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Nonlinear viscoelastic Petrovsky equation with fractional damped: Existence and blow-up

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Abstract

In this article, we study a nonlinear viscoelastic Petrovsky equation with fractional damping. First, we establish the existence of a local weak solution by using semigroup theory. Then, we prove the blow up of the solution under suitable conditions.

Keywords: Blow up, existence, fractional damping term, Petrovsky equation.
2010 MSC: 26A33, 35A01, 35B44, 35L75.

1. Introduction

In this article, we study the following Petrovsky equation with fractional damping

$$\begin{cases} u_{tt} + \Delta^2 u - \int\limits_0^{+\infty} h(s)\,\Delta^2 u(t-s)\,ds + \partial_t^{\alpha,\beta} u(t) = |u|^{q-2} u, & x \in \Omega,\ t > 0, \\ u(x,t) = \frac{\partial}{\partial \upsilon} u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n $(n > 1)$, with a smooth boundary $\partial\Omega$, υ is the unit outer normal to $\partial\Omega$. The constants $q > 2$ and $s > 0$. Also, (u_0, u_1) the initial data belong to a suitable function space. The kernel h is a function that will be specified later. The notation $\partial_t^{\alpha,\beta}$ stands for the generalized Caputo's fractional derivative (see [4, 5, 23]) defined by the following formula

$$\partial_t^{\alpha,\beta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} u_s(s)\,ds, \quad 0 < \alpha < 1,\ \beta > 0.$$

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- The fourth-order equation has its origin in the canonical model introduced by Petrovsky [19, 20]. This type equations arises in many branches in sciences such as acoustics, optics, geophysics and ocean acoustics [6].
- Fractional derivatives and integrals arise naturally in physics, biology, chemistry, ecology (see [16, 23, 24]).

Pişkin and Sancar [21] considered the following Petrovsky equation with a fractional time delay term

$$u_{tt} + \Delta^2 u + \alpha_1 \partial_t^{\alpha,\beta} u(t-\tau) + \alpha_2 u_t = |u|^{q-2} u.$$

They obtained the existence, decay and the blow-up of solutions under suitable conditions.

Kirane et al. [11] considered the following equation

$$u_{tt} - \Delta u + \int_0^{+\infty} g(\tau) \Delta u(t-\tau) \, d\tau + \partial_t^{\alpha,\beta} u(t) = |u|^{p-2} u.$$

They proved a non-linear wave equation with internal fractional damping, a polynomial source and an infinite memory.

Aounallah et al. [2] studied the following wave equation

$$u_{tt} - \Delta u + \alpha_1 \partial_t^{\alpha,\beta} u(t-\tau) + \alpha_2 u_t = |u|^{p-2} u.$$

They established the well-posedness, blow-up and asymptotic behavior for a wave equation with a time delay condition of fractional type.

Kirane and Tatar [10] studied the following equation

$$u_{tt} - \Delta u + \partial_t^{\alpha} u = |u|^{p-2} u.$$

They demonstrated the exponential growth for a fractionally damped wave equation.

Nicaise and Pignotti [15] considered as follows

$$u_{tt} - \Delta u + \alpha_1 u(t-s) + \alpha_2 u_t = f(u).$$

They demonstrated that the energy is exponentially stable when $\alpha_2 < \alpha_1$.

Kafini and Messaoudi [9] proved the following delayed wave equation with logarithmic source term

$$u_{tt} - \Delta u + \alpha_1 u(t-s) + \alpha_2 u_t = |u|^{p-2} u \ln |u|^k.$$

They investigated the local existence and blow-up of solutions.

Pişkin and Uysal [22] studied the following equations

$$u_{tt} + \Delta^2 u + \partial_t^{1+\alpha} u = |u|^{p-1} u.$$

They proved the blow-up of solution.

Georgiev and Todorova [7] studied the following equations

$$u_{tt} - \Delta u + a u_t |u_t|^{m-1} = bu |u|^{p-1}.$$

They considered the existence of a solution of the wave equation nonlinear damping and source term.

This study presents a solid mathematical foundation for analyzing the behavior of these systems and carries significant consequences for understanding and engineering viscoelastic materials and related physical phenomena. Additionally, the works by [1, 25, 26] can also be consulted for the literature review.

The purpose of this paper is to study the existence and the blow-up of solutions to problem (1.1).

Our study is divided into four parts.

- In part 2, we give some important lemmas.
- In part 3, we obtain the well-posedness by the semigroup method.
- In part 4, we prove the blow-up of solutions.

2. Preliminaries

This section presents the necessary materials required to support the proof of our findings. We outline the following assumptions:

(G1) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathbb{C}^1$ function such that

$$h(0) > 0, \quad h_0 = \int_0^{+\infty} h(s)\, ds = 1 - \lambda > 0,$$

(G2) there exists a positive constant θ such that:

$$h'(t) \leqslant -\theta h(t), \quad t \geqslant 0.$$

We assert the following statements without providing proof.

