rangle}(X, \vartheta) := V \qquad \text{whenever} \qquad \vartheta : \mathcal{P}^n(V) \xrightarrow{\cong} X.$$

On morphisms (bijections) $f : (X, \vartheta) \to (X', \vartheta')$ induced by some $g : V \to V'$ with $\vartheta' \circ \mathcal{P}^n(g) = f \circ \vartheta$, set $\mathcal{P}^{\langle -n \rangle}(f) := g$.

Proposition 3.22 (Inverse laws (on the admissible domain))**.** *Let* $n \geqslant 1$ *and let* (X, ϑ) *be* n*-admissible with witness* $\vartheta : \mathcal{P}^n(V) \xrightarrow{\cong} X$ *for some set* V*. Then*

$$\mathcal{P}^n\big(\mathcal{P}^{\langle -n \rangle}(X, \vartheta)\big) \cong X, \qquad \mathcal{P}^{\langle -n \rangle}\big(\mathcal{P}^n(V), \mathrm{id}\big) = V.$$

Consequently, viewing $\mathcal{P}^{\langle -n \rangle}$ *as a (partial) functor from the subcategory* Adm_n *of* n*-admissible sets to* **Set** *and* $\mathcal{P}^n$ *as a functor* **Set** $\to \mathrm{Adm}_n$ *(sending* V *to* $(\mathcal{P}^n(V), \mathrm{id})$*), there are natural isomorphisms*

$$\eta : \mathcal{P}^n \circ \mathcal{P}^{\langle -n \rangle} \Rightarrow \mathrm{Id}_{\mathrm{Adm}_n}, \qquad \varepsilon : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{P}^{\langle -n \rangle} \circ \mathcal{P}^n,$$

whose components are $\eta_{(X,\vartheta)} = \vartheta$ *and* $\varepsilon_V = \mathrm{id}_V$.

Proof. We first make precise the (partial) inverse construction. For an n-admissible object (X, ϑ) with $\vartheta : \mathcal{P}^n(V) \xrightarrow{\cong} X$, define on objects

$$\mathcal{P}^{\langle -n \rangle}(X, \vartheta) \;:=\; V.$$

For a morphism $f : (X, \vartheta_X) \to (Y, \vartheta_Y)$ in Adm_n we assume the standard compatibility: there exists a (necessarily unique) map $u : V_X \to V_Y$ such that the square

$$\vartheta_Y \circ \mathcal{P}^n(u) \;=\; f \circ \vartheta_X \tag{3.1}$$

commutes; then set

$$\mathcal{P}^{\langle -n \rangle}(f) \;:=\; u.$$

Uniqueness of u in (3.1) follows because ϑ_Y is bijective and $\mathcal{P}^n$ is faithful: if $\mathcal{P}^n(u) = \mathcal{P}^n(u')$, then $u = u'$. Indeed, let $\iota_0 : V \to \mathcal{P}(V)$ be $v \mapsto \{v\}$ and inductively define $\iota_{k+1} : V \to \mathcal{P}^{k+2}(V)$ by $\iota_{k+1}(v) := \{\iota_k(v)\}$. For every $v \in V$,

$$\mathcal{P}^n(u)\big(\iota_{n-1}(v)\big) \;=\; \iota_{n-1}\big(u(v)\big) \;=\; \mathcal{P}^n(u')\big(\iota_{n-1}(v)\big),$$

hence $u(v) = u'(v)$.

We now establish the two displayed identities.

1) Object-level isomorphism $\mathcal{P}^n\big(\mathcal{P}^{\langle -n \rangle}(X, \vartheta)\big) \cong X$. By definition $\mathcal{P}^{\langle -n \rangle}(X, \vartheta) = V$, so the left-hand side is $\mathcal{P}^n(V)$. The map

$$\eta_{(X,\vartheta)} \;:=\; \vartheta : \;\mathcal{P}^n(V) \xrightarrow{\;\cong\;} X$$

is a bijection by n-admissibility and thus furnishes the required isomorphism. Its inverse is $\vartheta^{-1} : X \to \mathcal{P}^n(V)$, and one checks explicitly

$$\eta_{(X,\vartheta)} \circ \vartheta^{-1} \;=\; \vartheta \circ \vartheta^{-1} \;=\; \mathrm{id}_X, \qquad \vartheta^{-1} \circ \eta_{(X,\vartheta)} \;=\; \vartheta^{-1} \circ \vartheta \;=\; \mathrm{id}_{\mathcal{P}^n(V)}.$$

2) Object-level identity $\mathcal{P}^{\langle -n \rangle}\big(\mathcal{P}^n(V), \mathrm{id}\big) \;=\; V$. This holds by the very definition of $\mathcal{P}^{\langle -n \rangle}$ when the chosen witness is the identity bijection $\mathrm{id} : \mathcal{P}^n(V) \to \mathcal{P}^n(V)$; in that case the underlying base set extracted is V exactly (not merely up to isomorphism). Hence we may set $\varepsilon_V := \mathrm{id}_V$.

3) Naturality and functoriality. We verify that the families $\eta_{(X,\vartheta)} = \vartheta$ and $\varepsilon_V = \mathrm{id}_V$ are natural. Let $f : (X, \vartheta_X) \to (Y, \vartheta_Y)$ in Adm_n with $\mathcal{P}^{\langle -n \rangle}(f) = u : V_X \to V_Y$ as in (3.1). Then

$$f \circ \eta_{(X,\vartheta_X)} \;=\; f \circ \vartheta_X \;=\; \vartheta_Y \circ \mathcal{P}^n(u) \;=\; \eta_{(Y,\vartheta_Y)} \circ \mathcal{P}^n\big(\mathcal{P}^{\langle -n \rangle}(f)\big),$$

which is the naturality square for η. For ε, naturality is tautological since each ε_V is the identity on V.

Finally, $\mathcal{P}^{\langle -n \rangle}$ is a functor: if $g : (Y, \vartheta_Y) \to (Z, \vartheta_Z)$ corresponds to $v : V_Y \to V_Z$ (i.e., $\vartheta_Z \circ \mathcal{P}^n(v) = g \circ \vartheta_Y$), then

$$\vartheta_Z \circ \mathcal{P}^n(v \circ u) \;=\; \vartheta_Z \circ \mathcal{P}^n(v) \circ \mathcal{P}^n(u) \;=\; g \circ \vartheta_Y \circ \mathcal{P}^n(u) \;=\; g \circ f \circ \vartheta_X,$$

so by uniqueness $\mathcal{P}^{\langle -n \rangle}(g \circ f) = v \circ u = \mathcal{P}^{\langle -n \rangle}(g) \circ \mathcal{P}^{\langle -n \rangle}(f)$. Also $\mathcal{P}^{\langle -n \rangle}(\mathrm{id}_{(X,\vartheta)}) = \mathrm{id}_V$ by (3.1) with $f = \mathrm{id}_X$.

Putting the above together, we have produced natural isomorphisms $\eta : \mathcal{P}^n \circ \mathcal{P}^{\langle -n \rangle} \Rightarrow \mathrm{Id}_{\mathrm{Adm}_n}$ and $\varepsilon : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{P}^{\langle -n \rangle} \circ \mathcal{P}^n$ with components $\eta_{(X,\vartheta)} = \vartheta$ and $\varepsilon_V = \mathrm{id}_V$, respectively. In particular,

$$\mathcal{P}^n\big(\mathcal{P}^{\langle -n \rangle}(X,\vartheta)\big) \;\cong\; X \quad \text{naturally in } (X,\vartheta), \qquad \mathcal{P}^{\langle -n \rangle}\big(\mathcal{P}^n(V),\mathrm{id}\big) \;=\; V \quad \text{strictly.}$$

If one adopts the standard identification convention via the chosen witnesses ϑ, the first isomorphism may be treated as an equality, yielding $\mathcal{P}^n \circ \mathcal{P}^{\langle -n \rangle} = \mathrm{Id}$ and $\mathcal{P}^{\langle -n \rangle} \circ \mathcal{P}^n = \mathrm{Id}$ on the admissible subcategory. ☐

Remark 3.23 (Well-defined up to isomorphism without a chosen presentation). If only X is given (no ϑ), $\mathcal{P}^{\langle -n \rangle}(X)$ is defined *up to bijection* as any V with $\mathcal{P}^n(V) \cong X$. For finite X, this is possible iff $|X| \in \{T_n(t) : t \in \mathbb{N}\}$, and then $|V|$ is uniquely determined by $t = \log_2^{(n)} |X|$.

Remark 3.24 (Composition rule (when defined)). For integers a, b (positive, zero, or negative) with the intermediate object admissible, one has the partial law

$$\mathcal{P}^{\langle a \rangle} \circ \mathcal{P}^{\langle b \rangle} \;\cong\; \mathcal{P}^{\langle a+b \rangle},$$

which specializes to $\mathcal{P}^2 \circ \mathcal{P}^{\langle -2 \rangle} = \mathrm{Id}$ on 2-admissible sets, as requested.

Example 3.25 (Verifying $\mathcal{P}^2 \circ \mathcal{P}^{\langle -2 \rangle} = \mathrm{Id}$ on a concrete universe)**.** Let $V = \{1,2\}$, so $|V| = 2$. Then

$$\mathcal{P}(V) = \big\{\emptyset,\{1\},\{2\},\{1,2\}\big\}, \quad |\mathcal{P}(V)| = 4, \qquad X := \mathcal{P}^2(V) = \mathcal{P}\big(\mathcal{P}(V)\big), \quad |X| = 2^4 = 16.$$

Equip X with the identity presentation $\vartheta := \mathrm{id}_{\mathcal{P}^2(V)}$. By definition,

$$\mathcal{P}^{\langle -2 \rangle}(X,\vartheta) = V.$$

Applying $\mathcal{P}^2$,

$$\mathcal{P}^2\big(\mathcal{P}^{\langle -2 \rangle}(X,\vartheta)\big) = \mathcal{P}^2(V) = X,$$

so $\mathcal{P}^2 \circ \mathcal{P}^{\langle -2 \rangle} = \mathrm{Id}$ on this X. Cardinal check: $|X| = 16 = T_2(2)$ and $|\mathcal{P}^{\langle -2 \rangle}(X,\vartheta)| = 2 = \log_2^{(2)}(16)$.

Example 3.26 (A (-1)th powerset on a 1-admissible universe)**.** Let $W = \{a,b,c,d\}$, so $|W| = 4$ and $U := \mathcal{P}(W)$ has $|U| = 2^4 = 16$. With the identity presentation $\phi = \mathrm{id}_{\mathcal{P}(W)}$,

$$\mathcal{P}^{\langle -1 \rangle}(U,\phi) = W, \qquad \mathcal{P}\big(\mathcal{P}^{\langle -1 \rangle}(U,\phi)\big) = \mathcal{P}(W) = U.$$

Again $|U| = T_1(4)$ and $|\mathcal{P}^{\langle -1 \rangle}(U,\phi)| = 4 = \log_2(16)$.

Example 3.27 (Mixing signs: $\mathcal{P}^{\langle -1 \rangle} \circ \mathcal{P}^2 = \mathcal{P}$ on any base)**.** Take any finite B. Then $\mathcal{P}^2(B)$ is 1-admissible via the presentation $\psi : \mathcal{P}(\mathcal{P}(B)) \xrightarrow{\mathrm{id}} \mathcal{P}^2(B)$. Therefore

$$\mathcal{P}^{\langle -1 \rangle}\big(\mathcal{P}^2(B),\psi\big) \;=\; \mathcal{P}(B), \qquad \Rightarrow \qquad \big(\mathcal{P}^{\langle -1 \rangle} \circ \mathcal{P}^2\big)(B) \;=\; \mathcal{P}(B).$$

Numerically, if $|B| = t$ then $|\mathcal{P}^2(B)| = 2^{2^t} = T_2(t)$, and the output size is $|\mathcal{P}(B)| = 2^t$.

Example 3.28 ((−1)th powerset in practice: recovering primitive audience tags from a complete segment lattice)**.** **Setting.** Let the primitive audience tags be

$$V = \{\textsf{New},\ \textsf{Returning},\ \textsf{VIP}\}.$$

A *segment* is any subset of tags, so the catalog of all segments is

$$X \;:=\; \mathcal{P}(V) \qquad (\text{every } A \subseteq V \text{ is a segment}).$$

Equip X with the identity presentation $\vartheta := \mathrm{id}_{\mathcal{P}(V)} : \mathcal{P}(V) \xrightarrow{\cong} X$. Then X is 1-admissible and the negative powerset returns the tag basis:

$$\mathcal{P}^{\langle -1\rangle}(X,\vartheta) \;=\; V.$$

Verification by sizes. $|V| = 3 \Rightarrow |X| = |\mathcal{P}(V)| = 2^3 = 8$; moreover

$$\mathcal{P}\big(\mathcal{P}^{\langle -1\rangle}(X,\vartheta)\big) \;=\; \mathcal{P}(V) \;=\; X,$$

so $\mathcal{P} \circ \mathcal{P}^{\langle -1\rangle} = \mathrm{Id}$ on this X.

Operational meaning. When a marketing data store contains the *complete* segment lattice (all subsets of tags), $\mathcal{P}^{\langle -1\rangle}$ "peels off" one subset layer to recover the primitive tag vocabulary. This supports taxonomy migration (identify the atomic tags that generated every segment), deduplication, and consistent roll-ups (any change to a primitive tag deterministically propagates to all segments).

Example 3.29 ((−2)th powerset in practice: recovering feature flags from a catalog of test suites)**.** **Setting.** Let feature flags be

$$V = \{\textsf{Search},\ \textsf{Share},\ \textsf{Sync}\}.$$

Level 1 objects are *configurations* (subsets of features): $\mathcal{P}(V)$. Level 2 objects are *test suites* (sets of configurations): $\mathcal{P}^2(V) = \mathcal{P}(\mathcal{P}(V))$. Assume the experimentation registry contains the *complete* catalog of suites:

$$X \;:=\; \mathcal{P}^2(V).$$

With the identity presentation $\vartheta := \mathrm{id}_{\mathcal{P}^2(V)}$ we have that X is 2-admissible and

$$\mathcal{P}^{\langle -2\rangle}(X,\vartheta) \;=\; V.$$

Verification by sizes. $|V| = 3 \Rightarrow |\mathcal{P}(V)| = 2^3 = 8$ and

$$|X| = |\mathcal{P}^2(V)| = 2^{|\mathcal{P}(V)|} = 2^8 = 256.$$

Applying the inverse law on the admissible domain,

$$\mathcal{P}^2\big(\mathcal{P}^{\langle -2\rangle}(X,\vartheta)\big) \;=\; \mathcal{P}^2(V) \;=\; X.$$

Operational meaning. From the full suite catalog (policies over configurations), the (-2)th powerset recovers the underlying *feature flag basis*. This enables capability-mining (which atomic flags generate all suites?), impact analysis (which suites are affected by toggling a base feature?), and principled pruning (remove a flag ⇒ determine exactly which configurations/suites vanish).

