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Existence Theory and Stability Analysis to a Class of Hybrid Differential Equations using Confirmable Fractal Fractional Derivative

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Abstract

This research work is related to study a class of hybrid differential equations (HDEs) using conformable fractal-fractional derivative (CFFD). To establish the condition of at least one solution to the said problem, Krasnoselskii's fixed point theorem is considered. Stability results are derived by the use of Ulam-Hyers (U-H) and U-H Rassias. At the end of the paper, we added two pertinent examples for the purpose of justification and strengthen of our derived results.

Keywords: Conformable calculus, Hybrid problem, Ulam-Hyers stability.
2010 MSC: put your Mathematics Subject Classification 2010 (MSC): 26A33;34A08.

1. Introduction

Boundary value problems (BVPs) are important for characterizing a wide range of real-world engineering and physical research challenges. Much research has been done in the field of BVPs, which correspond to both ordinary and fractional order differential equations. Using functional analytic tools for qualitative theory, researchers have examined many BVPs. We refer [1], [2] and references there in for more details about BVPs. In [3], authors accumulated various BVPs concerned to real world problems of mathematical physics. Applications for BVPs with integral boundary conditions can be found in a wide range of fields, including population dynamics, chemical engineering, thermo-elasticity, blood flow issues, physical systems, thermo-dynamics, and other dynamics (see [4]). In recent decades, FDEs have gained much attention due to their wide use in these branches of science. Being the generalization of classical derivatives,the importance of fractional calculus is started at the same time as ordinary calculus were [5]. Using such types of fractional order derivatives have numerous benefits [6] Various definitions were defined by some researchers but the prominent role is of Riemann-Liouvile and Coputo arbitrary order [7, 8].

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Recently, Atangana in [9] introduced the fractal-fractional derivatives (FFDs). Moreover, a relationship established between fractional and fractal calculus. However the application of Atangana Balenue Fractal derivatives in neuroscience has been considered [10]. The qualitative theory of fraction DE in CFD,s very recent. Using fixed point results, for developing sufficient conditions we must see [11]. For the study of existence and stability of CFD one can see [12]. For the justification of our results, Ulam-Hyers (U-H) and U-H Rassias (U-H-R) theory as tools are helpful. The various applications to the real world problems of the CFDs [13] can strengthen our problem. The first ever used of chain rule by Khalil introduce the definition known as confirmable fractional order derivative (CFD). The properties of CFD is much similar with integer order [14]. The two operators Confirmable Derivatives and Fractal Derivatives have prominent role in theory and applications [15]. Researchers have investigated various results for different problems of fractional calculus, we refer to [16, 17, 18, 19]. Combining the two operators confirmable and fractal fractional derivatives were used recently for different problems. See some details for fractals fractional derivatives, we refer to [20, 21]. For physical applications, we refer to [22, 23].

Hybrid problems and systems of FDEs have been analyzed by researchers using different tools of mathematical analysis. In this regards, the significant contributions are cited as [24, 25, 26, 27].

We discuss our following single hybrid problem using confirmable fractional derivative. We consider the following BVP of CFFDE with integral boundary condition given by

$$\begin{cases} {}^{CFF}D^{\alpha,\beta}[\mu(t) - f(t,\mu(t))] = g(t,\mu(t))], t \in [0,b] = \mathscr{J}, \\ \mu(0) = \int_0^b h(\mu(s))ds, \end{cases} \tag{1.1}$$

where $f, g : \mathscr{J} \times R \to R$ are continuous functions.

Adopting the renowned fixed point theorems of Banach and Krasnoselkii [27], adequate criteria are formulated for the uniqueness and existence of a solution. It is important to highlight that fixed point theory is a useful instrument for examining a range of issues related to the qualitative analysis. Through applications, the stability problem has attracted significant attention in a number of research domain [28], [29], and [30]. Furthermore, stability has recently been extensively investigated as a useful tool for various problems of U-H type. Recently, stability analysis of multiple types by employing various fractional differential operators has been explored by authors in [31] and [32] utilizing the U-H idea. Considering the importance of the aforementioned stability, we also look at some results for U-H and generalized U-H stabilities for our topic under consideration. Examples are given to demonstrate the results of this study.

2. Preliminaries

Here, we accumulate some definitions and results regarding conformable fractal fractional derivatives required in our analysis.