Lemma 2.1. *The following inequality holds:*

$$\int_\Omega \left(\int_0^{+\infty} h(s) \Delta w(s)\, ds \right)^2 dx \leqslant (1-\lambda) \int_0^{+\infty} h(s) \|\Delta w(s)\|^2 ds.$$

Lemma 2.2. *[13] Let η be the function:*

$$\eta(\xi) = |\xi|^{\frac{2\alpha-1}{2}}, \quad \xi \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Then the relationship between the "input" U and the "output" O of the system

$$\begin{cases} \phi_t(x,\xi,t) + \left(\xi^2+\beta\right)\phi(x,\xi,t) - U(x,t)\eta(\xi) = 0, \ \xi \in \mathbb{R}, \ t>0, \ \beta>0, \\ \phi(x,\xi,0) = 0, \\ O(t) = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \phi(x,\xi,t)\eta(\xi)\, d\xi \end{cases} \tag{2.1}$$

is given by

$$O = I^{1-\alpha,\beta} U,$$

where

$$I^{\alpha,\beta} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\beta(t-s)} u(s)\, ds.$$

Lemma 2.3. *[3] For all* $\lambda \in D_{\beta=} \{\lambda \in \mathbb{C} : Re\lambda + \beta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$,

$$\int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\lambda + \beta + \xi^2} d\xi = \frac{\pi}{\sin(\alpha\pi)} (\lambda + \beta)^{\alpha - 1}.$$

Now, we introduce, as in [14, 17] the new variable

$$z_t(x,s) = u(x,t) - u(x,t-s), \tag{2.2}$$

where z is the relative history of u that satisfies

$$z_t(x,s) - u_t(x,t) + z_s(x,s) = 0, \quad x \in \Omega,\ t,s > 0. \tag{2.3}$$

Therefore, by (2.2) and using Lemma 2, problem (1.1) is equivalent to

$$\begin{cases} u_{tt} + \lambda\Delta^2 u + \int\limits_0^{+\infty} h(s)\,\Delta^2 z(x,s)\,ds + b\int_{-\infty}^{+\infty} \phi(x,\xi,t)\,\eta(\xi)\,d\xi = |u|^{q-2}u, & x \in \Omega,\ t > 0, \\ \phi_t(x,\xi,t) + \left(\xi^2 + \beta\right)\phi(x,\xi,t) - u_t(x,t)\,\eta(\xi) = 0, & x \in \Omega,\ \xi \in R,\ t > 0, \\ z_t(x,s) + z_s(x,s) = u_t(x,t), & x \in \Omega,\ t,s > 0, \\ u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ z(x,0) = 0, & x \in \Omega, \\ z_0(x,s) = u_0(x) - u_0(x,-s), & x \in \Omega,\ t,s > 0, \\ \phi(x,\xi,0) = 0, & x \in \Omega,\ \xi \in R, \end{cases} \tag{2.4}$$

where $b = \frac{\sin(\alpha\pi)}{\pi}$.

Lemma 2.4. *For* $z \in L^2(\Omega)$ *and* $\xi\phi \in L^2(\Omega \times (-\infty, +\infty))$, *we have*

$$\begin{aligned} \left| \int_\Omega z(x,\rho,t) \int_{-\infty}^{+\infty} \eta(\xi)\,\phi(x,\xi,t)\,d\xi dx \right| &\leqslant A_0 \int_\Omega |z(x,\rho,t)|^2\,dx \\ &\quad + \frac{1}{4} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi(x,\xi,t)|^2\,d\xi dx \end{aligned}$$

for a positive constant A_0.

Proof. Applying the Cauchy-Schwarz inequality, we get

$$\left| \int_{-\infty}^{+\infty} \eta(\xi)\,\phi(x,\xi,t)\,d\xi dx \right| \leqslant \left(\int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\xi^2 + \beta} d\xi \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi(x,\xi,t)|^2\,d\xi \right)^{\frac{1}{2}}.$$

Using Young's inequality, we have

$$\begin{aligned} \left| \int_\Omega z(x,\rho,t) \int_{-\infty}^{+\infty} \eta(\xi)\,\phi(x,\xi,t)\,d\xi dx \right| &\leqslant A_0 \int_\Omega |z(x,\rho,t)|^2\,dx \\ &\quad + \frac{1}{4} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi(x,\xi,t)|^2\,d\xi dx \end{aligned}$$

with

$$A_0 = \int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\xi^2 + \beta} d\xi.$$

This completes the proof. □

Lemma 2.5. *The energy*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2\, d\xi dx + \frac{\lambda}{2}\|\Delta u\|^2 \\ & -\frac{1}{q}\|u\|_q^q + \frac{1}{2}\int_0^{+\infty} h(s)\,\|\Delta z(s)\|^2\, ds, \end{aligned} \tag{2.5}$$

satisfies

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{1}{2}\int_0^{+\infty} h'(s)\,\|\Delta z(s)\|^2\, ds \\ & -b\int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2\, d\xi dx \\ &\leqslant 0. \end{aligned} \tag{2.6}$$