3.4. Negative SuperHyperStructure

Negative SuperHyperStructure removes m lifts: if $(\mathcal{P}^m(Y),\ \mu^{[m]}) \cong (X, \circ)$, return (Y, μ); inverse inside lift towers on admissible presentations.

Definition 3.30 ((k, m)-admissible superhyperstructure and m-presentation)**.** Let $k \geqslant 0$ and $m \geqslant 1$. A superhyperstructure $(X, \circ)$ is called (k, m)-*admissible* if there exist a base set S, a hyperoperation μ on S, and an isomorphism of hyperstructures

$$\Phi :\ \big(\mathcal{P}^k(S),\ \mu^{[k]}\big)\ \xrightarrow{\cong}\ (X, \circ).$$

When, in addition, $k \geqslant m$, we call the tuple (S, μ, k, Φ) an m-*presentation* of $(X, \circ)$.

Remark 3.31 (Cardinal constraints (finite case)). If $|S| = s$ and $T_0(s) := s$, $T_{i+1}(s) := 2^{T_i(s)}$, then $|\mathcal{P}^k(S)| = T_k(s)$. Hence a finite $(X, \circ)$ can admit an m-presentation with level k only if $|X| = T_k(s)$ for some s; in that case any negative reduction by m levels will have carrier size $T_{k-m}(s)$.

Definition 3.32 (Negative SuperHyperStructure of order m)**.** Fix $m \geqslant 1$. On the groupoid of (k, m)-admissible superhyperstructures with chosen m-presentations, define the *Negative SuperHyperStructure of order* m by

$$\mathrm{NegSH}_m\big((X, \circ);\ S, \mu, k, \Phi\big)\ :=\ \big(\mathcal{P}^{k-m}(S),\ \mu^{[k-m]}\big), \qquad (k \geqslant m).$$

On a morphism $f : (X, \circ) \to (X', \circ')$ that is compatible with presentations via some $g : S \to S'$ in the sense that

$$\Phi' \circ \mathcal{P}^k(g)\ =\ f\ \circ \Phi \quad \text{and} \quad (\mathcal{P}^k(g) \times \mathcal{P}^k(g)) \text{ intertwines } \mu^{[k]} \text{ with } (\mu')^{[k]},$$

set $\mathrm{NegSH}_m(f) := \mathcal{P}^{k-m}(g)$.

Proposition 3.33 (Inverse laws within a lift tower)**.** *Let* $m \geqslant 1$ *and suppose* $(X, \circ)$ *admits an* m*-presentation* (S, μ, k, Φ) *with* $k \geqslant m$*; that is,* $\Phi : \mathcal{P}^k(S) \xrightarrow{\cong} X$ *is a bijection such that for all* $U, V \in \mathcal{P}^k(S)$,

$$\Phi\big(\mu^{[k]}(U, V)\big)\ =\ \Phi(U)\ \circ\ \Phi(V),$$

where $\mu^{[0]} := \mu$ *and* $\mu^{[i+1]} := \mathrm{Lift}\big(\mu^{[i]}\big)$ *for* $i \geqslant 0$*, and* $\mathcal{SH}^{(i)}(S, \mu) := \big(\mathcal{P}^i(S), \mu^{[i]}\big)$.

Then:

$$\mathrm{Lift}^{\,m}\Big(\mathrm{NegSH}_m\big((X, \circ); S, \mu, k, \Phi\big)\Big)\ \cong\ (X, \circ),$$

and for any (S, μ) *and any* $k \geqslant m$,

$$\mathrm{NegSH}_m\big(\mathcal{SH}^{(k)}(S, \mu);\ S, \mu, k, \mathrm{id}\,\big)\ =\ \mathcal{SH}^{(k-m)}(S, \mu).$$

In particular, on the admissible domain of m*-presented objects,*

$$\mathrm{Lift}^{\,m} \circ \mathrm{NegSH}_m\ \cong\ \mathrm{Id}, \qquad \mathrm{NegSH}_m \circ \mathrm{Lift}^{\,m}\ =\ \mathrm{Id}.$$

Proof. We first make explicit the constructions and their types.

By definition of the lift tower,

$$\mu^{[k]} = \underbrace{\mathrm{Lift}\circ\cdots\circ\mathrm{Lift}}_{m\text{ times}}\big(\mu^{[k-m]}\big) = \mathrm{Lift}^{m}\big(\mu^{[k-m]}\big), \qquad \mathcal{P}^{k}(S) = \mathcal{P}^{m}\big(\mathcal{P}^{k-m}(S)\big).$$

By the given m-presentation (S,μ,k,Φ), the operation $\circ$ on X is the transport of $\mu^{[k]}$ along Φ:

$$\forall\, U,V\in\mathcal{P}^{k}(S):\quad \Phi\big(\mu^{[k]}(U,V)\big) = \Phi(U)\circ\Phi(V). \qquad (\star)$$

Definition of the lowering functor. For an m-presentation (S,μ,k,Φ) of $(X,\circ)$ we set

$$\mathrm{NegSH}_{m}\big((X,\circ);S,\mu,k,\Phi\big) := \mathcal{SH}^{(k-m)}(S,\mu) = \big(\mathcal{P}^{k-m}(S),\ \mu^{[k-m]}\big).$$

Thus NegSH_{m} extracts the unique m-th "root" inside the fixed lift tower.

Claim 1. There is a (canonical) isomorphism

$$\mathrm{Lift}^{m}\Big(\mathrm{NegSH}_{m}\big((X,\circ);S,\mu,k,\Phi\big)\Big) \cong (X,\circ).$$

Proof of Claim 1. Applying m successive lifts to $\big(\mathcal{P}^{k-m}(S),\mu^{[k-m]}\big)$ yields by definition

$$\mathrm{Lift}^{m}\Big(\mathcal{P}^{k-m}(S),\mu^{[k-m]}\Big) = \Big(\mathcal{P}^{k}(S),\ \mathrm{Lift}^{m}\big(\mu^{[k-m]}\big)\Big) = \big(\mathcal{P}^{k}(S),\ \mu^{[k]}\big) = \mathcal{SH}^{(k)}(S,\mu).$$

The bijection $\Phi:\mathcal{P}^{k}(S)\xrightarrow{\cong}X$ is an isomorphism of hyperstructures by $(\star)$, since for all $U,V\in\mathcal{P}^{k}(S)$,

$$\Phi\big(\mu^{[k]}(U,V)\big) = \Phi(U)\circ\Phi(V),$$

and hence $\Phi:\big(\mathcal{P}^{k}(S),\mu^{[k]}\big)\to(X,\circ)$ preserves the operation and is bijective. Therefore

$$\mathrm{Lift}^{m}\Big(\mathrm{NegSH}_{m}\big((X,\circ);S,\mu,k,\Phi\big)\Big) \cong (X,\circ). \qquad \square_{\text{Claim 1}}$$

Claim 2. For any (S,μ) and $k\geqslant m$,

$$\mathrm{NegSH}_{m}\big(\mathcal{SH}^{(k)}(S,\mu);\ S,\mu,k,\mathrm{id}\big) = \mathcal{SH}^{(k-m)}(S,\mu).$$

Proof of Claim 2. Here the m-presentation of $\mathcal{SH}^{(k)}(S,\mu)$ is $(S,\mu,k,\mathrm{id}_{\mathcal{P}^{k}(S)})$, so by the very definition of NegSH_{m} we obtain

$$\mathrm{NegSH}_{m}\big(\mathcal{SH}^{(k)}(S,\mu);S,\mu,k,\mathrm{id}\big) = \big(\mathcal{P}^{k-m}(S),\ \mu^{[k-m]}\big) = \mathcal{SH}^{(k-m)}(S,\mu),$$

as a strict equality of hyperstructures (same carrier, same operation).

Finally, the stated "in particular" follows immediately: on m-presentable objects $(X,\circ)$ we have a canonical isomorphism $\mathrm{Lift}^{m}\circ\mathrm{NegSH}_{m}(X,\circ)\cong(X,\circ)$ witnessed by Φ (Claim 1), and on objects in the canonical tower $\mathcal{SH}^{(k-m)}(S,\mu)$ we have the strict identity $\mathrm{NegSH}_{m}\circ\mathrm{Lift}^{m}=\mathrm{Id}$ (Claim 2).

This completes the proof.

Remark 3.34 (Presentation-free version (up to isomorphism)). If only $(X, \circ)$ is given, define $\mathrm{NegSH}_m(X, \circ)$ *up to isomorphism* as any $(Y, \star)$ such that

$$\big(\mathcal{P}^m(Y),\ \star^{[m]}\big)\ \cong\ (X, \circ).$$

Existence demands both a carrier cardinal tower condition ($|X| = T_m(|Y|)$ in the finite case) and that the operation $\circ$ be lift-generated up to isomorphism.

Example 3.35 (Order-1 negative on a level-1 superhyperstructure)**.** Let $S = \{x, y, z\}$ and define a base hyperoperation

$$\mu(a, b) := \{a, b\}\ \subseteq S \qquad (a, b \in S).$$

The level-1 superhyperstructure is

$$(X, \circ) := \mathcal{SH}^{(1)}(S, \mu)\ =\ \big(\mathcal{P}(S),\ \mu^{[1]}\big),$$

where

$$\mu^{[1]}(A, B)\ =\ \big\{\ \{a, b\}\ \big|\ a \in A,\ b \in B\ \big\}\ \in\ \mathcal{P}\big(\mathcal{P}(S)\big).$$

Take the 1-presentation $(S, \mu, k{=}1, \Phi{=}\mathrm{id})$. Then

$$\mathrm{NegSH}_1\big((X, \circ);\ S, \mu, 1, \mathrm{id}\big)\ =\ \mathcal{SH}^{(0)}(S, \mu)\ =\ (S, \mu).$$

Concrete computation: choose $A = \{x\}$, $B = \{y, z\}$. Then

$$\mu^{[1]}(A, B) = \big\{\{x, y\}, \{x, z\}\big\}.$$

Applying one lift to the negative result (S, μ) reproduces this output, hence $\mathrm{Lift}\big(\mathrm{NegSH}_1(X, \circ)\big) \cong (X, \circ)$. Cardinals: $|S| = 3$, $|\mathcal{P}(S)| = 2^3 = 8$; one negative step reduces $8 \mapsto 3$ at the carrier level.

Example 3.36 (Order-2 negative on a level-2 superhyperstructure (explicit numbers))**.** Let $S = \{0, 1\}$, so $|S| = 2$. Define $\mu(u, v) := \{u, v\}$ on S. Then

$$\mathcal{P}(S) = \big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\big\}, \quad |\mathcal{P}(S)| = 4,$$

and

$$(X, \circ) := \mathcal{SH}^{(2)}(S, \mu) = \big(\mathcal{P}^2(S), \mu^{[2]}\big), \qquad |X| = |\mathcal{P}^2(S)| = 2^4 = 16.$$

Take the 2-presentation $(S, \mu, k{=}2, \Phi{=}\mathrm{id})$. By definition,

$$\mathrm{NegSH}_2\big((X, \circ);\ S, \mu, 2, \mathrm{id}\big)\ =\ \big(\mathcal{P}^0(S), \mu^{[0]}\big)\ =\ (S, \mu),$$

and

$$\mathrm{Lift}^2\big(\mathrm{NegSH}_2(X, \circ)\big)\ =\ \mathrm{Lift}^2(S, \mu)\ =\ \big(\mathcal{P}^2(S), \mu^{[2]}\big)\ =\ (X, \circ).$$

A concrete input:

$$\mathcal{A} = \big\{\{0\}, \{0, 1\}\big\}, \qquad \mathcal{B} = \big\{\{1\}\big\}\ \in\ \mathcal{P}\big(\mathcal{P}(S)\big).$$

Then

$$\mu^{[2]}(\mathcal{A}, \mathcal{B})\ =\ \big\{\ \mu^{[1]}(A, B)\ \big|\ A \in \mathcal{A},\ B \in \mathcal{B}\ \big\}\ =\ \Big\{\ \{\{0, 1\}\},\ \{\{0, 1\}, \{1\}\}\ \Big\}\ \in\ \mathcal{P}^3(S).$$

Descending two levels by NegSH_2 recovers the base (S, μ); ascending back by two lifts reproduces the same output as above. Cardinals verify the identities:

$$2 \xrightarrow{\mathcal{P}} 4 \xrightarrow{\mathcal{P}} 16, \qquad 16 \xrightarrow{\mathrm{NegSH}_2} 2, \qquad 2 \xrightarrow{\mathrm{Lift}^2} 16.$$

Example 3.37 (Presentation via transport of structure (isomorphic realization)). Let Y be a 16-element set and fix a bijection $b : \mathcal{P}^2(S) \xrightarrow{\cong} Y$ from Example 2. Transport $\mu^{[2]}$ along b:

$$\alpha \circ \beta \;:=\; b\Big(\mu^{[2]}\big(b^{-1}(\alpha),\, b^{-1}(\beta)\big)\Big), \qquad \alpha, \beta \in Y.$$

Then $(Y, \circ)$ admits the 2-presentation $(S, \mu, k{=}2, \Phi{=}b)$, hence

$$\mathrm{NegSH}_2\big((Y,\circ);\; S, \mu, 2, b\big) \;=\; (S, \mu),$$

and $\mathrm{Lift}^2\big(\mathrm{NegSH}_2(Y,\circ)\big) \;\cong\; (Y,\circ)$. This shows presentation-independence up to isomorphism.

Example 3.38 (Order-2 Negative SuperHyperStructure for access-governance playbooks). **Setting.** Let the primitive permissions be

$$S = \{\text{read } (r),\ \text{write } (w),\ \text{deploy } (d)\}.$$

Define the base hyperoperation $\mu : S \times S \to \mathcal{P}(S)$ by

$$\mu(s,t) := \{s,t\} \qquad (s, t \in S).$$

Its canonical lifts are $\mu^{[k]} : \mathcal{P}^k(S) \times \mathcal{P}^k(S) \to \mathcal{P}^{k+1}(S)$.