Definition 2.1. [9] The conformable derivative of a function $f(t)$ defined on the interval $[0,\infty)$ and for all $t > 0$, with order $\alpha \in (0,1]$, is defined as follows:

$${}_0^C D_t^\alpha f(t) = \lim_{\xi \to 0} \frac{f(t + \xi t^{1-\alpha}) - f(t)}{\xi}$$

Importantly, when the function f is differentiable, we have:

$$ {}_0^C D_t^\alpha f(t) = \lim_{\xi \to 0} \frac{f(t + \xi t^{1-\alpha}) - f(t)}{\xi} = t^{1-\alpha} f'(t). $$

Definition 2.2. [9] The fractal derivative of a function f with order α is given by:

$$ {}_0^F D_t^\alpha f(t) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha}. $$

The most general case is given as:

$$ {}_0^F D_t^{\alpha,\beta} f(t) = \lim_{t \to t_1} \frac{f^\beta(t) - f^\beta(t_1)}{t^\alpha - t_1^\alpha}, \quad \alpha, \beta > 0. $$

For differentiable mapping f, one has

$$ \begin{aligned} {}_0^F D_t^\alpha f(t) &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha} = \frac{t^{1-\alpha}}{\alpha} f'(t) \\ &= \frac{1}{\alpha} {}_0^C D_t^\alpha f(t). \end{aligned} $$

Related integration is described as follows:

$$ {}_0^F I_t^\alpha f(t) = \alpha \int_0^t t^{\alpha-1} f(t) dt. $$

Lemma 2.3. *[21] Let $h \in L \in L[0, T]$, then the solution of*

$$ \begin{cases} {}^{CFF}D^{\alpha,\beta} \mu(t) = h(t), \; t \in \mathscr{J}, \\ \mu(0) = \mu_0 \end{cases} $$

is given by

$$ \mu(t) = \mu_0 + \int_0^t \beta s^{\beta-1} (t-s)^{\alpha-1} h(s) ds. $$

3. Main Results

In this section, we present our main results.

3.1. Integral representation of Problem (2.1)

The CFFDE BVP (2.1) is equivalent to the following integral equation

$$ \mu(t) = \mu(0) + \int_0^t \beta s^{\beta-1} (t-s)^{\alpha-1} g(s, \mu(s)) ds + f(t, \mu(t)), \quad t \in \mathscr{J}. \tag{3.1} $$

Let us define a Banach space say $X = C(\mathscr{J})$ with norm defined by

$$ \|\mu\| = \max_{t \in [a,b]} |\, \mu(t) \,|\, . $$

Now, we present some hypothesis which are helpful in building our main existence results.
(H_1) There exists constants $l_g, l_f > 0$ and $\mu, \bar{\mu} \in X$ such that

$$|f(t,\mu(t)) - f(t,\bar{\mu}(t))| \leqslant L_f|\mu - \bar{\mu}|$$
$$\text{and} \quad |g(t,\mu(t)) - g(t,\bar{\mu}(t))| \leqslant L_g|\mu - \bar{\mu}|.$$

(H_2) There exists constants $C_g, D_g > 0$ such that

$$|g(t,\mu(t))| \leqslant C_g|\mu(t)| + D_g.$$

(H_3) For $\mu, \bar{\mu}$ there exists constants $C_h > 0$ such that

$$|h(\mu) - h(\bar{\mu})| \leqslant C_h|\mu - \bar{\mu}|.$$

Theorem 3.1. *The problem* (2.1) *has a unique solution, if*

$$\Omega_{\alpha,\beta,L_f,L_g,C_h} = C_h + \beta L_g B(\alpha,\beta) b^{\alpha+\beta-1} + L_f < 1. \tag{3.2}$$

Proof. Let $N : X \to X$ be the operator defined by

$$N[\mu(t)] = \int_0^b h(\mu(s))ds + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} g(s,\mu(s))ds + f(t,\mu(t)).$$

Let $\mu, \bar{\mu} \in X$, then

$$\begin{aligned}
\|N(\mu) - N(\bar{\mu})\| \; = \; & \max_{t\in[0,b]} \bigg| \int_0^b h(\mu(s))ds + \int_o^t \beta s^{\beta-1}(t-s)^{\alpha-1} g(s,\mu(s))ds + f(t,\mu(t)) \\
& - \int_0^b h(\bar{\mu}(s))ds - \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} g(s,\bar{\mu}(s))ds - f(t,\bar{\mu}(t)) \bigg| \\
\leqslant & \int_0^b \big|h(\mu(s)) - h(\bar{\mu}(s))\big| ds + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} \big|g(s,\mu(s)) - g(s,\bar{\mu}(s)\big| ds \\
& + |f(t,\mu(t)) - f(t,\bar{\mu}(t)| \\
\leqslant & C_h\|\mu - \bar{\mu}\| b + L_g\|\mu - \bar{\mu}\| \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} ds + L_f\|\mu - \bar{\mu}\| \\
\leqslant & \Omega_{\alpha,\beta,L_f,L_g,C_h}\|\mu - \bar{\mu}\|.
\end{aligned}$$

Hence, by inequality (3.2) operator N is a contraction. So by Banach contraction principle, the problem (2.1) has a unique solution. □