Proof. Multiply u_t with the first equation of (2.4) and integrating by parts over Ω, we obtain

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{\lambda}{2}\|\Delta u\|^2 - \frac{1}{q}\|u\|_q^q\right] \\ & +b\int_\Omega u_t \int_{-\infty}^{+\infty} \eta(\xi)\,\phi(x,\xi,t)\, d\xi dx \\ & +\int_\Omega u_t \int_0^{+\infty} h(s)\,\Delta^2 u(s)\, ds dx = 0. \end{aligned}$$

We apply the method in (2.3) to modify the final term in (2.7) in the following manner:

$$\begin{aligned} \int_\Omega u_t \int_0^{+\infty} h(s)\,\Delta^2 u(s)\, ds\, dx &= \int_0^{+\infty} h(s)\int_\Omega (z_t+z_s)\,\Delta^2 u(s)\, dx\, ds \\ &= \int_0^{+\infty} h(s)\int_\Omega z_t\,\Delta^2 u(s)\, dx\, ds \\ &\quad + \int_0^{+\infty} h(s)\int_\Omega z_s\,\Delta^2 u(s)\, dx\, ds, \end{aligned}$$

and integrating by parts, we have

$$\begin{aligned} \int_\Omega u_t \int_0^{+\infty} h(s)\,\Delta^2 u(s)\, ds dx &= \frac{d}{dt}\left[\frac{1}{2}\int_0^{+\infty} h(s)\,\|\Delta z(s)\|^2\, ds\right] \\ & -\frac{1}{2}\int_0^{+\infty} h'(s)\,\|\Delta z(s)\|^2\, ds. \end{aligned} \tag{2.7}$$

By substituting (2.7) in (2.7), we get

$$\begin{aligned} & \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{\lambda}{2}\|\Delta u\|^2 - \frac{1}{q}\|u\|_q^q + \frac{1}{2}\int_0^{+\infty} h(s)\,\|\Delta z(s)\|^2\, ds\right) \\ & -\frac{1}{2}\int_0^{+\infty} h'(s)\,\|\Delta z(s)\|^2\, ds + b\int_\Omega u_t \int_{-\infty}^{+\infty} \eta(\xi)\,\phi(x,\xi,t)\, d\xi dx \\ =\;& 0. \end{aligned} \tag{2.8}$$

Now multiply $b\phi$ with the second equation of (2.4) and integrating by parts over $\Omega \times R$, we have

$$\begin{aligned} & \frac{d}{dt}\left(\frac{b}{2}\int_{\Omega}\int_{-\infty}^{+\infty}\left|\phi\left(x,\xi,t\right)\right|^{2}d\xi dx\right) \\ & +b\int_{\Omega}\int_{-\infty}^{+\infty}\left(\xi^{2}+\beta\right)\left|\phi\left(x,\xi,t\right)\right|^{2}d\xi dx \\ & -b\int_{\Omega}u_{t}\int_{-\infty}^{+\infty}\eta\left(\xi\right)\phi\left(x,\xi,t\right)d\xi dx \\ = & \; 0. \end{aligned} \tag{2.9}$$

Utilizing references (2.5), (2.8) and (2.9), we derive the result in (2.6), thus completing the proof of the lemma. □

3. Well-posedness

In this section, we establish the result for the local existence of the problem (2.4). Firstly, we present the function of vector

$$U = \left(u, v, \phi, z\right)^{T}$$

and a new dependent variable

$$v = u_t.$$

So, (2.4) can be rewritten as follows:

$$\begin{cases} U_t\left(t\right) + \mathcal{A}U\left(t\right) = J\left(\Phi\left(t\right)\right), \\ U\left(0\right) = U_0, \end{cases} \tag{3.1}$$

where the operator $\mathcal{A} : D\left(\mathcal{A}\right) \to \mathcal{H}$ is defined by

$$\mathcal{A}U = \mathcal{A}\begin{pmatrix} u \\ v \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} -v \\ \lambda\Delta^2 u + \int\limits_0^{+\infty} h\left(s\right)\Delta^2 u\left(s\right)ds + b\int_{-\infty}^{+\infty}\phi\left(x,\xi\right)\eta\left(\xi\right)d\xi \\ \left(\xi^2+\beta\right)\phi\left(x,\xi\right) - v\left(x\right)\eta\left(\xi\right) \\ z_s - v \end{pmatrix}$$

$$J\left(U\right) = \left(0, |u|^{p-2}u, 0, 0\right)^{T}, \tag{3.2}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = H_0^2\left(\Omega\right) \times L^2\left(\Omega\right) \times L^2\left(\Omega \times \left(-\infty, +\infty\right)\right) \times L_g^2\left(R_+, H_0^2\left(\Omega\right)\right)$$

such that

$$L_h^2\left(R_+, H_0^2\left(\Omega\right)\right) = \left\{ w : R_+ \to H_0^2\left(\Omega\right), \int\limits_0^{+\infty} h\left(s\right)\left\|\Delta w\left(s\right)\right\|^2 ds < \infty \right\};$$

the space $L_h^2\left(R_+, H_0^2(\Omega)\right)$ is endowed with the inner product:

$$\langle w_1, w_2\rangle_{L_h^2(R_+,H_0^2(\Omega))} = \int\limits_0^{+\infty} h(s)\int_\Omega \Delta w_1(s)\,\Delta w_2(s)\,dxds.$$