Given superlevel (depth $k = 2$). A *playbook* is a set of roles (roles are subsets of S), so the carrier is $\mathcal{P}^2(S)$ and the level-2 superhyperstructure is

$$(X, \circ) \;:=\; \mathcal{SH}^{(2)}(S,\mu) \;=\; \big(\mathcal{P}^2(S),\ \mu^{[2]}\big).$$

Take two concrete playbooks

$$\mathcal{A} = \big\{\{r\},\{w\}\big\}, \qquad \mathcal{B} = \big\{\{w\},\{d\}\big\} \;\in\; \mathcal{P}^2(S).$$

Then

$$\mu^{[2]}(\mathcal{A},\mathcal{B}) \;=\; \big\{\, \mu^{[1]}(\{r\},\{w\}),\ \mu^{[1]}(\{r\},\{d\}),\ \mu^{[1]}(\{w\},\{w\}),\ \mu^{[1]}(\{w\},\{d\}) \,\big\},$$

and since $\mu^{[1]}(A,B) = \{\{s,t\} \mid s \in A,\ t \in B\}$, one obtains the explicit family

$$\mu^{[2]}(\mathcal{A},\mathcal{B}) \;=\; \big\{\, \{\{r,w\}\},\ \{\{r,d\}\},\ \{\{w\}\},\ \{\{w,d\}\} \,\big\} \;\in\; \mathcal{P}^3(S).$$

Order-2 negative step. Equip $(X, \circ)$ with the identity presentation $\Phi = \mathrm{id}_{\mathcal{P}^2(S)}$. The order-2 Negative SuperHyperStructure (within the lift tower) is

$$\mathrm{NegSH}_2\big((X,\circ);\, S, \mu, 2, \Phi\big) \;=\; \big(\mathcal{P}^0(S), \mu^{[0]}\big) \;=\; (S,\mu).$$

Verification. By construction,

$$\mathrm{Lift}^2\Big(\mathrm{NegSH}_2(X,\circ)\Big) \;=\; \mathcal{SH}^{(2)}(S,\mu) \;=\; (X,\circ).$$

Operational meaning. From a complete catalog of playbooks (depth 2), the order-2 negative step recovers the *atomic permission hyperstructure* that generated them. This enables (i) schema recovery of primitive capabilities, (ii) separation-of-duties checks at the atomic layer with results lifted upward, and (iii) precise impact analysis for proposed edits to S (which roles/playbooks are affected).

Example 3.39 (Order-3 Negative SuperHyperStructure for menu-program governance). **Setting.** Let $S = \{\mathsf{Prep},\ \mathsf{Cook},\ \mathsf{Serve}\}$ be atomic kitchen tasks. Define the base hyperoperation

$$\mu(A, B) := \{A,\ B,\ A \cup B\} \qquad (A, B \subseteq S),$$

encoding non-deterministic choices: keep either task set or merge them.

Hierarchy. Level 1 (*recipes*) are bundles of tasks: elements of $\mathcal{P}(S)$. Level 2 (*menus*) are sets of recipes: elements of $\mathcal{P}^2(S)$. Level 3 (*seasonal programs*) are sets of menus: elements of $\mathcal{P}^3(S)$. Thus the depth-3 superhyperstructure is

$$(Y, \star) \ := \ \mathcal{SH}^{(3)}(S, \mu) \ = \ \big(\mathcal{P}^3(S),\ \mu^{[3]}\big).$$

For concreteness, take two seasonal programs

$$\mathcal{M}_{\text{Spring}} = \big\{\{\{\mathsf{Prep}\}, \{\mathsf{Cook}\}\},\ \{\{\mathsf{Serve}\}\}\big\}, \quad \mathcal{M}_{\text{Summer}} = \big\{\{\{\mathsf{Cook}\}\},\ \{\{\mathsf{Prep}\}, \{\mathsf{Serve}\}\}\big\},$$

each an element of $\mathcal{P}^3(S)$ (sets of menus; each menu is a set of recipes; each recipe is a subset of S). The level-3 operation $\mu^{[3]}(\mathcal{M}_{\text{Spring}}, \mathcal{M}_{\text{Summer}})$ produces a *family of families of menus* collecting outcomes like keeping either program or merging corresponding menus.

Order-3 negative step. With the identity presentation $\Phi = \mathrm{id}_{\mathcal{P}^3(S)}$, the order-3 Negative SuperHyperStructure gives

$$\mathrm{NegSH}_3\big((Y, \star);\, S, \mu, 3, \Phi\big) \ = \ (S, \mu).$$

Verification. Applying three lifts recovers the given seasonal-program layer:

$$\mathrm{Lift}^3\Big(\mathrm{NegSH}_3(Y, \star)\Big) \ = \ \mathcal{SH}^{(3)}(S, \mu) \ = \ (Y, \star).$$

Operational meaning. From a registry of seasonal programs (policy over menus over recipes), the order-3 negative step identifies the *atomic task hyperstructure*. This supports: (i) workforce planning by tracing each program back to required atomic tasks; (ii) compliance auditing—e.g., ensure Prep and Serve are never merged without Cook; (iii) controlled refactoring of programs by editing at the atomic task layer and lifting changes deterministically to menus and programs.

3.5. Complex-height powerset

A *Complex-height powerset* extends iterated powersets to complex exponents using operator theory, enabling continuous interpolation—including fractional and imaginary layers—between discrete subset hierarchies.

Definition 3.40 (Iterated powerset and a canonical encoding). Let $U = \{u_1, \dots, u_h\}$ be a nonempty finite set. For integers $k \geqslant 0$ define

$$\mathcal{P}^0(U) := U, \qquad \mathcal{P}^{k+1}(U) := \mathcal{P}\big(\mathcal{P}^k(U)\big).$$

Fix, for each k, a *canonical bijection* $\phi_k : \mathcal{P}^k(U) \xrightarrow{\cong} \{0, 1, \dots, T_k(h) - 1\}$ as follows:

- $\phi_0(u_j) := j - 1$.

- Given ϕ_{k-1}, encode each $A \in \mathcal{P}^k(U)$ (a subset of $\mathcal{P}^{k-1}(U)$) by the binary integer

$$\phi_k(A) \;:=\; \sum_{x \in \mathcal{P}^{k-1}(U)} \mathbf{1}_A(x)\, 2^{\phi_{k-1}(x)}.$$

Here $T_0(h) := h$ and $T_{i+1}(h) := 2^{T_i(h)}$, so $|\mathcal{P}^k(U)| = T_k(h)$. Set $\Sigma_k := \{0, \dots, T_k(h) - 1\}$ and let $\Sigma := \bigsqcup_{k \geqslant 0} \Sigma_k$.

Definition 3.41 (Level-raising map and Koopman isometry)**.** Define the *level-raising map* $S : \Sigma \to \Sigma$ by

$$S\big(\phi_k(x)\big) \;:=\; \phi_{k+1}\big(\,\mathcal{P}(x)\,\big), \qquad x \in \mathcal{P}^k(U).$$

Thus S maps a code of x at level k to the code of its full powerset $\mathcal{P}(x)$ at level $k+1$. S is everywhere-defined and injective (but not surjective). Let $\ell^2(\Sigma)$ denote the Hilbert space of square-summable complex functions on Σ (counting measure). The *Koopman isometry* associated to S is

$$\mathbf{K} :\ \ell^2(\Sigma) \to \ell^2(\Sigma), \qquad (\mathbf{K}f)(\sigma) = f\big(S^{-1}\sigma\big) \quad (\text{with } f\restriction_{\Sigma \setminus S(\Sigma)} := 0).$$

Definition 3.42 (Complex-height powerset via unitary dilation (including $\mathcal{P}^{\,i}$))**.** By the Sz.-Nagy dilation theorem, there exist a Hilbert space $\mathcal{H} \supset \ell^2(\Sigma)$, an isometric embedding $J : \ell^2(\Sigma) \hookrightarrow \mathcal{H}$, and a unitary $\mathbf{U} : \mathcal{H} \to \mathcal{H}$ such that

$$\mathbf{K}^n \;=\; J^* \, \mathbf{U}^n \, J \qquad \text{for all } n \in \mathbb{N}.$$

Fix the principal functional calculus for $\mathbf{U}$. For any $z \in \mathbb{C}$ we define the *complex-height powerset operator* on observables by

$$\mathcal{P}^{\,z} \ \ (\text{on observables}) \ \ := \ J^* \, \mathbf{U}^{\,z} \, J \ : \ \ell^2(\Sigma) \to \ell^2(\Sigma)$$

and on finite signed measures by the adjoint $(\mathcal{P}^{\,z})^* = J^*(\mathbf{U}^{\bar{z}})^* J$. This extension satisfies the semigroup law $\mathcal{P}^{\,z+w} = \mathcal{P}^{\,z} \circ \mathcal{P}^{\,w}$ and agrees with the classical iterates on nonnegative integers: $\mathcal{P}^{\,n} = \mathbf{K}^n$ for all $n \in \mathbb{N}$. In particular,

$$\mathcal{P}^{\,it} \;=\; J^* \, e^{\,it \log \mathbf{U}} \, J \quad (t \in \mathbb{R})$$

gives the *imaginary-height powerset* as a bounded normal operator interpolating continuously between powerset levels.

Example 3.43 (Fully explicit encoding and action for $U = \{a, b\}$)**.** Let $U = \{a, b\}$ $(h = 2)$. Then

$$\mathcal{P}^0(U) = \{a, b\}, \quad \mathcal{P}^1(U) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \quad \mathcal{P}^2(U) = \mathcal{P}(\mathcal{P}(U)) \text{ (size 16).}$$

Canonical codes:

$$\phi_0(a) = 0,\ \phi_0(b) = 1; \qquad \phi_1(\emptyset) = 0,\ \phi_1(\{a\}) = 1,\ \phi_1(\{b\}) = 2,\ \phi_1(\{a, b\}) = 3.$$

For $A \subseteq \mathcal{P}(U)$, $\phi_2(A) = \sum_{Y \in \mathcal{P}(U)} \mathbf{1}_A(Y)\, 2^{\phi_1(Y)}$. The level-raising map S sends

$$S\big(\phi_0(a)\big) = \phi_1\big(\mathcal{P}(a)\big) = \phi_1\big(\{\emptyset, \{a\}\}\big) = 1 + 2^0 = 2,$$

$$S\big(\phi_1(\{a\})\big) = \phi_2\big(\mathcal{P}(\{a\})\big) = \phi_2\big(\{\emptyset,\{\{a\}\}\}\big) = 1 + 2^0 = 2,$$

and similarly for b and $\{a,b\}$. In particular S is injective and raises the layer by one while encoding complete powersets. On observables $f \in \ell^2(\Sigma)$, the classical integer iterate $\mathcal{P}^2 = \mathbf{K}^2$ acts as

$$(\mathbf{K}^2 f)\big(\phi_2(A)\big) \;=\; f\big(\phi_0(x)\big) \quad \text{whenever } A = \mathcal{P}(\mathcal{P}(x)).$$

To apply the imaginary-height operator $\mathcal{P}^{\,i}$ concretely, choose a unitary dilation $(\mathcal{H}, \mathbf{U}, J)$. A standard choice realizes $\mathcal{H} = \ell^2(\mathbb{Z})$ with the bilateral shift $(\mathbf{U}g)(n) = g(n-1)$ and embeds the *powerset orbit* of a seed $x \in U$ as the chain

$$\phi_0(x) \xrightarrow{S} \phi_1(\mathcal{P}(x)) \xrightarrow{S} \phi_2(\mathcal{P}^2(x)) \xrightarrow{S} \cdots,$$

mapping its codes to the basis vectors $\{e_0, e_1, e_2, \dots\}$ of $\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Z})$. Then, for any $t \in \mathbb{R}$ and indicator $\delta_{\phi_0(x)}$, one obtains the explicit formula

$$\mathcal{P}^{\,it}\,\delta_{\phi_0(x)} \;=\; J^*\, e^{\,it\log \mathbf{U}}\, J\,\delta_{\phi_0(x)} \;=\; \sum_{m \geqslant 0} k_t(m)\;\delta_{\phi_m(\mathcal{P}^m(x))},$$

where $k_t(m)$ are the (computable) matrix coefficients of $e^{\,it\log \mathbf{U}}$ in the shift basis (e.g. via Fourier transform on the unit circle). Thus $\mathcal{P}^{\,it}$ produces a *structured, norm-controlled mixture* over the successive powersets $\mathcal{P}^m(x)$ with weights $k_t(m)$ depending continuously on t.

Example 3.44 (Detailed real-life scenario: staged rollouts across powerset layers)**.** Consider a feature platform with base features $U = \{\mathsf{Search}, \mathsf{Share}, \mathsf{Sync}\}$. Interpret levels:

$$\underbrace{\mathcal{P}^0(U)}_{\text{individual features}} \;\xrightarrow{\mathcal{P}}\; \underbrace{\mathcal{P}^1(U)}_{\text{user configurations}} \;\xrightarrow{\mathcal{P}}\; \underbrace{\mathcal{P}^2(U)}_{\text{test suites}} \;\xrightarrow{\mathcal{P}}\; \underbrace{\mathcal{P}^3(U)}_{\text{catalogs of suites}}\,.$$

Suppose a product team currently evaluates two configurations

$$C_1 = \{\mathsf{Search}, \mathsf{Share}\}, \qquad C_2 = \{\mathsf{Search}, \mathsf{Sync}\}$$

and two test suites

$$S_1 = \{C_1\}, \qquad S_2 = \{C_1, C_2\}.$$

With the canonical encoding ϕ_k, these are concrete codes in Σ_1 and Σ_2. Let $w \in \ell^1(\Sigma)$ summarize *traffic weights*: for example

$$w = \underbrace{0.40\,\delta_{\phi_1(C_1)} + 0.40\,\delta_{\phi_1(C_2)}}_{\text{config-level traffic}} \;+\; \underbrace{0.20\,\delta_{\phi_2(S_2)}}_{\text{suite-level traffic}}\,.$$

(Here δ_σ is the unit mass at code σ.) To *gently* ramp experimentation one layer upward without an abrupt jump, evolve w by the imaginary-height powerset on measures:

$$w_t \;:=\; (\mathcal{P}^{\,it})^* w \;=\; J^*\, e^{-\,it\log \mathbf{U}}\, J\, w \qquad (t \geqslant 0).$$

Operationally this performs a smooth, reversible reallocation that *spreads* mass from Σ_1 toward Σ_2 (and, for larger t, into Σ_3), with exact, reproducible weights given by the matrix coefficients of $e^{-it\log \mathbf{U}}$ in the chosen dilation. In a staged rollout one may set, e.g.,

$$t = 0.2: \text{ tiny spillover to suites,}$$

$$t = 0.5: \text{ balanced weight between configurations and suites,}$$

$$t = 1.0: \text{ stronger emphasis on suite-level evaluation.}$$

Because $\|\mathcal{P}^{it}\| = 1$ (as a compression of a unitary), total traffic is preserved exactly: $\|w_t\|_1 = \|w\|_1$, and the dependence on t is continuous, enabling fine-grained control.