Theorem 3.2. *Under the hypothesis* $(H_1 - H_3)$ *the problem* (2.1) *has at least one solution.*

Proof. Let defined the space $E = \{\mu \in X : \| \mu \| \leqslant \rho\}$ and taking two operators $P, Q : X \to X$ defined by

$$Q\mu(t) = f(t,\mu(t)),$$

$$P\mu(t) = \int_0^b h(\mu(s))ds + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} g(s,\mu(s))ds.$$

Then for $\mu, \bar{\mu} \in X$, one has

$$\begin{aligned}\|Q\mu - Q\bar{\mu}\| &= \max_{t \in \mathscr{J}} |Q\mu(t) - Q\bar{\mu}(t)| \\ &= \max_{t \in \mathscr{J}} |f(t, \mu(t)) - f(t, \bar{\mu}(t))| \\ &\leqslant L_f \|\mu - \bar{\mu}\|,\end{aligned}$$

then the operator Q is a contraction. Now, we show that the operator P is bounded and continuous. Let $\mu \in E$, then

$$\begin{aligned}\|\mu\| = \max_{t \in [0,1]} &\left| \int_0^b h(\mu(s))ds + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} g(s, \mu(s))ds \right| \\ &\leqslant \left[\rho C_h + C_0 + \beta B(\alpha, \beta) b^{\alpha+\beta-1}(\rho C_g + D_g)\right],\end{aligned}$$

where $C_0 = \max_{t \in [0,1]} \int_0^b h(0)ds$. Hence

$$\|\mu\| \leqslant (C_0 + \beta B(\alpha, \beta) b^{\alpha+\beta-1} C_g)\rho + C_0 + \beta B(\alpha, \beta) b^{\alpha+\beta-1} D_g \leqslant \rho, \tag{4}$$

where

$$\rho \geqslant \frac{C_g + \beta B(\alpha, \beta) b^{\alpha+\beta-1} D_g}{1 - (C_0 + \beta B(\alpha, \beta) b^{\alpha+\beta-1} C_g)}.$$

Hence $\|\mu\| \leqslant \rho \Rightarrow \mu \in E$ is Bounded. In the same way , we have $\|P\mu\| \leqslant \rho$ which mean $P(E) \leqslant E$.

If $t_1 < t_2$, then

$$\begin{aligned}|P\mu(t_2) - P\mu(t_1)| =& \left| \int_0^{t_2} \beta s^{\beta-1}(t_2-s)^{\alpha-1} g(s, \mu(s))ds - \int_0^{t_1} \beta s^{\beta-1}(t_1-s)^{\alpha-1} g(s, \mu(s))ds \right| \\ =& \left| \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]\beta s^{\beta-1} g(s, \mu(s))ds \right. \\ &\left. + \int_{t_1}^{t_2} [t_2-s]^{\alpha-1} \beta s^{\beta-1} g(s, \mu(s))ds \right| \\ \leqslant& \int_0^{t_1} ([(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]\beta s^{\beta-1} |g(s, \mu(s))|ds \\ &+ \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \beta s^{\beta-1} |g(s, \mu(s))|ds \\ \leqslant& \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]\beta s^{\beta-1} [C_g\|\mu\| + D_g]ds \\ &+ \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \beta s^{\beta-1} [C_g\|\mu\| + D_g]ds \\ =& (C_g\rho + D_g)\left[\int_0^{t_2} \beta s^{\beta-1}(t_2-s)^{\alpha-1} ds + \int_0^{t_1} \beta s^{\beta-1}(t_1-s)^{\alpha-1} ds \right] \\ =& \beta(C_g\rho + D_g)B(\alpha, \beta)[t_2^{\alpha+\beta} - t_1^{\alpha+\beta}].\end{aligned}$$

Now as $t_1 \to t_2$, we see that right side gives us zero. Thus $\|\rho\mu(t_2) - \rho\mu(t_1)\| \to 0$ as $t_1 \to t_2$. Hence P is equi-continuous. Thus by Arzelá-Ascoli theorem the operator P is compact. Consequently, it follows that problem (2.1) has at least one solution. □

4. Stability Theory

Let us define a function $\pi : \mathscr{J} \to R$ independent of μ, such that for any $\epsilon > 0$,

Remark 4.1. (i)

$$|\pi(t)| \leqslant \epsilon.$$

(ii)

$$\begin{cases} {}^{CFF}D^{\alpha,\beta}[\mu(t) - f(t,\mu(t))] = g(t,\mu(t))] + \pi(t), \\ \mu(0) = \int_0^b h(\mu(s))ds, \end{cases} \tag{4.1}$$

where $f, g : \mathscr{J} \times R \to R$ are continuous functions.
The solution of (4.1) is given by

$$\mu(t) = f(t,\mu(t)) + \int_0^b h(\mu(s)ds) + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} g(s,\mu(s))ds + \int_0^t \beta s^{\beta-1}(t-s)^{(\alpha-1)}\pi(s)ds. \tag{4.2}$$