For any $U=(u,v,\phi,z)^T \in \mathcal{H}$ and $\widetilde{U}=\left(\widetilde{u},\widetilde{v},\widetilde{\phi},\widetilde{z}\right)^T \in \mathcal{H}$, we equip with the inner product defined by

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_\Omega [\lambda\Delta u\Delta\widetilde{u} + v\widetilde{v}]\,dx + b\int_\Omega\int_{-\infty}^{+\infty}\phi(x,\xi)\,\widetilde{\phi}(x,\xi)\,d\xi dx \\ &\quad + \int\limits_0^{+\infty} h(s)\int_\Omega \Delta z(s)\,\Delta\widetilde{z}(s)\,dxds.\end{aligned}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{\begin{array}{l} U=(u,v,\phi,z)^T \in \mathcal{H} : u\in H_0^2(\Omega),\ v\in H_0^2(\Omega), \\ \left(\xi^2+\beta\right)\phi - v\eta(\xi)\in L^2(\Omega\times(-\infty,+\infty)), \\ |\xi|\,\phi \in L^2(\Omega,R),\ z_s \in L_h^2\left(R_+, H_0^2(\Omega)\right). \end{array}\right\}$$

Now, we are able to give the result of following existence.

Theorem 3.1. *Assume that*

$$\begin{cases} 2<q<\infty, & \text{if } n=1,2,3,4; \\ 2<q\leqslant \frac{2n}{n-4}, & \text{if } n\geqslant 5. \end{cases} \tag{3.3}$$

Suppose further that

$$U_0 \in \mathcal{H}, \tag{3.4}$$

then the problem (2.4) has a unique local solution

$$U \in C([0,T), \mathcal{H}). \tag{3.5}$$

Proof. The demonstration is founded on [8, 12, 18]. We begin by showing that $\mathcal{A}$ is a maximal monotone operator on $\mathcal{H}$. We begin by demonstrating that the operator $\mathcal{A}$ is monotone. For, for any $U\in D(\mathcal{A})$, using (3.1), we get

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= b\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi)|^2\,d\xi dx \\ &\quad -\frac{1}{2}\int_0^{+\infty} h'(s)\,\|\Delta z(s)\|^2\,ds \\ &\geqslant 0. \end{aligned}\tag{3.6}$$

So, the operator $\mathcal{A}$ is a monotone . Next, we show that the operator $I+\mathcal{A}$ is surjective. For, given $F=(f_1,f_2,f_3,f_4)^T \in \mathcal{H}$, we will show that there exist $U=(u,v,\phi,z)^T \in D(\mathcal{A})$ satisfying

$$(I+\mathcal{A})\,U = F,$$

that is,

$$\begin{cases} u - v = f_1 \in H_0^2(\Omega), \\ v + \lambda\Delta^2 u + \int\limits_0^{+\infty} h(s)\,\Delta^2 u(s)\,ds + b\int_{-\infty}^{+\infty} \phi(\xi)\,\eta(\xi)\,d\xi = f_2 \in L^2(\Omega), \\ \phi + \left(\xi^2 + \beta\right)\phi - v\eta(\xi) = f_3(\xi) \in L^2(\Omega, R), \\ z + z_s - v = f_4(s) \in L_h^2\left(R_+, H_0^2(\Omega)\right). \end{cases} \tag{3.7}$$

Using the third equation in (3.7), we have

$$\phi = \frac{f_3 + v\eta(\xi)}{\xi^2 + \beta + 1}. \tag{3.8}$$

Conversely, a unique solution can be found for the fourth equation in (3.7)

$$z = \left(\int_0^s e^{\sigma}\left(f_4(\sigma) + u - f_1\right) d\sigma\right) e^{-s}. \tag{3.9}$$

Inserting $v = u - f_1$, (3.8) and (3.9) in the second equation in (3.7), we have

$$Ku + \widetilde{\lambda}\Delta^2 u = G, \tag{3.10}$$

here

$$\begin{cases} K = 1 + b\int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\xi^2+\beta+1} d\xi > 0, \\ \widetilde{\lambda} = \lambda + \int\limits_0^{+\infty} h(s)\,e^{-s}\left(\int_0^s e^{\sigma} d\sigma\right) ds \\ = 1 - \int\limits_0^{+\infty} h(s)\,e^{-s} ds > 0, \\ G = f_2 + Kf_1 - b\int_{-\infty}^{+\infty} \frac{\eta(\xi) f_3(\xi)}{\xi^2+\beta+1} d\xi \\ + \int\limits_0^{+\infty} h(s)\,e^{-s}\left(\int_0^s e^{\sigma}\Delta^2\left(f_4(\sigma) - f_1\right) d\sigma\right) ds. \end{cases}$$