Example 3.45 (Hierarchical privacy/aggregation with explicit counts)**.** Let event counts be collected at three adjacent layers for the same cohort:

$$c_0 = \sum_{x\in\mathcal{P}^0(U)} \alpha_x\, \delta_{\phi_0(x)}, \quad c_1 = \sum_{A\in\mathcal{P}^1(U)} \beta_A\, \delta_{\phi_1(A)}, \quad c_2 = \sum_{B\in\mathcal{P}^2(U)} \gamma_B\, \delta_{\phi_2(B)}.$$

Set $f := c_0 + c_1 + c_2 \in \ell^2(\Sigma)$ (after appropriate scaling). Choose a modest $t > 0$ and form

$$\tilde{f} := \mathcal{P}^{it} f = J^* e^{it\log \mathbf{U}} J f.$$

Then $\tilde{f}$ is an *automatically balanced* aggregation that diffuses signal upward (reducing variance) while retaining a calibrated contribution from lower layers; increasing t yields stronger aggregation. If regulation requires reporting only at the configuration layer, simply take $\tilde{f}\restriction_{\Sigma_1}$, which is obtained by reading off the corresponding coordinates in $\tilde{f}$—no ad hoc smoothing kernel needs to be hand-designed.

3.6. Complex-height SuperHyperStructure

A *Complex-height SuperHyperStructure* extends superhyperstructures to complex exponents, enabling fractional and imaginary lifts, smoothly interpolating between hierarchical powerset layers with preserved hyperoperations.

Definition 3.46 (Complex-height SuperHyperStructure)**.** Let S be a nonempty finite base set and let $\mu : S \times S \to \mathcal{P}(S)$ be a base hyperoperation. Write $\mu^{[k]} : \mathcal{P}^k(S) \times \mathcal{P}^k(S) \to \mathcal{P}^{k+1}(S)$ for its canonical k-fold lift (defined on Dirac inputs by collecting the set-valued outputs and then extended bilinearly). For each level $k \geqslant 0$ define the bilinear map

$$\Omega_k : \ell^2(\Sigma_k) \times \ell^2(\Sigma_k) \longrightarrow \ell^2(\Sigma_{k+1}) \quad \text{by} \quad \Omega_k(\delta_A, \delta_B) := \sum_{C\in\mu^{[k]}(A,B)} \delta_C,$$

and extend by bilinearity and continuity.

For any complex $z \in \mathbb{C}$, the *Complex-height SuperHyperStructure* generated by (S, μ) is the pair

$$\mathcal{SH}^{(z)}(S,\mu) := \big(\, \ell^2(\Sigma)\, ,\, \odot^{[z]}\, \big),$$

whose (observable-level) hyperoperation $\odot^{[z]} : \ell^2(\Sigma) \times \ell^2(\Sigma) \to \ell^2(\Sigma)$ is

$$\odot^{[z]}(F, G) := \mathcal{T}_{k+1-z}\Big(\, \Omega_k\big(\, \Pi_k \mathcal{T}_{z-k} F\, ,\, \Pi_k \mathcal{T}_{z-k} G\, \big)\, \Big), \quad k \in \mathbb{Z}_{\geqslant 0}.$$

This definition is independent of the chosen k (by $\mathcal{T}_{a+b} = \mathcal{T}_a\mathcal{T}_b$ and the lift law $\mu^{[k+1]} = \mathrm{Lift}(\mu^{[k]})$), and for integer heights $z = n \in \mathbb{Z}_{\geqslant 0}$ it reduces to the classical $k = n$ level operation:

$$\odot^{[n]}(F, G) \;=\; \Omega_n\big(\Pi_n F, \Pi_n G\big) \in \ell^2(\Sigma_{n+1}).$$

Example 3.47 (Blue/green release blending at imaginary height $z = it$)**.** Let $S = \{\text{Auth}, \text{Billing}, \text{Analytics}\}$ be deployable services and define the base hyperoperation

$$\mu(A, B) := \{A,\ B,\ A \cup B\} \subseteq \mathcal{P}(S) \qquad (A, B \subseteq S).$$

Thus $\mu^{[1]}$ acts on bundles of services, producing a family of candidate outcomes in $\mathcal{P}^2(S)$.

Scenario. Two candidate bundles are

$$A = \{\text{Auth}\}, \qquad B = \{\text{Billing}, \text{Analytics}\} \;\in\; \mathcal{P}(S).$$

We regard the (unit-mass) observables $F = \delta_A$ and $G = \delta_B$ as elements of $\ell^2(\Sigma)$ concentrated on level $k = 1$. Choose a purely imaginary height $z = it$ $(t > 0)$ to *soften* decisions between levels without an abrupt jump.

Computation. With $k = 1$ in the boxed formula,

$$\odot^{[it]}(F, G) \;=\; \mathcal{T}_{1-it}\Big(\, \Omega_1\big(\, \Pi_1\mathcal{T}_{it-1}\delta_A \,,\; \Pi_1\mathcal{T}_{it-1}\delta_B \,\big) \Big).$$

By the Complex-height powerset, $\Pi_1\mathcal{T}_{it-1}\delta_A = \sum_{m\geqslant 0} c_m(t)\, \delta_{A_m}$ and $\Pi_1\mathcal{T}_{it-1}\delta_B = \sum_{n\geqslant 0} d_n(t)\, \delta_{B_n}$, where A_0=A, B_0=B and the coefficients $c_m(t), d_n(t)$ are the matrix entries of $\mathcal{T}_{it-1}$ restricted to level 1 (computable from the chosen dilation). Bilinearity gives

$$\Omega_1\Big(\sum_m c_m\delta_{A_m}, \sum_n d_n\delta_{B_n}\Big) \;=\; \sum_{m,n} c_m d_n \sum_{C\in\mu^{[1]}(A_m, B_n)} \delta_C \;\in\; \ell^2(\Sigma_2).$$

Finally, $\mathcal{T}_{1-it}$ transports this family back to the complex height $z = it$. In words, $\odot^{[it]}$ outputs a *structured mixture of candidate playbooks* $\{A_m, B_n, A_m \cup B_n\}$, with weights depending continuously on t; for $t \to 0$ we recover the classical set $\{A, B, A \cup B\}$, while larger t yields a smoother, more conservative blending across adjacent layers. Operationally, this provides a principled knob to interpolate between *keeping* either bundle and *merging* them, while diffusing decision mass across nearby hierarchy levels (roles ↔ policies).

Example 3.48 (Incident response across hierarchy at mixed height $z = \alpha + it$)**.** Let $S = \{\text{Investigate}, \text{Patch}, \text{Rollback}\}$ be base actions. Define the base hyperoperation

$$\mu(A, B) := \{\, A,\ B,\ A \cup B,\ A \cap B \,\} \qquad (A, B \subseteq S),$$

so that at level $k = 0$ it proposes *parallel*, *merged*, or *common* actions. We want a playbook that balances individual actions (level 0), bundles (level 1), and sets of bundles (level 2).

Pick a complex height $z = \alpha + it$ with $0 < \alpha < 1$ and $t \in \mathbb{R}$. Given two incident cues represented as level-0 Dirac observables $F = \delta_{\{\text{Investigate}\}}$ and $G = \delta_{\{\text{Patch}\}}$, use $k = 0$ in the definition:

$$\odot^{[\alpha+it]}(F, G) \;=\; \mathcal{T}_{1-(\alpha+it)}\Big(\, \Omega_0\big(\, \Pi_0\mathcal{T}_{\alpha+it}\, F,\; \Pi_0\mathcal{T}_{\alpha+it}\, G \,\big) \Big).$$

Here $\Pi_0\mathcal{T}_{\alpha+it}$ spreads each atomic cue into a controlled combination of nearby level-0 atoms (imaginary component t) while softly biasing toward level 1 (real component α). Applying Ω_0 enumerates the candidate action families $\{\mathsf{Investigate}\}$, $\{\mathsf{Patch}\}$, $\{\mathsf{Investigate}, \mathsf{Patch}\}$, and $\emptyset$ (with complex-analytic weights). The final transport $\mathcal{T}_{1-(\alpha+it)}$ lifts them to the target height, yielding a *graded playbook observable* that can be thresholded into concrete procedures.

Small α keeps suggestions close to single actions; larger α favors bundle-level playbooks. The parameter t adds reversible smoothing across adjacent levels, stabilizing recommendations when signals are noisy. When $z \in \mathbb{Z}_{\geqslant 0}$, we exactly recover the classical superhyperstructural recommendations at that integer depth.

Notation 3.49. Throughout, (S, μ) is a finite hyperalgebra with canonical lifts $\mu^{[k]} : \mathcal{P}^k(S) \times \mathcal{P}^k(S) \to \mathcal{P}^{k+1}(S)$, the level sets $\Sigma_k := \mathcal{P}^k(S)$, the Hilbert grading $\ell^2(\Sigma) := \bigoplus_{k\geqslant 0} \ell^2(\Sigma_k)$, and the bilinear maps $\Omega_k : \ell^2(\Sigma_k) \times \ell^2(\Sigma_k) \to \ell^2(\Sigma_{k+1})$ defined on Dirac masses by

$$\Omega_k(\delta_A, \delta_B) := \sum_{C \in \mu^{[k]}(A,B)} \delta_C, \quad \text{and extended bilinearly/continuously.}$$

We also fix a height–shift family $(\mathcal{T}_z)_{z\in\mathbb{C}}$ on $\ell^2(\Sigma)$ such that

$$\mathcal{T}_0 = \mathrm{Id}, \qquad \mathcal{T}_{a+b} = \mathcal{T}_a\mathcal{T}_b, \qquad \mathcal{T}_1\big(\ell^2(\Sigma_k)\big) \subseteq \ell^2(\Sigma_{k+1}),$$

and Π_k denotes the orthogonal projection onto $\ell^2(\Sigma_k)$. The complex–height product $\odot^{[z]}$ is

$$\odot^{[z]}(F, G) := \mathcal{T}_{k+1-z}\Big(\Omega_k\big(\Pi_k \mathcal{T}_{z-k} F \,,\, \Pi_k \mathcal{T}_{z-k} G \big) \Big), \quad k \in \mathbb{Z}_{\geqslant 0},$$

which is independent of k and reduces to Ω_n at integer heights $z = n \in \mathbb{Z}_{\geqslant 0}$.

Theorem 3.50 (Boundedness and well-definedness)**.** *For each $z \in \mathbb{C}$, the map*

$$\odot^{[z]} : \ell^2(\Sigma) \times \ell^2(\Sigma) \longrightarrow \ell^2(\Sigma)$$

is a continuous bilinear operator. More precisely, if

$$d_k := \max_{A,B\in\Sigma_k} \big|\mu^{[k]}(A, B)\big| \qquad (k \geqslant 0),$$

then for every $k \in \mathbb{Z}_{\geqslant 0}$ and $z \in \mathbb{C}$,

$$\big\|\odot^{[z]}(F, G)\big\|_2 \leqslant \|\mathcal{T}_{k+1-z}\| \, \|\mathcal{T}_{z-k}\|^2 \sqrt{d_k} \, \|F\|_2 \, \|G\|_2.$$

In particular, $\odot^{[z]}$ is bounded and independent of the index k used in its definition.

Proof. Fix k. Since S is finite, Σ_k and Σ_{k+1} are finite sets, hence all linear maps between the corresponding finite–dimensional Hilbert spaces are bounded. Define the linearization $\widetilde{\Omega}_k : \ell^2(\Sigma_k) \otimes_2 \ell^2(\Sigma_k) \to \ell^2(\Sigma_{k+1})$ by $\widetilde{\Omega}_k(\delta_A \otimes \delta_B) := \Omega_k(\delta_A, \delta_B)$ and extend linearly; its columns (indexed by (A, B)) have at most d_k nonzero entries, each equal to 1. Therefore

$$\|\widetilde{\Omega}_k\| \leqslant \sqrt{d_k},$$

e.g. by the column–sparsity bound for operator norms (each column has ℓ^2–norm $\sqrt{d_k}$ and the operator norm is bounded by the maximal column ℓ^2–norm). By bilinearity, for $f, g \in \ell^2(\Sigma_k)$ we have

$$\|\Omega_k(f,g)\|_2 \ \leqslant\ \|\widetilde{\Omega}_k\|\, \|f\|_2\, \|g\|_2 \ \leqslant\ \sqrt{d_k}\, \|f\|_2\, \|g\|_2.$$

Applying this with $f = \Pi_k \mathcal{T}_{z-k} F$ and $g = \Pi_k \mathcal{T}_{z-k} G$ and then composing with $\mathcal{T}_{k+1-z}$ yields

$$\left\|\odot^{[z]}(F,G)\right\|_2 \ \leqslant\ \|\mathcal{T}_{k+1-z}\|\, \sqrt{d_k}\, \|\Pi_k \mathcal{T}_{z-k}\|\, \|F\|_2\, \|\Pi_k \mathcal{T}_{z-k}\|\, \|G\|_2.$$

Since $\|\Pi_k\| \leqslant 1$ and $\|\Pi_k \mathcal{T}_{z-k}\| \leqslant \|\mathcal{T}_{z-k}\|$, the displayed bound follows. Independence of k is part of the construction (the two expressions agree by the lift law $\mu^{[k+1]} = \mathrm{Lift}(\mu^{[k]})$ and the semigroup identity $\mathcal{T}_{a+b} = \mathcal{T}_a \mathcal{T}_b$). Continuity in each argument follows from boundedness; bilinearity is immediate from bilinearity of Ω_k and linearity of $\mathcal{T}_\cdot, \Pi_k$. ☐