To solve (3.10), we consider the following variational formulation:

$$B(u, w) = L(w), \quad \forall w \in H_0^2(\Omega), \tag{3.11}$$

here B is the bilinear form defined by

$$B(u, w) = \mu\int_{\Omega} uw\,dx + \widetilde{\lambda}\int_{\Omega} \Delta u \Delta w\,dx \tag{3.12}$$

and L is the linear functional given by

$$L(w) = \int_{\Omega} Gw\,dx. \tag{3.13}$$

It can be easily confirmed that L is bounded, B is both coercive and bounded. So, applying the Lax-Milgram theorem, we deduce that for all $w \in H_0^2(\Omega)$, the linear eliptic equation (3.10) has a unique solution $u \in H_0^2(\Omega)$. Substituting u into the first equation in (3.7) gives $v \in H_0^2(\Omega)$. Inserting v in (3.7) and bearing in mind the third equation in (3.7), we obtain

$$\phi \in L^2(\Omega, R).$$

Similarly, we have

$$z \in L_h^2\left(R_+, H_0^2(\Omega)\right).$$

Using (3.10), we get

$$K\int_\Omega uw dx + \widetilde{\lambda}\int_\Omega \Delta u \Delta w dx = \int_\Omega G w dx. \tag{3.14}$$

The elliptic regularity theory, then, implies that $u \in H^2(\Omega)$. So, $I + \mathcal{A}$ is surjective. Now, we prove that the operator defined in (3.2) is locally Lipschitz in $\mathcal{H}$. For $U, \widetilde{U} \in \mathcal{H}$, we have

$$\begin{aligned}
\left\| J(U) - J(\widetilde{U}) \right\|_{\mathcal{H}} &= \left\| 0, u\,|u|^{q-2} - \widetilde{u}\,|\widetilde{u}|^{q-2}, 0, 0 \right\|_{\mathcal{H}} \\
&= \left\| u\,|u|^{q-2} - \widetilde{u}\,|\widetilde{u}|^{q-2} \right\|_{L^2(\Omega)} \\
&= \left\| |u|^q - |\widetilde{u}|^q \right\|_{L^2(\Omega)} \\
&= \left\| (u-\widetilde{u})\left(|u|^{q-1} + u^{q-2}\widetilde{u} + \dots + \widetilde{u}^{q-1} \right) \right\|_{L^2(\Omega)} \\
&= C\left\| (u-\widetilde{u})\left(u^{q-1} + \widetilde{u}^{q-1} \right) \right\|_{L^2(\Omega)} \\
&\leqslant C\left(\int_\Omega \left(|u-\widetilde{u}|^2 \right)\left(|u|^{q-1} + |\widetilde{u}|^{q-1} \right)^2 dx \right)^2.
\end{aligned}$$

Using Hölder's inequality, we get

$$\left\| J(U) - J(\widetilde{U}) \right\|_{\mathcal{H}} \leqslant C\left(\int_\Omega |u-\widetilde{u}|^{2\gamma} dx \right)^{\frac{1}{2\gamma}} \left(\int_\Omega \left(|u|^{q-1} + |\widetilde{u}|^{q-1} \right)^{2\delta} dx \right)^{\frac{1}{2\delta}}, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1,$$

with $\gamma = \frac{n}{n-4}$ and $\delta = \frac{n}{4}$. So, we get

$$\begin{aligned}
\left\| J(U) - J(\widetilde{U}) \right\|_{\mathcal{H}} &\leqslant C\left(\int_\Omega |u-\widetilde{u}|^{\frac{2n}{n-4}\gamma} dx \right)^{\frac{n-4}{2n}} \left(\int_\Omega \left(|u|^{q-1} + |\widetilde{u}|^{q-1} \right)^n dx \right)^{\frac{1}{n}} \\
&\leqslant C\left(\int_\Omega |u-\widetilde{u}|^{\frac{2n}{n-4}\gamma} dx \right)^{\frac{n-4}{2n}} \left(\int_\Omega \left(|u|^{n(q-1)} + |\widetilde{u}|^{n(q-1)} \right) dx \right)^{\frac{1}{n}} \\
&\leqslant C\left\| u-\widetilde{u} \right\|_{L^{\frac{2n}{n-4}}(\Omega)} \left[\left(\int_\Omega |u|^{n(q-1)} dx \right)^{\frac{1}{n}} + \left(\int_\Omega |\widetilde{u}|^{n(q-1)} dx \right)^{\frac{1}{n}} \right] \\
&\leqslant C\left\| u-\widetilde{u} \right\|_{L^{\frac{2n}{n-4}}(\Omega)} \left[\|u\|_{L^{n(q-1)}(\Omega)}^{q-1} + \|\widetilde{u}\|_{L^{n(q-1)}(\Omega)}^{q-1} \right].
\end{aligned} \tag{3.15}$$