Theorem 3.51 (Height–covariance (conjugacy law)). *For every $z \in \mathbb{C}$ and every integer $r \in \mathbb{Z}$ one has*

$$\mathcal{T}_r\big(F \odot^{[z]} G\big) \ =\ \big(\mathcal{T}_r F\big) \ \odot^{[z+r]}\ \big(\mathcal{T}_r G\big), \qquad F, G \in \ell^2(\Sigma).$$

Proof. Fix any $k \geqslant 0$ and compute using the definition with index k on the left and index $k + r$ on the right. Using $\mathcal{T}_{a+b} = \mathcal{T}_a \mathcal{T}_b$ and $\Pi_{k+r} \mathcal{T}_r = \mathcal{T}_r \Pi_k$ (the latter is the grading–compatibility of the shift), we obtain

$$\begin{aligned}
\mathcal{T}_r\big(F \odot^{[z]} G\big) &= \mathcal{T}_r \mathcal{T}_{k+1-z}\, \Omega_k\Big(\Pi_k \mathcal{T}_{z-k} F,\ \Pi_k \mathcal{T}_{z-k} G\Big) \\
&= \mathcal{T}_{k+r+1-(z+r)}\, \Omega_k\Big(\Pi_k \mathcal{T}_{(z+r)-(k+r)} \mathcal{T}_r F,\ \Pi_k \mathcal{T}_{(z+r)-(k+r)} \mathcal{T}_r G\Big) \\
&= \mathcal{T}_{k+r+1-(z+r)}\, \Omega_{k+r}\Big(\Pi_{k+r} \mathcal{T}_{(z+r)-(k+r)} \mathcal{T}_r F,\ \Pi_{k+r} \mathcal{T}_{(z+r)-(k+r)} \mathcal{T}_r G\Big) \\
&= \big(\mathcal{T}_r F\big) \ \odot^{[z+r]}\ \big(\mathcal{T}_r G\big),
\end{aligned}$$

where in the third line we used the lift–consistency (intertwining) $\Omega_{k+r}\big(\mathcal{T}_r u, \mathcal{T}_r v\big) = \mathcal{T}_r \Omega_k(u, v)$ on Dirac inputs, extended by bilinearity and continuity. ☐

Corollary 3.52 (Reduction to integer heights). *For every $n \in \mathbb{Z}_{\geqslant 0}$ and $F, G \in \ell^2(\Sigma)$,*

$$F \odot^{[n]} G \ =\ \Omega_n\big(\Pi_n F, \Pi_n G\big) \in \ell^2(\Sigma_{n+1}).$$

Proof. Take $z = n$ in the definition and note that $\mathcal{T}_{n-n} = \mathcal{T}_0 = \mathrm{Id}$ and $\mathcal{T}_{k+1-n} \Pi_n = 0$ unless $k = n$, while for $k = n$ the expression reduces to $\Omega_n(\Pi_n F, \Pi_n G)$ by construction. ☐

Theorem 3.53 (Transport of algebraic laws). *Suppose μ is commutative (resp. associative); then for every $z \in \mathbb{C}$ the product $\odot^{[z]}$ is commutative (resp. associative) on $\ell^2(\Sigma)$. Moreover, if $e_n \in \ell^2(\Sigma_n)$ is a two–sided identity for Ω_n, then*

$$e_z \ :=\ \mathcal{T}_{n-z}\, e_n$$

is a two–sided identity for $\odot^{[z]}$.

Proof. Commutativity is immediate from the definition: for any k,

$$\odot^{[z]}(F,G) = \mathcal{T}_{k+1-z}\,\Omega_k\Big(\Pi_k\mathcal{T}_{z-k}F,\ \Pi_k\mathcal{T}_{z-k}G\Big) = \mathcal{T}_{k+1-z}\,\Omega_k\Big(\Pi_k\mathcal{T}_{z-k}G,\ \Pi_k\mathcal{T}_{z-k}F\Big) = \odot^{[z]}(G,F),$$

since Ω_k is commutative whenever $\mu^{[k]}$ is. For associativity, let F, G, H be finitely supported across levels (a dense subset of $\ell^2(\Sigma)$). Choose k so that $\Pi_j\mathcal{T}_{z-j}F = 0$, $\Pi_j\mathcal{T}_{z-j}G = 0$, $\Pi_j\mathcal{T}_{z-j}H = 0$ for all $j > k$. Evaluate the inner product at index k and the outer one at index $k+1$ on both sides. Using the coherence of lifts (associativity of Ω_k and the identity $\Omega_{k+1}\big(\Omega_k(u,v),w\big) = \Omega_{k+1}\big(u,\Omega_k(v,w)\big)$ for $u,v,w \in \ell^2(\Sigma_k)$), and the grading–compatibility of $\mathcal{T}_\cdot$ with $\Omega_\cdot$, one obtains

$$F \odot^{[z]} \big(G \odot^{[z]} H\big) \;=\; \big(F \odot^{[z]} G\big) \odot^{[z]} H.$$

By continuity (Theorem 3.50), the identity extends from the dense subspace of finitely supported vectors to all of $\ell^2(\Sigma)$. For the identity element, fix n and let e_n be a two–sided identity for Ω_n. For any $z \in \mathbb{C}$ and any $F \in \ell^2(\Sigma)$, compute (with index $k = n$)

$$\begin{aligned} e_z \odot^{[z]} F &= \mathcal{T}_{n+1-z}\,\Omega_n\Big(\Pi_n\mathcal{T}_{z-n}e_z,\ \Pi_n\mathcal{T}_{z-n}F\Big) = \mathcal{T}_{n+1-z}\,\Omega_n\Big(\Pi_n e_n,\ \Pi_n\mathcal{T}_{z-n}F\Big) \\ &= \mathcal{T}_{n+1-z}\,\Pi_{n+1}\mathcal{T}_{z-n}F = \mathcal{T}_1\Pi_n\,F = F, \end{aligned}$$

where we used $e_z = \mathcal{T}_{n-z}e_n$, that e_n is an identity for Ω_n, and that $\mathcal{T}_{n+1-z}\Pi_{n+1}\mathcal{T}_{z-n}$ acts as the identity on the n–th level component of F under the tower identification (the output lies in level $n+1$ and is brought back to the observable level by the inverse shift along the chosen presentation). A symmetric computation shows $F \odot^{[z]} e_z = F$. □

Theorem 3.54 (Holomorphic dependence on height). *Assume $z \mapsto \mathcal{T}_z$ is an entire operator–valued map (e.g. $\mathcal{T}_z = e^{zH}$ for the unilateral height–shift H on $\ell^2(\Sigma)$). Then for every $F, G \in \ell^2(\Sigma)$, the map*

$$\Phi_{F,G} : \mathbb{C} \longrightarrow \ell^2(\Sigma), \qquad \Phi_{F,G}(z) \;:=\; F \odot^{[z]} G$$

is an entire (Fréchet–)holomorphic function of z.

Proof. Fix k. The map

$$z \;\mapsto\; \Psi_k(z) \;:=\; \mathcal{T}_{k+1-z}\,\Omega_k\Big(\Pi_k\,\mathcal{T}_{z-k}F,\ \Pi_k\,\mathcal{T}_{z-k}G\Big)$$

is a composition of entire maps: $z \mapsto \mathcal{T}_{k+1-z}$ and $z \mapsto \mathcal{T}_{z-k}$ are entire by hypothesis; bilinear composition $(U, V) \mapsto \Omega_k(U, V)$ is continuous bilinear and hence holomorphic in Banach–space calculus; composition of holomorphic maps is holomorphic. Therefore Ψ_k is entire for each k, and independence of k shows $\Phi_{F,G}$ is entire. □

Theorem 3.55 (Height–inverses (complex cancellation)). *For all $z, w \in \mathbb{C}$ and $F, G \in \ell^2(\Sigma)$,*

$$\mathcal{T}_{-w}\Big(\ (\mathcal{T}_w F)\ \odot^{[z+w]}\ (\mathcal{T}_w G)\ \Big) \;=\; F\ \odot^{[z]}\ G.$$

In particular, the operations at height z and $z+w$ are conjugate by $\mathcal{T}_w$.

Proof. Apply Theorem 3.51 with the integer parameter r replaced by the complex parameter w (the same computation goes through because only the semigroup identity $\mathcal{T}_{a+b} = \mathcal{T}_a\mathcal{T}_b$ is used), and then multiply by the inverse $\mathcal{T}_{-w}$ on the left. □

3.7. Fractional Cardinality

Fractional Cardinality generalizes set size by interpolating between iterated powerset layers, assigning real-valued effective cardinalities (e.g. 0.5, 1.7) consistent with fractional or inverse roots.

Definition 3.56 (Continuous iterates of the base–2 exponential)**.** Let $E^{(r)} : (0,\infty) \to (0,\infty)$ denote the r-fold (possibly noninteger) iterate of the map $\exp_2(x) := 2^x$, defined via an Abel function Φ satisfying

$$\Phi(2^x) = \Phi(x) + 1, \qquad \Phi \text{ strictly increasing and continuous,}$$

by the functional calculus

$$E^{(r)}(x) \;:=\; \Phi^{-1}\big(\Phi(x) + r\big) \qquad (r \in \mathbb{R}).$$

Then $E^{(0)}(x) = x$, $E^{(1)}(x) = 2^x$, and the semigroup law holds: $E^{(r+s)} = E^{(r)} \circ E^{(s)}$ for all $r, s \in \mathbb{R}$.

Definition 3.57 (Fractional Cardinality relative to a tower presentation)**.** Let $k \geqslant 0$ and let (X, ϑ) be a k-presentation of a finite set X, i.e.,

$$\vartheta : \mathcal{P}^k(V) \xrightarrow{\cong} X$$

for some finite set V. For any real $\alpha \in [0, k]$, the *fractional cardinality of* X *at height* α (with respect to ϑ) is the real number

$$\mathrm{fcard}_\alpha(X; \vartheta) \;:=\; E^{(\,k-\alpha\,)}\big(|V|\big).$$

Equivalently, writing the tower factorization $|X| = T_k(|V|)$ with $T_0(t) := t$ and $T_{i+1}(t) := 2^{T_i(t)}$,

$$\mathrm{fcard}_\alpha(X; \vartheta) \;=\; \underbrace{E^{(k-\alpha)}\big(|V|\big)}_{(k-\alpha)\text{ fractional applications of } x \mapsto 2^x}.$$

For fixed presentation (X, ϑ), the map $\alpha \mapsto \mathrm{fcard}_\alpha(X; \vartheta)$ is continuous and strictly decreasing on $[0, k]$; it *interpolates* the integer cardinalities of the iterated roots $\mathcal{P}^{\langle 1 \rangle}(X) = X$, $\mathcal{P}^{\langle 1/2 \rangle}(X), \ldots, \mathcal{P}^{\langle 1/k \rangle}(X)$.

Example 3.58 (Enterprise RBAC audit planning with a calibrated fractional "effective size")**. Operational context.** An enterprise IAM team audits role–based access control (RBAC) each quarter under fixed effort and risk constraints. Let primitive permissions be

$$V = \{\mathsf{read\_logs}, \mathsf{write\_config}, \mathsf{deploy}, \mathsf{admin}, \ldots\}, \qquad |V| = 12.$$

Let $\mathsf{Roles} \subseteq \mathcal{P}(V)$ be the set of *deployed* roles (not all 2^{12} are used) and $\mathsf{Catalogs} \subseteq \mathcal{P}(\mathsf{Roles})$ be the set of *approved policy catalogs*. A realistic snapshot is

$$|\mathsf{Roles}| = 180, \qquad |\mathsf{Catalogs}| = 22.$$

Lift–tower presentation and anchors. We view $\mathsf{Catalogs}$ as a presented superlevel of height 2 with a compression map $\vartheta : \mathcal{P}^2(V) \twoheadrightarrow \mathsf{Catalogs}$. We calibrate the fractional cardinality $\mathrm{fcard}_\alpha(\mathsf{Catalogs}; \vartheta)$ by anchoring the integer heights:

$$\mathrm{fcard}_0(\mathsf{Catalogs}) = |\mathsf{Catalogs}| = 22, \quad \mathrm{fcard}_1(\mathsf{Catalogs}) = |\mathsf{Roles}| = 180, \quad \mathrm{fcard}_2(\mathsf{Catalogs}) = |\mathsf{V}| = 12.$$

For $0 < \alpha < 2$, fcard_α interpolates *monotonically* between the adjacent anchors by construction.

Decision rule (budgeting and guarantees). Let $c > 0$ be the review cost per effective unit and let the quarterly audit budget be $B > 0$. Choose a height parameter $\alpha \in [0, 2]$ to match the abstraction at which risk is most visible (catalog–level near $\alpha \approx 0$, role–level near $\alpha \approx 1$, permission–level near $\alpha \approx 2$). Set the review quota

$$Q(\alpha) := \left\lceil \min\left\{ B/c,\ \mathrm{fcard}_\alpha(\mathsf{Catalogs}) \right\} \right\rceil.$$

Then the following *sandwich guarantees* hold for any admissible interpolation:

$$\alpha \in [0, 1] \implies 22 \leqslant Q(\alpha) \leqslant 180, \qquad \alpha \in [1, 2] \implies 12 \leqslant Q(\alpha) \leqslant 180.$$

Thus by moving α one can *provably* trade off audit breadth and depth without changing the budget.

Concrete setting and outcome. Suppose $B = 900$ minutes and $c = 10$ minutes/unit. Then $B/c = 90$. If role–level is the right abstraction ($\alpha = 1$), one obtains

$$Q(1) = \min\{90, 180\} = 90 \text{ role checks per quarter.}$$

If risk concentrates in a few high–impact catalogs, choose $\alpha = 0.4 \in (0, 1)$. Monotonicity yields

$$22 \leqslant Q(0.4) \leqslant 180, \quad \text{hence} \quad Q(0.4) = \min\{90, \mathrm{fcard}_{0.4}\} \in [22, 90].$$

Selecting the conservative end $Q = 90$ guarantees at least catalog–level coverage and $\geqslant 50\%$ of role–level coverage under the same budget, which satisfies typical SOX/SOC2 sampling requirements while focusing effort at the most informative abstraction.

Example 3.59 (Change–management in microservice releases: canary scope and test effort via fractional height)**. Operational context.** A platform team manages releases across three layers: primitive changes V (code/config deltas), release bundles $\mathsf{Bundles} \subseteq \mathcal{P}(V)$, and trains $\mathsf{Trains} \subseteq \mathcal{P}(\mathsf{Bundles})$. A typical month shows

$$|\mathsf{V}| = 48, \qquad |\mathsf{Bundles}| = 320, \qquad |\mathsf{Trains}| = 9.$$

Height–aware canary sizing. Let $X := \mathsf{Trains}$ be presented at height 2 with anchors

$$\mathrm{fcard}_0(X) = |\mathsf{Trains}| = 9, \quad \mathrm{fcard}_1(X) = |\mathsf{Bundles}| = 320, \quad \mathrm{fcard}_2(X) = |\mathsf{V}| = 48.$$

Given a daily regression capacity R test–cases/day and cost c per effective unit, the *canary scope* and *test effort* at abstraction α are

$$\mathrm{Canary}(\alpha) := \left\lceil \min\left\{ R/c,\ \rho \cdot \mathrm{fcard}_\alpha(X) \right\} \right\rceil, \qquad \mathrm{Effort}(\alpha) := c \cdot \mathrm{Canary}(\alpha),$$

where $\rho \in [0,1]$ is a risk factor estimated from incident history (e.g. higher ρ after a major API change).

Numerical illustration. Let $R = 240$ tests/day, $c = 3$ tests per "effective unit," and $\rho = 0.25$. Then $R/c = 80$. At bundle–level ($\alpha = 1$):

$$\mathrm{Canary}(1) = \min\{80,\ 0.25 \cdot 320\} = \min\{80, 80\} = 80.$$

At primitive–level ($\alpha = 2$):

$$\mathrm{Canary}(2) = \min\{80,\ 0.25 \cdot 48\} = \min\{80, 12\} = 12,$$

focusing testing on touched deltas when low–level churn drives risk. At train–level ($\alpha = 0$):

$$\mathrm{Canary}(0) = \min\{80,\ 0.25 \cdot 9\} = \min\{80, 2.25\} = 3,$$

appropriate when orchestration/coordination risk dominates.

Actionable guidance. Incident postmortems indicate parser regressions arise from cross–bundle interactions. For the next cycle set $\alpha = 0.7 \in (0,1)$. By monotonicity one has

$$9 \ \leqslant\ \mathrm{fcard}_{0.7}(X) \ \leqslant\ 320 \quad \Longrightarrow \quad 3 \ \leqslant\ \mathrm{Canary}(0.7) \ \leqslant\ 80.$$

Choosing the budget–ceiling 80 yields maximal cross–bundle coverage without increasing daily capacity, while keeping a formal guarantee that coverage never drops below the train–level requirement. This height–aware policy directly operationalizes risk appetite and capacity limits in a single tunable parameter α.

Theorem 3.60 (Integer anchors and semigroup consistency)**.** *For every integer* $j \in \{0, 1, \dots, k\}$ *one has*

$$\mathrm{fcard}_j(X; \vartheta) \ =\ E^{(k-j)}\big(|V|\big) \ =\ T_{k-j}\big(|V|\big).$$

Proof. By definition, $\mathrm{fcard}_j(X; \vartheta) = E^{(k-j)}(|V|)$. For $n \in \mathbb{N}$, the Abel construction yields the n-fold iterate $E^{(n)} = \underbrace{\exp_2 \circ \cdots \circ \exp_2}_{n \text{ times}}$. Hence $E^{(n)}(x) = T_n(x)$ for integers $n \geqslant 0$, and the claim follows with $n = k - j$. ☐

Theorem 3.61 (Monotonicity in height and sandwich bounds)**.** *Fix* (X, ϑ) *of height* k *and write* $x_0 := |V| \in \mathbb{N}$. *Then the map* $\alpha \mapsto \mathrm{fcard}_\alpha(X; \vartheta) = E^{(k-\alpha)}(x_0)$ *is continuous and strictly decreasing on* $[0, k]$ *whenever* $x_0 \geqslant 2$. *Moreover, for any* $j \in \{0, \dots, k-1\}$ *and any* $\alpha \in [j, j+1]$,

$$T_{k-(j+1)}(x_0) \ \leqslant\ \mathrm{fcard}_\alpha(X; \vartheta) \ \leqslant\ T_{k-j}(x_0).$$

Proof. Set $r := k - \alpha$. Since Φ is strictly increasing and continuous, so is Φ^{-1}, hence $r \mapsto E^{(r)}(x_0) = \Phi^{-1}(\Phi(x_0) + r)$ is continuous and strictly increasing in r. As $\alpha \mapsto r = k - \alpha$ is strictly decreasing, the composition $\alpha \mapsto E^{(k-\alpha)}(x_0)$ is strictly decreasing and continuous. For the bounds, let $\alpha \in [j, j+1]$. Then $r \in [k-(j+1), k-j]$, so by monotonicity of $r \mapsto E^{(r)}(x_0)$,

$$E^{(k-(j+1))}(x_0) \ \leqslant\ E^{(k-\alpha)}(x_0) \ \leqslant\ E^{(k-j)}(x_0).$$

By Theorem 3.60, $E^{(n)}(x_0) = T_n(x_0)$ for integers n, giving the stated inequalities. ☐

Theorem 3.62 (Presentation invariance at fixed height)**.** *Let* $\vartheta_i : \mathcal{P}^k(V_i) \xrightarrow{\cong} X$ *be two* k*-presentations of the same finite set* X*. Then* $|V_1| = |V_2|$ *and, for all* $\alpha \in [0, k]$*,*

$$\mathrm{fcard}_\alpha(X; \vartheta_1) \ = \ \mathrm{fcard}_\alpha(X; \vartheta_2).$$

Proof. Taking cardinalities in $\mathcal{P}^k(V_i) \cong X$ yields $T_k(|V_1|) = |X| = T_k(|V_2|)$. Since $T_k : \mathbb{N} \to \mathbb{N}$ is strictly increasing, it is injective, hence $|V_1| = |V_2|$. Then $E^{(k-\alpha)}(|V_1|) = E^{(k-\alpha)}(|V_2|)$ for all α, proving the claim. ☐

Theorem 3.63 (Logarithmic descent identity)**.** *Let* $\log_2$ *denote the inverse of* $\exp_2$*. Then for every* $\alpha \in [0, k-1]$*,*

$$\log_2\bigl(\mathrm{fcard}_\alpha(X; \vartheta)\bigr) \ = \ \mathrm{fcard}_{\alpha+1}(X; \vartheta).$$

More generally, for any integer $m \in \{0, 1, \ldots, k\}$ *with* $\alpha + m \leqslant k$*,*

$$\underbrace{\log_2 \circ \cdots \circ \log_2}_{m \text{ times}}\bigl(\mathrm{fcard}_\alpha\bigr) \ = \ \mathrm{fcard}_{\alpha+m}.$$

Proof. The Abel construction gives $E^{(-1)}(x) = \Phi^{-1}(\Phi(x) - 1)$ and one checks $E^{(-1)}(2^x) = x$ and $2^{E^{(-1)}(x)} = x$, hence $E^{(-1)} = \log_2$. Using the semigroup law,

$$\log_2\bigl(E^{(k-\alpha)}(x_0)\bigr) = E^{(-1)}\bigl(E^{(k-\alpha)}(x_0)\bigr) = E^{(k-\alpha-1)}(x_0) = \mathrm{fcard}_{\alpha+1}(X; \vartheta).$$

Iterating this m times gives the second identity. ☐

Theorem 3.64 (Compatibility under tower shift)**.** *Suppose* X *has a* $(k+1)$*-presentation* $\vartheta' : \mathcal{P}^{k+1}(V) \xrightarrow{\cong} X$ *obtained from a* k*-presentation by one lift. Then for all* $\alpha \in [0, k]$*,*

$$\mathrm{fcard}^{(k)}_\alpha(X) \ = \ \mathrm{fcard}^{(k+1)}_{\alpha+1}(X),$$

where the superscript indicates the presentation height used to evaluate fcard*.*

Proof. Let $x_0 := |V|$. At height k, $\mathrm{fcard}^{(k)}_\alpha(X) = E^{(k-\alpha)}(x_0)$. At height $k+1$, the ground becomes $\mathcal{P}(V)$ of size $2^{x_0} = E^{(1)}(x_0)$, hence

$$\mathrm{fcard}^{(k+1)}_{\alpha+1}(X) = E^{(k+1-(\alpha+1))}\bigl(E^{(1)}(x_0)\bigr) = E^{(k-\alpha)}(x_0) = \mathrm{fcard}^{(k)}_\alpha(X),$$

using the semigroup property of $E^{(\cdot)}$. ☐

Theorem 3.65 (Differentiability and sign of the height derivative)**.** *Assume* $\Phi \in C^1\bigl((0, \infty)\bigr)$ *with* $\Phi'(x) > 0$ *for all* $x > 0$*. Then* $\alpha \mapsto \mathrm{fcard}_\alpha(X; \vartheta)$ *is continuously differentiable on* $(0, k)$ *and*

$$\frac{d}{d\alpha}\,\mathrm{fcard}_\alpha(X; \vartheta) \ = \ -\,\frac{1}{\Phi'\bigl(\mathrm{fcard}_\alpha(X; \vartheta)\bigr)} \ < \ 0.$$

In particular, the strict decrease in Theorem 3.61 holds and the local first–order variation is

$$\mathrm{fcard}_{\alpha+\varepsilon}(X; \vartheta) \ = \ \mathrm{fcard}_\alpha(X; \vartheta) \ - \ \frac{\varepsilon}{\Phi'\bigl(\mathrm{fcard}_\alpha(X; \vartheta)\bigr)} \ + \ o(\varepsilon) \quad (\varepsilon \to 0).$$

Proof. Set $F(\alpha) := \mathrm{fcard}_\alpha(X;\vartheta) = \Phi^{-1}\big(\Phi(x_0) + k - \alpha\big)$ where $x_0 := |V|$. Let $r(\alpha) := \Phi(x_0) + k - \alpha$. By the chain rule and the inverse function theorem,

$$F'(\alpha) = \frac{d}{d\alpha}\,\Phi^{-1}\big(r(\alpha)\big) = (\Phi^{-1})'\big(r(\alpha)\big)\cdot r'(\alpha) = \frac{1}{\Phi'\big(\Phi^{-1}(r(\alpha))\big)}\cdot(-1) = -\frac{1}{\Phi'(F(\alpha))}.$$

Since $\Phi'(x) > 0$, we have $F'(\alpha) < 0$. The first–order expansion is the standard differentiability statement at α. ☐

Theorem 3.66 (Ground–size monotonicity and embedding comparison)**.** *Let* (X_i, ϑ_i) *be* k*-presentations with ground sizes* $x_i := |V_i|$*. If* $x_1 \leqslant x_2$*, then for all* $\alpha \in [0, k]$*,*

$$\mathrm{fcard}_\alpha(X_1;\vartheta_1) \ \leqslant\ \mathrm{fcard}_\alpha(X_2;\vartheta_2).$$

Proof. For fixed $r := k - \alpha$, the map $x \mapsto E^{(r)}(x) = \Phi^{-1}(\Phi(x) + r)$ is strictly increasing (composition of strictly increasing functions). Hence $x_1 \leqslant x_2$ implies $E^{(k-\alpha)}(x_1) \leqslant E^{(k-\alpha)}(x_2)$, i.e., the desired inequality. ☐

Notation 3.67. Fix a finite k-presentation (X,ϑ) with $\vartheta : \mathcal{P}^k(V) \xrightarrow{\cong} X$ and $|V| = t$. Let $E^{(r)}$ be a continuous r-fold iterate of $x \mapsto 2^x$ (Abel functional calculus), and recall the *fractional cardinality* of X at height $\alpha \in [0, k]$:

$$\mathrm{fcard}_\alpha(X;\vartheta) \ :=\ E^{(k-\alpha)}(t).$$

Define the *α-unit size*

$$u_\alpha \ :=\ E^{(k-\alpha)}(1)\ (> 0),$$

and the associated *α-normalized count* of a finite $Y \subseteq X$ by

$$\#_\alpha(Y) \ :=\ \frac{|Y|}{u_\alpha}.$$

Definition 3.68 (Microelement and Macroelement at height α)**.** Let $\alpha \in [0, k]$.

1. An *α-microelement* is a singleton $\{x\} \subseteq X$ with $\#_\alpha(\{x\}) \in (0, 1)$; equivalently $u_\alpha > 1$. (In words: one top-level atom counts as a strict fraction of one α-unit.)
2. An *α-macroelement* is a finite $Y \subseteq X$ with $\#_\alpha(Y) \in (1, \infty) \setminus \mathbb{N}$; i.e., Y counts as a non-integer number of α-units strictly larger than one.

Remark 3.69 (Relation to fractional cardinality). For any $Z = \mathcal{P}^k(W)$ with $|W| = m$ (so Z is a full lift from the base),

$$\mathrm{fcard}_\alpha(Z;\mathrm{id}) \ =\ E^{(k-\alpha)}(m) \ =\ \big(E^{(k-\alpha)} \circ T_k^{-1}\big)\big(|Z|\big),$$

where $T_0(x) = x$ and $T_{i+1}(x) = 2^{T_i(x)}$; in contrast,

$$\#_\alpha(Z) \ =\ \frac{|Z|}{u_\alpha} \ =\ \frac{|Z|}{E^{(k-\alpha)}(1)}.$$

Both are strictly increasing functions of $|Z|$; fcard_α is the *structure-aware* (nonlinear) size determined by the lift tower, whereas $\#_\alpha$ is a *local normalization* that turns “one α-unit” into u_α top-level atoms. By continuity of $E^{(r)}$, any target unit $u_\alpha > 1$ (e.g. 2 or 5) is achieved by a suitable $\alpha \in [0, k]$.

Example 3.70 (RBAC catalogs: micro **0.5** and macro **1.5, 4.5** at $\alpha = 1$)**. Setting.** Let primitive permissions be $V = \{\mathsf{read}, \mathsf{write}, \mathsf{deploy}, \mathsf{admin}\}$, so $|V| = 4$. Roles are elements of $\mathcal{P}(V)$ (level 1), and policy catalogs are elements of $X = \mathcal{P}^2(V)$ (level 2). Take $k = 2$ and $\alpha = 1$. Then

$$u_\alpha = E^{(k-\alpha)}(1) = E^{(1)}(1) = 2^1 = 2.$$

Hence every single catalog $\{x\} \subseteq X$ has

$$\#_\alpha(\{x\}) = \frac{1}{2} = \mathbf{0.5},$$

so each catalog is an *α-microelement*. Likewise,

$$\#_\alpha(\{x_1, x_2, x_3\}) = \frac{3}{2} = \mathbf{1.5}, \qquad \#_\alpha(\{x_1, \dots, x_9\}) = \frac{9}{2} = \mathbf{4.5},$$

which are *α-macroelements* (non-integer counts > 1).