The Sobolev embedding theorem gives

$$\|u-\widetilde{u}\|_{L^{\frac{2n}{n-4}}(\Omega)} \leqslant C\,\|u-\widetilde{u}\|_{L^2(\Omega)} \leqslant C\left\| U - \widetilde{U} \right\|_{\mathcal{H}}. \tag{3.16}$$

The necessity to estimate $\|u\|_{n(q-1)}$ by the energy form $\|U\|_{\mathcal{H}}$ requires to consider different ranges of q. Namely, we need $n(q-1) \leqslant \frac{2n}{n-4}$ and this coincides with the cut in our assumption $q \leqslant \frac{n}{n-4}$. Thus, the Sobolev embedding theorem

$$H_0^2(\Omega) \hookrightarrow L^{\frac{n}{n-4}}(\Omega),$$

it holds

$$\|u\|_{L^{n(q-1)}}^{q-1}(\Omega) \leqslant C\|u\|_{H_0^2}^{q-1}(\Omega). \tag{3.17}$$

Therefore, by combining (3.16) and (3.17), we obtain

$$\left\|U-\tilde{U}\right\|_{\mathcal{H}} \leqslant C\left(\|u\|_{H_0^2}^{q-1}(\Omega),\ \|\tilde{u}\|_{H_0^2}^{q-1}(\Omega)\right)\left\|U-\tilde{U}\right\|_{\mathcal{H}}.$$

So, J is locally Lipschitzian. □

4. Blow up

In this part, we employ a carefully chosen Lyapunov functional to demonstrate that certain solutions may blow up in finite time. To accomplish this, we first require the following lemma.

Lemma 4.1. *Suppose that $q \geqslant 2$. Then, there exists a positive constant $C > 1$ such that*

$$\|u\|_q^l \leqslant C_2\left[\|u\|_q^q + \|\Delta u\|^2\right] \tag{4.1}$$

for any $u \in H_0^2(\Omega)$ and $2 \leqslant l \leqslant q$.

Proof. If $\|u\|_q \geqslant 1$ then $\|u\|_q^l \leqslant \|u\|_q^q$. If $\|u\|_q \leqslant 1$ then $\|u\|_q^l \leqslant \|u\|_q^2 \leqslant C_*\|\Delta u\|^2$ by Sobolev embedding theorems. □

Let

$$\begin{aligned} H(t) &= -E(t) = \frac{1}{q}\|u\|_q^q - \frac{1}{2}\|u_t\|^2 - \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2\,d\xi dx \\ &\quad -\frac{\lambda}{2}\|\Delta u\|^2 - \frac{1}{2}\int_0^{+\infty} h(s)\|\Delta z(s)\|^2\,ds. \end{aligned} \tag{4.2}$$

Theorem 4.2. *Suppose that $q > 4$ satisfies (3.3). Assume further that (G1)*

$$h_0 = \int_0^{+\infty} h(s)\,ds < \frac{q-4}{q} \tag{4.3}$$

and

$$E(0) \leqslant 0. \tag{4.4}$$

Then the solution of system (2.4) blows up in finite time.

Proof. Using (2.6), we get

$$E(t) \leqslant E(0) \leqslant 0. \tag{4.5}$$

So, we have

$$\begin{aligned} H^{'}(t) &= -E^{'}(t) = -\frac{1}{2}\int_0^{+\infty} h'(s)\|\Delta z(s)\|^2 ds \\ &+b\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx \\ &\geqslant 0. \end{aligned} \tag{4.6}$$

In addition, we get

$$0 < H(0) \leqslant H(t) \leqslant \frac{1}{q}\|u\|_q^q. \tag{4.7}$$

Let

$$\varphi(t) = H^{1-\gamma}(t) + \varepsilon\int_\Omega uu_t dx, \tag{4.8}$$

where $\varepsilon > 0$ to be specified later and

$$0 < \gamma < \frac{q-2}{2q}. \tag{4.9}$$

By applying differentiation to (4.8) and utilizing (2.4), we derive

$$\begin{aligned} \varphi^{'}(t) &= (1-\gamma)H^{-\gamma}H^{'}(t) + \varepsilon\|u_t\|^2 - \varepsilon\lambda\|\Delta u\|^2 \\ &-b\varepsilon\int_\Omega u\int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t)d\xi dx + \varepsilon\|u\|_q^q \\ &-\varepsilon\int_\Omega \Delta u\int_0^{+\infty} h(s)\Delta z(x,s)dsdx. \end{aligned} \tag{4.10}$$

Utilizing Young's inequality in conjunction with Lemma 2.1, we derive

$$\begin{aligned} &\int_\Omega \Delta u\int_0^{+\infty} h(s)\Delta z(x,s)dsdx \\ \leqslant &\frac{1}{4}\int_0^{+\infty} h(s)\|\Delta z(s)\|^2 ds + (1-\lambda)\|\Delta u\|^2. \end{aligned} \tag{4.11}$$

here, if we assume that for the first term

$$a = \sqrt{2}(1-\lambda)\int_\Omega \Delta u dx$$

and for the second term

$$b = \frac{\int_0^{+\infty} h(s)\Delta z(x,s)ds}{\sqrt{2}(1-\lambda)}$$

we obtain (4.11). Substituting (4.11) in (4.10), we obtain

$$\begin{aligned} \varphi'(t) \geqslant\ & (1-\gamma)H^{-\gamma}H'(t)+\varepsilon\|u_t\|^2-\varepsilon\|\Delta u\|^2 \\ & -b\varepsilon\int_\Omega u\int_{-\infty}^{+\infty}\eta(\xi)\,\phi(x,\xi,t)\,d\xi dx \\ & +\varepsilon\|u\|_q^q-\frac{\varepsilon}{4}\int_0^{+\infty}h(s)\,\|\Delta z(s)\|^2\,ds. \end{aligned} \tag{4.12}$$

Utilizing Young's inequality in conjunction with (4.6), we conclude that

$$\begin{aligned} & b\int_\Omega u\int_{-\infty}^{+\infty}\eta\,(\xi)\,\phi\,(x,\xi,t)\,d\xi dx \\ \leqslant\ & \delta C_1\,\|u\|^2+\frac{b}{4\delta}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi\,(x,\xi,t)|^2\,d\xi dx \\ \leqslant\ & \delta C_1\,\|u\|^2+\frac{1}{4\delta}H'\,(t)\,, \end{aligned} \tag{4.13}$$

for $C_1=b\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta}d\xi$ and $\delta>0$, which may depend on t. Substituting (4.13) in (4.12), we get

$$\begin{aligned} \varphi'(t) \geqslant\ & \left((1-\gamma)\,H^{-\gamma}-\frac{\varepsilon}{4\delta}\right)H'\,(t) \\ & +\varepsilon\,\|u_t\|^2-\varepsilon\,\|\Delta u\|^2-\varepsilon\delta C_1\,\|u\|^2 \\ & +\varepsilon\,\|u\|_q^q-\frac{\varepsilon}{4}\int_0^{+\infty}h\,(s)\,\|\Delta z\,(s)\|^2\,ds. \end{aligned} \tag{4.14}$$

Next, we choose an appropriate δ as follows:

$$\frac{1}{4\delta}=kH^{-\gamma}\,(t)\,, \tag{4.15}$$

where k is a positive constant that will be defined later. Substituting (4.15) into (4.14), we get

$$\begin{aligned} \varphi'(t) \geqslant\ & ((1-\gamma)-\varepsilon k)\,H^{-\gamma}H'\,(t)+\varepsilon\,\|u_t\|^2 \\ & -\varepsilon\,\|\Delta u\|^2-\frac{\varepsilon C_1}{4k}H^\gamma\,(t)\,\|u\|^2 \\ & +\varepsilon\,\|u\|_q^q-\frac{\varepsilon}{4}\int_0^{+\infty}h\,(s)\,\|\Delta z\,(s)\|^2\,ds. \end{aligned} \tag{4.16}$$

Utilizing (4.7), we get

$$H^\gamma\,(t)\leqslant\frac{1}{q^\gamma}\,\|u\|_q^{q\gamma}\,. \tag{4.17}$$

Consequently, we get

$$C_1H^\gamma\,(t)\,\|u\|^2\leqslant C_2\,\|u\|_q^{q\gamma+2}\,, \tag{4.18}$$

for some $C_2 > 0$. Combining (4.16) and (4.18), we obtain

$$\begin{aligned}
\varphi'(t) \geqslant\ & ((1-\gamma)-\varepsilon k) H^{-\gamma}H'(t)+\varepsilon\left(\frac{q}{4}+1\right)\|u_t\|^2 \\
&+\frac{\varepsilon}{2}\|u\|_q^q+\varepsilon\left(\frac{\lambda q}{4}-1\right)\|\Delta u\|^2 \\
&+\frac{\varepsilon b q}{4}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2\,d\xi dx \\
&\varepsilon\left(\frac{q}{2}H(t)-\frac{C_2}{4k}(t)\,\|u\|_q^{q\gamma+2}\right) \\
&+\varepsilon\left(\frac{q-1}{4}\right)\int_0^{+\infty}h(s)\,\|\Delta z(s)\|^2\,ds. \qquad (4.19)
\end{aligned}$$