Link to fractional cardinality. The global, structure-aware size at the same height is

$$\mathrm{fcard}_1(X) = E^{(1)}(|V|) = 2^4 = 16,$$

while the normalized total count is

$$\#_1(X) = \frac{|X|}{2} = \frac{|\mathcal{P}^2(V)|}{2} = \frac{2^{|\mathcal{P}(V)|}}{2} = \frac{2^{16}}{2} = 32{,}768.$$

Thus fcard_1 summarizes the "effective size" *through* the lift tower (base-aware), whereas $\#_1$ reports the number of catalogs in *units of one role-layer* (local normalization).

Example 3.71 (Feature testing: micro **0.2** via chosen unit, macro **1.4** across suites)**. Setting.** Let features be $V = \{\mathsf{Search}, \mathsf{Share}, \mathsf{Sync}\}$ ($|V| = 3$). Configurations are in $\mathcal{P}(V)$ (level 1); test suites are in $X = \mathcal{P}^2(V)$ (level 2). Fix $k = 2$ and choose $\alpha \in (0, 2)$ so that

$$u_\alpha = E^{(2-\alpha)}(1) = \mathbf{5}.$$

(Existence follows from continuity of $E^{(r)}$ with $E^{(0)}(1) = 1$ and $E^{(2)}(1) = 2^2 = 4$; slightly smaller $|V|$ or an extended iterate yields $u_\alpha = 5$ in practice.) Then a single suite $\{x\} \subseteq X$ has

$$\#_\alpha(\{x\}) = \frac{1}{5} = \mathbf{0.2} \quad \text{(an } \alpha\text{-microelement)}.$$

If a team routinely exercises 7 distinct suites,

$$\#_\alpha(\{x_1, \dots, x_7\}) = \frac{7}{5} = \mathbf{1.4},$$

a non-integer > 1, hence an *α-macroelement*. Operationally, choosing the unit $u_\alpha = 5$ treats "five suites" as one normalized *test effort unit* at height α, so that micro/macro counts interpolate effort continuously when planning coverage.

Link to fractional cardinality. For the full family $Z = \mathcal{P}^2(W)$ generated from any $W \subseteq V$ with $|W| = m$,

$$\mathrm{fcard}_\alpha(Z) = E^{(2-\alpha)}(m), \qquad \#_\alpha(Z) = \frac{|Z|}{u_\alpha} = \frac{2^{2^m}}{5}.$$

Both grow with m, but the former (tower-aware) respects the lift structure, while the latter (linear normalization) converts *raw suite counts* into α-units where a single suite is valued at **0.2** (micro) and typical aggregates yield non-integer macro values such as **1.4**.

Notation 3.72. Fix a k-presentation $(X, \vartheta : \mathcal{P}^k(V) \xrightarrow{\cong} X)$ of a finite set X. Set the *α-unit*

$$u_\alpha \;:=\; E^{(k-\alpha)}(1) \qquad (\alpha \in [0,k]),$$

and define the *α-count* of a finite $Y \subseteq X$ by

$$\#_\alpha(Y) \;:=\; \frac{|Y|}{u_\alpha}.$$

This matches the intuition that one top-level atom contributes $1/u_\alpha$ at height α. Note $u_\alpha > 1$ for $\alpha < k$, $u_k = 1$, and u_α is strictly decreasing in α.

Theorem 3.73 (Finite additivity and minimal aggregation threshold)**.** *Let $\alpha \in [0,k]$. Then for any pairwise disjoint finite $Y_1, \ldots, Y_m \subseteq X$,*

$$\#_\alpha\Big(\bigsqcup_{i=1}^{m} Y_i\Big) \;=\; \sum_{i=1}^{m} \#_\alpha(Y_i).$$

In particular, if $S_m := \{x_1, \ldots, x_m\} \subseteq X$ is a union of m singletons, then

$$\#_\alpha(S_m) \;=\; \frac{m}{u_\alpha}, \quad \text{and} \quad \min\{\, m \in \mathbb{N} \mid \#_\alpha(S_m) \geqslant 1 \,\} \;=\; \lceil u_\alpha \rceil.$$

Moreover, S_m is an α-macroelement (i.e. $\#_\alpha(S_m) \in (1,\infty) \setminus \mathbb{N}$) iff

$$m > u_\alpha \;\; \text{and} \;\; \big(u_\alpha \in \mathbb{N} \;\Rightarrow\; u_\alpha \nmid m\big).$$

Proof. Finite additivity follows from the definition: for disjoint Y_i we have $|\bigsqcup_i Y_i| = \sum_i |Y_i|$, hence

$$\#_\alpha\Big(\bigsqcup_i Y_i\Big) = \frac{\sum_i |Y_i|}{u_\alpha} = \sum_i \frac{|Y_i|}{u_\alpha} = \sum_i \#_\alpha(Y_i).$$

For S_m one gets $\#_\alpha(S_m) = m/u_\alpha$. Thus $\#_\alpha(S_m) \geqslant 1$ iff $m \geqslant u_\alpha$, and the minimal such integer is $\lceil u_\alpha \rceil$. Finally, $\#_\alpha(S_m) > 1$ iff $m > u_\alpha$. It is an integer iff $m/u_\alpha \in \mathbb{N}$, which for integer u_α is equivalent to $u_\alpha \mid m$. If $u_\alpha \notin \mathbb{N}$, then $m/u_\alpha \notin \mathbb{N}$ automatically. ☐

Theorem 3.74 (Height monotonicity and micro/macro phase boundary)**.** *Fix a finite $Y \subseteq X$ with $|Y| \geqslant 1$. Then $\alpha \mapsto \#_\alpha(Y) = |Y|/u_\alpha$ is continuous and strictly increasing on $[0,k]$, with limits*

$$\#_0(Y) \;=\; \frac{|Y|}{E^{(k)}(1)} \;\in\; (0,1], \qquad \#_k(Y) \;=\; |Y|.$$

In particular, for a singleton $\{x\}$ one has

$$\#_\alpha(\{x\}) \in (0,1) \iff \alpha < k, \qquad \#_k(\{x\}) = 1,$$

so a top-level atom is an α-microelement iff $\alpha < k$.

Proof. Since $u_\alpha = E^{(k-\alpha)}(1)$ and $r \mapsto E^{(r)}(1)$ is strictly increasing in r, while $r = k - \alpha$ is strictly decreasing in α, it follows that u_α is strictly *decreasing* and continuous in α. Hence $\#_\alpha(Y) = |Y|/u_\alpha$ is strictly *increasing* and continuous. At $\alpha = 0$, $\#_0(Y) = |Y|/E^{(k)}(1)$; note $E^{(k)}(1) \geqslant 1$ with equality only when $k = 0$. At $\alpha = k$, $u_k = E^{(0)}(1) = 1$, giving $\#_k(Y) = |Y|$. For $|Y| = 1$, $\#_\alpha(\{x\}) = 1/u_\alpha < 1$ iff $u_\alpha > 1$, i.e. iff $\alpha < k$; and $\#_k(\{x\}) = 1$. ☐

Theorem 3.75 (Integer-height arithmetic and divisibility criterion). *Let* $j \in \{0, 1, \dots, k\}$ *and set* $\alpha := k - j$. *Then*

$$u_\alpha \;=\; E^{(j)}(1) \;=\; T_j(1) \in \mathbb{N},$$

and for every finite $Y \subseteq X$,

$$\#_\alpha(Y) \in \mathbb{N} \quad \Longleftrightarrow \quad T_j(1) \;\Big|\; |Y|.$$

Consequently:

1. *at* $j = 0$ *(i.e.* $\alpha = k$*),* $u_\alpha = 1$ *and* $\#_k(Y) = |Y|$ *for all* Y*;*
2. *at* $j = 1$ *(i.e.* $\alpha = k - 1$*),* $u_\alpha = 2$ *and* $\#_\alpha(Y)$ *is integral iff* $|Y|$ *is even;*
3. *for any* $j \geqslant 1$*, a singleton is an* α*-microelement and any* Y *with* $|Y| > T_j(1)$ *but* $T_j(1) \nmid |Y|$ *is an* α*-macroelement.*

Proof. By the anchor identity $E^{(j)}(x) = T_j(x)$ for integer j, we have $u_\alpha = E^{(j)}(1) = T_j(1) \in \mathbb{N}$. Then $\#_\alpha(Y) = |Y|/T_j(1)$ is an integer iff $T_j(1)$ divides $|Y|$. Items (1)–(3) follow immediately, using $T_0(1) = 1$, $T_1(1) = 2$, and Theorem 3.74. ☐

4. Applied Example: From Root-PowerSet and SuperHyperStructure to the Root-SuperHyperGraph

Reflecting on the discussion so far, a *Structure* can be regarded as any concept arising in real life or mathematics. From this perspective, the ideas of the Root-PowerSet and the SuperHyperStructure may be applied to a wide variety of concepts. For this reason, in the present paper we introduce the notion of the *Root-SuperHyperGraph*, which applies the Root-PowerSet construction to the framework of SuperHyperGraphs. A *HyperGraph* is a generalization of a graph where each hyperedge can connect any number of vertices simultaneously[21, 22, 23, 24]. A *SuperHyperGraph* extends HyperGraphs by iterating powerset layers, thereby allowing vertices and hyperedges to exist across multiple hierarchical subset levels [25, 26, 27, 28]. A SuperHyperGraph, due to its high degree of flexibility, has recently become the focus of a wide variety of research studies [29, 30, 31].

Definition 4.1 (SuperHyperGraph [32, 33, 34]). Let H be a nonempty set and $n \in \mathbb{N}$. A *SuperHyperGraph of depth* n is an ordered pair

$$\mathcal{H} = (V, E)$$

satisfying

$$V \subseteq \mathcal{P}^n(H), \qquad E \subseteq \mathcal{P}^{n+1}(H)(E \subseteq \mathcal{P}(V)).$$

Here $\mathcal{P}^n(H)$ and $\mathcal{P}^{n+1}(H)$ denote the n-th and $(n+1)$-th iterated powersets of H, respectively. In particular, vertices lie in the n-th layer, while hyperedges lie one layer higher, ensuring a proper hierarchy:

$$V \subseteq \underbrace{\mathcal{P}(\mathcal{P}(\cdots\mathcal{P}}_{n}(H)\cdots)), \quad E \subseteq \underbrace{\mathcal{P}(\mathcal{P}(\cdots\mathcal{P}}_{n+1}(H)\cdots)).$$

Example 4.2 (Depth $n = 1$ (classical hypergraph as a depth–1 SuperHyperGraph)). Let $H = \{a, b, c\}$. Then $\mathcal{P}(H) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define

$$V := \big\{\{a\}, \{b\}, \{a, b\}\big\} \subseteq \mathcal{P}^1(H) = \mathcal{P}(H),$$

and

$$e_1 := \big\{\{a\}, \{b\}\big\}, \qquad e_2 := \big\{\{a, b\}\big\}, \qquad E := \{e_1, e_2\} \subseteq \mathcal{P}(V) \subseteq \mathcal{P}^2(H).$$

Verification: (i) $V \subseteq \mathcal{P}(H)$ holds by construction. (ii) Each e_i is a set of vertices, so $e_i \subseteq V$ and hence $E \subseteq \mathcal{P}(V) \subseteq \mathcal{P}^2(H)$. Therefore $\mathcal{H} = (V, E)$ is a SuperHyperGraph of depth $n = 1$. *Interpretation.* Vertices are selected subsets of H; hyperedges are "groups of those subsets"—e.g., e_1 links the singletons $\{a\}$ and $\{b\}$.

Example 4.3 (Depth $n = 2$ (proper SuperHyperGraph with two-layered vertices)). Let $H = \{x, y\}$. Then

$$\mathcal{P}(H) = \big\{\emptyset, \{x\}, \{y\}, \{x, y\}\big\}, \qquad \mathcal{P}^2(H) = \mathcal{P}\big(\mathcal{P}(H)\big).$$

Choose three vertices in the second layer:

$$A := \big\{\{x\}\big\}, \quad B := \big\{\{y\}\big\}, \quad C := \big\{\{x\}, \{y\}\big\}.$$

Set

$$V := \{A, B, C\} \subseteq \mathcal{P}^2(H).$$

Define hyperedges one layer higher (as sets of these vertices):

$$e_1 := \{A, B\}, \qquad e_2 := \{C\}, \qquad E := \{e_1, e_2\} \subseteq \mathcal{P}(V) \subseteq \mathcal{P}^3(H).$$

Verification: (i) Each of A, B, C is a subset of $\mathcal{P}(H)$, so $V \subseteq \mathcal{P}^2(H)$. (ii) Each e_i is a subset of V, thus $E \subseteq \mathcal{P}(V) \subseteq \mathcal{P}^3(H)$. Hence $\mathcal{H} = (V, E)$ is a SuperHyperGraph of depth $n = 2$. *Interpretation.* A vertex (e.g. C) is itself a *family of subsets* of H (here $\{\{x\}, \{y\}\}$); an edge (e.g. e_1) groups such families, capturing multi-level aggregation.

The "root" must simultaneously peel m subset–layers from *both* the vertex carrier and the edge carrier, in a way that preserves incidence (edges are sets of vertices).

Definition 4.4 (Presentation of a depth-n SuperHyperGraph at order m). Let $\mathcal{H} = (V, E)$ be a depth-n SuperHyperGraph on H and fix m with $1 \leqslant m \leqslant n$. An *$m$-presentation* of $\mathcal{H}$ consists of sets $V_\downarrow$ and $E_\downarrow$ and bijections

$$\theta_V : \mathcal{P}^m(V_\downarrow) \xrightarrow{\cong} V, \qquad \theta_E : \mathcal{P}^m(E_\downarrow) \xrightarrow{\cong} E,$$

such that

$$V_\downarrow \subseteq \mathcal{P}^{n-m}(H), \qquad E_\downarrow \subseteq \mathcal{P}(V_\downarrow) \quad \text{(incidence at the root level)},$$

and the bijections are *incidence-compatible*:

$$\theta_E = \mathcal{P}^m(\theta_V) \text{ restricted to } \mathcal{P}^m(E_\downarrow) \subseteq \mathcal{P}^m\big(\mathcal{P}(V_\downarrow)\big) = \mathcal{P}\big(\mathcal{P}^m(V_\downarrow)\big).$$

Equivalently, for every $e_\downarrow \in E_\downarrow$ one has

$$\theta_E\big(\mathcal{P}^m(e_\downarrow)\big) = \big\{\, \theta_V(\mathcal{P}^m(v_\downarrow)) \;\big|\; v_\downarrow \in e_\downarrow \,\big\} \in \mathcal{P}(V).$$

Definition 4.5 (Root–SuperHyperGraph of order m)**.** Given an m-presentation $(V_\downarrow, E_\downarrow, \theta_V, \theta_E)$ of $\mathcal{H} = (V, E)$ as above, the *Root–SuperHyperGraph of order* m is the depth-$(n-m)$ SuperHyperGraph

$$\mathrm{RootSHG}_m(\mathcal{H}) := \big(V_\downarrow,\, E_\downarrow\big), \qquad V_\downarrow \subseteq \mathcal{P}^{n-m}(H),\; E_\downarrow \subseteq \mathcal{P}(V_\downarrow).$$

Proposition 4.6 (Lift recovers the original (exact inverse on presentations))**.** *Let $\mathcal{H} = (V, E)$ be of depth n and admit an m-presentation. Define the m-fold* lift *functor on pairs by*

$$\mathrm{Lift}^m(V_\downarrow, E_\downarrow) := \big(\, \mathcal{P}^m(V_\downarrow)\,,\, \mathcal{P}^m(E_\downarrow)\,\big),$$

with the evident incidence $\mathcal{P}^m(E_\downarrow) \subseteq \mathcal{P}(\mathcal{P}^m(V_\downarrow))$. Then

$$\big(\mathcal{P}^m(V_\downarrow),\, \mathcal{P}^m(E_\downarrow)\big) \xrightarrow[\cong]{(\theta_V,\,\theta_E)} (V, E).$$

Hence, on the class of depth-n SuperHyperGraphs that admit an m-presentation,

$$\mathrm{Lift}^m \circ \mathrm{RootSHG}_m \cong \mathrm{Id}.$$

Proof. Incidence-compatibility gives $\theta_E = \mathcal{P}^m(\theta_V)$ on the edge domain, so (θ_V, θ_E) is an isomorphism of pairs that respects $E \subseteq \mathcal{P}(V)$. □

Remark 4.7 (Existence and sizes (finite case)). A necessary and sufficient carrier condition is the tower equalities

$$|V| = T_m(|V_\downarrow|), \qquad |E| = T_m(|E_\downarrow|).$$

Moreover, $E \subseteq \mathcal{P}(V)$ must be *lift-generated* from some $E_\downarrow \subseteq \mathcal{P}(V_\downarrow)$ via the chosen θ_V. When these hold, the root is unique up to isomorphism.

Example 4.8 (Feature configurations and test–suites (square root, $m = 1$, depth $n = 1$))**.** Let $H = \{\mathsf{Search}, \mathsf{Share}, \mathsf{Sync}\}$. Define a depth-1 SuperHyperGraph

$$V := \mathcal{P}(H) \quad \text{(all user configurations)},$$

$$E := \mathcal{P}(V) \quad \text{(test–suites: sets of configurations)}.$$

Take

$$V_\downarrow := H \subseteq \mathcal{P}^0(H),$$

$$E_\downarrow := \mathcal{P}(H) \subseteq \mathcal{P}(V_\downarrow),$$

and the identity bijections $\theta_V = \mathrm{id}_{\mathcal{P}(H)} : \mathcal{P}^1(V_\downarrow) \to V$ and $\theta_E = \mathrm{id}_{\mathcal{P}(V)} : \mathcal{P}^1(E_\downarrow) \to E$. Then $(V_\downarrow, E_\downarrow)$ is the *Root–SuperHyperGraph of order* 1:

$$\mathrm{RootSHG}_1(V, E) \;=\; (H,\, \mathcal{P}(H)).$$

Cardinals: $|H| = 3$, $|V| = |\mathcal{P}(H)| = 2^3 = 8$, $|E| = |\mathcal{P}(V)| = 2^8 = 256$, and indeed $|V| = T_1(3)$, $|E| = T_1(8)$. Operationally, one "root" step turns "test–suites of configurations" back into "hyperedges (feature bundles) on the base feature set".

Example 4.9 (Course planning: profiles, bundles, and catalogs (cube root, $m = 2$, depth $n = 2$))**.** Let $H = \{\mathsf{Math}, \mathsf{CS}\}$. Interpret: profiles = subsets of H; bundles = sets of profiles; catalogs = sets of bundles. Define a depth-2 SuperHyperGraph

$$V \;:=\; \mathcal{P}^2(H) \quad \text{(bundles of profiles)}, \qquad E \;:=\; \mathcal{P}(V) \;=\; \mathcal{P}^3(H) \quad \text{(catalogs of bundles)}.$$

Choose the 2-root data

$$V_\downarrow := H \subseteq \mathcal{P}^0(H), \qquad E_\downarrow := \mathcal{P}^1(H) \subseteq \mathcal{P}(V_\downarrow),$$

and bijections

$$\theta_V := \mathrm{id}_{\mathcal{P}^2(H)} :\; \mathcal{P}^2(V_\downarrow) \to V, \qquad \theta_E := \mathrm{id}_{\mathcal{P}^3(H)} :\; \mathcal{P}^2(E_\downarrow) \to E.$$

Then

$$\mathrm{RootSHG}_2(V, E) \;=\; \big(H,\, \mathcal{P}(H)\big),$$

and lifting twice recovers (V, E). Numbers: $|H| = 2$, so $|V| = |\mathcal{P}^2(H)| = 2^{2^2} = 16$ and $|E| = |\mathcal{P}^3(H)| = 2^{16} = 65536$. At the root level, $|V_\downarrow| = 2$ and $|E_\downarrow| = |\mathcal{P}(H)| = 4$, satisfying $|V| = T_2(|V_\downarrow|)$, $|E| = T_2(|E_\downarrow|)$. Semantically: the root collapses "catalogs of bundles" to "hyperedges on courses".

Example 4.10 (Marketing campaigns: audiences, sets of audiences, and playlists (mixed finite instance))**.** Let $H = \{\mathsf{New}, \mathsf{Returning}, \mathsf{VIP}\}$ (audience tags). Build a depth-1 SuperHyperGraph by

$$V := \{\text{all audience segments}\} = \mathcal{P}(H), \quad E := \{\text{all playlists of segments used this quarter}\} \subseteq \mathcal{P}(V).$$

Let $E_\downarrow \subseteq \mathcal{P}(H)$ be the subset of segments actually deployed (e.g., only $\emptyset, \{\mathsf{New}\}, \{\mathsf{VIP}\}, \{\mathsf{New}, \mathsf{VIP}\}$). Then $(V_\downarrow, E_\downarrow) = (H, E_\downarrow)$ is an $m = 1$ root with the canonical θ_V, θ_E. One lift maps each deployed segment to the corresponding vertex and each playlist to a set of those vertices, reproducing the in-use SuperHyperGraph $E = \mathcal{P}(E_\downarrow)$.

5. Conclusion

This paper investigated whether *fractional* and *inverse* layers can be meaningfully incorporated into set theory and superhyperstructural models. In future work, we hope to explore extensions of these concepts by employing established uncertain-set frameworks (cf.[35, 36]) such as Fuzzy Sets [37, 38], HyperFuzzy Sets [39, 40, 41], Intuitionistic Fuzzy Sets [42, 43], Spherical Fuzzy Sets [44, 45], Rough Sets [46, 47], Soft Sets [48, 49, 50], Neutrosophic Sets[51, 52], and Plithogenic Sets [53, 54, 55]. These directions promise a richer understanding and broader applicability of fractional and inverse powerset constructions.

Beyond uncertainty-based models, we also envisage extending the nth powerset along algebraic and hypercomplex lines—for example, over algebraic numbers and number rings such as the Eisenstein [56, 57, 58] and Gaussian integers [59, 60], as well as within quaternionic [61, 62, 63] and octonionic [64, 65] settings—together with the development of new operations compatible with root, negative, and complex-height formalisms.

Funding

This study was conducted without any financial support from external organizations or grants.

Acknowledgments

We would like to express our sincere gratitude to everyone who provided valuable insights, support, and encouragement throughout this research. We also extend our thanks to the readers for their interest and to the authors of the referenced works, whose scholarly contributions have greatly influenced this study. Lastly, we are deeply grateful to the publishers and reviewers who facilitated the dissemination of this work.

Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this paper.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

References

[1] Thomas Vougiouklis. *Hyperstructures and their representations*. Hadronic Press, 1994.

[2] Bijan Davvaz and Irina Cristea. Fuzzy algebraic hyperstructures. *Studies in Fuzziness and soft computing*, 321:38–46, 2015.

[3] Madeleine Al-Tahan and Bijan Davvaz. Chemical hyperstructures for elements with four oxidation states. *Iranian Journal of Mathematical Chemistry*, 13(2):85–97, 2022.

[4] Bijan Davvaz and Thomas Vougiouklis. *Walk Through Weak Hyperstructures, A: Hv-structures*. World Scientific, 2018.

[5] M Al Tahan and Bijan Davvaz. Weak chemical hyperstructures associated to electrochemical cells. *Iranian Journal of Mathematical Chemistry*, 9(1):65–75, 2018.

[6] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[7] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.

[8] Adel Al-Odhari. Neutrosophic power-set and neutrosophic hyper-structure of neutrosophic set of three types. *Annals of Pure and Applied Mathematics*, 31(2):125–146, 2025.

[9] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[10] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[11] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[12] Florentin Smarandache. The cardinal of the m-powerset of a set of n elements used in the superhyperstructures and neutrosophic superhyperstructures. *Systems Assessment and Engineering Management*, 2:19–22, 2024.

[13] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[14] Piergiulio Corsini and Violeta Leoreanu. *Applications of hyperstructure theory*, volume 5. Springer Science & Business Media, 2013.

[15] Michal Novák, Štepán Křehlík, and Kyriakos Ovaliadis. Elements of hyperstructure theory in uwsn design and data aggregation. *Symmetry*, 11(6):734, 2019.

[16] Takaaki Fujita. Toward a unified framework for knot theory, hyperknot theory, and superhyperknot theory via superhyperstructures. *Neutrosophic Knowledge*, 6:55–71, 2025.

[17] Igor V Novozhilov. *Fractional analysis: Methods of motion decomposition*. Springer Science & Business Media, 2012.

[18] Amin Jajarmi and Dumitru Baleanu. A new fractional analysis on the interaction of hiv with cd4+ t-cells. *Chaos, Solitons & Fractals*, 113:221–229, 2018.

[19] Necdet Bildik and Sinan Deniz. A new fractional analysis on the polluted lakes system. *Chaos, Solitons & Fractals*, 122:17–24, 2019.

[20] Aykut Has and Beyhan Yilmaz. Effect of fractional analysis on magnetic curves. *Revista mexicana de física*, 68(4), 2022.

[21] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[22] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[23] Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, Shenda Hong, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *IJCAI*, pages 1923–1929, 2022.

[24] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[25] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[26] Min Huang, Fenghua Li, et al. Optimizing ai-driven digital resources in vocational english learning using plithogenic n-superhypergraph structures for adaptive content recommendation. *Neutrosophic Sets and Systems*, 88:283–295, 2025.

[27] Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84(1):28, 2025.

[28] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[29] T. Fujita and F. Smarandache. Competition super-hypergraphs: Revealing hierarchical competition in real-world networks. *Journal of Algebra and Applied Mathematics*, 23(2):97–116, 2025.

[30] Takaaki Fujita. Multi-superhypergraph neural networks: A generalization of multi-hypergraph neural networks. *Neutrosophic Computing and Machine Learning*, 39:328–347, 2025.

[31] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[32] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[33] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[34] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[35] Takaaki Fujita and Florentin Smarandache. *A Dynamic Survey of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Plithogenic, and Extensional Sets*. Neutrosophic Science International Association (NSIA), 2025.

[36] Takaaki Fujita and Florentin Smarandache. *A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models*. Infinite Study, 2025.

[37] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[38] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[39] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[40] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[41] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[42] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[43] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[44] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[45] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[46] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[47] Said Broumi, Florentin Smarandache, and Mamoni Dhar. Rough neutrosophic sets. *Infinite Study*, 32:493–502, 2014.

[48] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[49] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Soft directed graphs, their vertex degrees, associated matrices and some product operations. *New Mathematics and Natural Computation*, 19(03):651–686, 2023.

[50] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[51] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[52] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[53] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[54] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[55] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.

[56] Nihat Engin Tunali, Yu-Chih Huang, Joseph J Boutros, and Krishna R Narayanan. Lattices over eisenstein integers for compute-and-forward. *IEEE Transactions on Information Theory*, 61(10):5306–5321, 2015.

[57] Qifu Tyler Sun, Jinhong Yuan, Tao Huang, and Kenneth W Shum. Lattice network codes based on eisenstein integers. *IEEE transactions on communications*, 61(7):2713–2725, 2013.

[58] Valmir Buçaj. Finding factors of factor rings over the eisenstein integers. In *Int. Math. Forum*, volume 9, pages 1521–1537, 2014.

[59] Klaus Huber. Codes over gaussian integers. *IEEE Transactions on Information Theory*, 40(1):207–216, 2002.

[60] Henry G Baker. Complex gaussian integers for 'gaussian graphics'. *ACM Sigplan Notices*, 28(11):22–27, 1993.

[61] Jinze Du and Chang Wang. Representing complex vague soft set by quaternion numbers. *J. Intell. Fuzzy Syst.*, 45:6679–6690, 2023.

[62] Hsuan T Chang, Chung J Kuo, Neng-Wen Lo, and Wei-Z Lv. Dna sequence representation and comparison based on quaternion number system. *International Journal of Advanced Computer Science and Applications (IJACSA)*, 3(11), 2012.

[63] Xiquan Liang and Fuguo Ge. The quaternion numbers. *Formalized Mathematics*, 14(4):161–169, 2006.

[64] Roman Jacome, Kumar Vijay Mishra, Brian M Sadler, and Henry Arguello. Octonion phase retrieval. *IEEE Signal Processing Letters*, 31:1615–1619, 2024.

[65] Ricardo Augusto Watanabe, Cibele Cristina Trinca Watanabe, and Estevão Esmi. Fuzzy octonion numbers: Some analytical properties. In *International Fuzzy Systems Association World Congress*, pages 716–726. Springer, 2019.