By Lemma 4.1 and (4.9), for $l = q\gamma+2 \leqslant q$, we find

$$\begin{aligned}
\varphi'(t) \geqslant\ & ((1-\gamma)-\varepsilon k) H^{-\gamma}H'(t)+\varepsilon\left(\frac{q}{4}+1\right)\|u_t\|^2 \\
&+\frac{\varepsilon}{2}\left(1-\frac{C_3}{2k}\right)\|u\|_q^q+\frac{\varepsilon}{4}\left(\lambda q-4-\frac{C_3}{k}\right)\|\Delta u\|^2 \\
&+\frac{\varepsilon b q}{4}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2\,d\xi dx+\frac{q\varepsilon}{2}H(t) \\
&+\varepsilon\left(\frac{q-1}{4}\right)\int_0^{+\infty}h(s)\,\|\Delta z(s)\|^2\,ds. \qquad (4.20)
\end{aligned}$$

where $C_3 = CC_2$. Using (4.3) and (G1), we get $q\lambda - 4 > 0$

At this point, we choose k large enough such that

$$1-\frac{C_3}{2k} > 0, \quad q\lambda-4-\frac{C_3}{k} > 0.$$

When k is fixed, we pick ε small enough such that

$$(1-\gamma)-\varepsilon k > 0, \quad H(0)+\varepsilon\int_\Omega u_0u_1\,dx > 0.$$

Therefore, there exists a positive constant C_4 such that

$$\varphi'(t) \geqslant C_4\left(H(t)+\|u_t\|^2+\|\Delta u\|^2+\|u\|_q^q\right). \qquad (4.21)$$

Furthermore, we get

$$\varphi(t) \geqslant \varphi(0) > 0, \quad t \geqslant 0. \qquad (4.22)$$

By Hölder's inequality and the embedding inequalities, we have

$$\begin{aligned}
\int_\Omega uu_t\,dx &\leqslant \|u\|_2\,\|u_t\|_2 \\
&\leqslant d\,\|u\|_q\,\|u_t\|_2,
\end{aligned}$$

where $d > 0$ is the best embedding constant. Using Young's inequality, we find

$$\left(\int_\Omega uu_t\,dx\right)^{\frac{1}{1-\gamma}} \leqslant d_1\left(\|u_t\|^{\frac{\theta'}{1-\gamma}}+\|u\|_q^{\frac{\theta}{1-\gamma}}\right), \qquad (4.23)$$

where d_1 is a constant and $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Using Lemma 4.1, for $\theta'=2\left(1-\gamma\right)$, we obtain

$$\frac{\theta}{1-\gamma}=\frac{2}{1-2\gamma}\leqslant q.$$

Thus, for $l=\frac{2}{1-2\gamma}$, we obtain

$$\left(\int_{\Omega}uu_t\,dx\right)^{\frac{1}{1-\gamma}}\leqslant d_2\left(\left\|u_t\right\|^2+\left\|\Delta u\right\|^2+\left\|u\right\|_q^q\right), \tag{4.24}$$

where $d_2>0$ is a constant. Consequently, by (4.24), we have

$$\begin{aligned}\varphi^{\frac{1}{1-\gamma}}\left(t\right) &\leqslant \left(H^{1-\gamma}\left(t\right)+\int_{\Omega}uu_t\,dx\right)^{\frac{1}{1-\gamma}}\\ &\leqslant d_3\left(H\left(t\right)+\left(\int_{\Omega}uu_t\,dx\right)^{\frac{1}{1-\gamma}}\right)\\ &\leqslant d_3\left(H\left(t\right)+\left\|u_t\right\|^2+\left\|\Delta u\right\|^2+\left\|u\right\|_q^q\right),\quad t>0,\end{aligned} \tag{4.25}$$

where d_3 is a positive constant. Combining (4.21) and (4.25), we obtain

$$\varphi'\left(t\right)\geqslant d_4\varphi^{\frac{1}{1-\gamma}}\left(t\right),\quad t\geqslant 0, \tag{4.26}$$

where d_3 is a positive constant. Integrating (4.26) over $\left(0,t\right)$, we get

$$\varphi\left(t\right)\geqslant\frac{1}{\varphi^{\frac{-\gamma}{1-\gamma}}\left(0\right)-\frac{\gamma d_4 t}{1-\gamma}}. \tag{4.27}$$

So, $\varphi\left(t\right)$ blows up in time

$$T\leqslant T^*=\frac{1-\gamma}{d_4\gamma\varphi^{\frac{\gamma}{1-\gamma}}\left(0\right)}.$$

The proof is completed. □

5. Conclusion

The researchers have investigated a nonlinear wave equation incorporating fractional damping and an infinite memory term, which is relevant to the analysis of viscoelastic material behavior. To the best of our knowledge, there has been no prior research addressing the existence and blow-up of solutions to the Petrovsky equation that includes a delay term with a fractional damping and a polynomial source term. In this study, it is shown that, under suitable conditions, the equation admits both the existence and finite-time blow-up of solutions in a bounded domain.

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