In view of Theorem 3.1, and using Remark 4.1, one has from (4.2)

$$\begin{aligned} \mu(t) &= N(\mu(t)) + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1}\pi(s)ds \\ |\mu(t) - N(\mu(t))| &\leqslant \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1}|\pi(s)|ds \\ |\mu(t) - N(\mu(t))| &\leqslant b^{\alpha+\beta-1}\beta B(\alpha,\beta)\epsilon \\ \|\mu(t) - N(\mu(t))\| &\leqslant \Delta\epsilon, \end{aligned}$$

where

$$\Delta = b^{\alpha+\beta-1}\beta B(\alpha,\beta).$$

Theorem 4.2. *The solution of* (2.1) *is U-H stable and consequently generalized U-H stable if the condition* $\Omega_{\alpha,\beta,L_f,L_g,C_h} < 1$ *holds.*

Proof. Consider μ be any solution of (2.1) and $\bar{\mu}$ be a unique solution of (2.1), then taking

$$\begin{aligned} \|\mu - \bar{\mu}\| &= \max_{t\in\mathscr{J}} |\mu(t) - \bar{\mu}(t)| \\ &\leqslant \max_{t\in\mathscr{J}} |\mu(t) - N\mu(t)| + \max_{t\in\mathscr{J}} |N\mu(t) - \bar{\mu}(t)|. \end{aligned}$$

Using Theorem 3.1, we have

$$\|\mu - \bar{\mu}\| \leqslant \Delta\epsilon + \Omega_{\alpha,\beta,L_f,L_g,C_h}\|\mu - \bar{\mu}\|.$$

After, rearrangement one has from above relation

$$\|\mu - \bar{\mu}\| \leqslant \frac{\Delta}{1 - \delta_{\alpha,\beta}}\epsilon.$$

Hence the solution of (2.1) is U-H stable.
In addition, let there exist a non-decreasing function $\omega : \mathscr{J} \to R$ such that $\omega(0) = 0$, then from above inequality we have

$$\|\mu - \bar{\mu}\| \leqslant \frac{\Delta}{1-\delta_{\alpha,\beta}}\omega(\epsilon).$$

where $\omega(\epsilon) = \epsilon$, we see that the condition of generalized U-H stability is also holds. □

Consider the given remark

Remark 4.3. For function $\pi : [0, b] \to R$ independent of μ, we have

$$|\pi(t)| \leqslant \lambda(t)\epsilon,$$

then the solution of

$$^{CFF}D^{\alpha,\beta}[\mu(t) - f(t, \mu(t))] = g(t, \mu(t))] + \pi(t) \tag{4.3}$$

$$\mu(0) = \int_0^b h(\mu(s))ds \tag{4.4}$$

has

$$\mu(t) = f(t, \mu(t)) + \int_0^b h(\mu(s))ds + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1} g(s, \mu(s))ds + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1}\pi(s)ds.$$

Proof. In view of Theorem 3.1 yields

$$\mu(t) = N(\mu(t)) + \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1}\pi(s)ds, \ t \in \mathscr{J}.$$

$$\begin{aligned} |\mu(t) - N(\mu(t))| &\leqslant \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1}|\pi(s)|ds \\ &\leqslant \int_0^t \beta s^{\beta-1}(t-s)^{\alpha-1}\lambda(s)\epsilon ds \\ &= \epsilon\int_0^b \beta s^{\beta-1}(b-s)^{\alpha-1}\lambda(s) \ ds \\ &= \epsilon\lambda_{\alpha,\beta,b}, \end{aligned}$$

where $\lambda_{\alpha,\beta,b} = \int_0^b \beta s^{\beta-1}(b-s)^{\alpha-1}\lambda(s)$ ds. Hence, one has

$$|\mu(t) - N(\mu(t))| \leqslant \epsilon\lambda_{\alpha,\beta,b}.$$

□

Theorem 4.4. *The solution of (2.1) is U-H Rassias stable if* $\Omega_{\alpha,\beta,L_f,L_g,C_h} < 1$.

Proof. Consider $\mu, \bar{\mu} \in X$, then using Theorem 3.1, we have

$$\begin{aligned} \|\mu - \bar{\mu}\| &= \max_{t \in \mathcal{J}} |\mu(t) - \bar{\mu}(t)| \\ &\leqslant \max_{t \in \mathcal{J}} |\mu(t) - N\mu(t)| + \max_{t \in \mathcal{J}} |N\mu(t) - \bar{\mu}(t)| \\ &\leqslant \epsilon\lambda_{\alpha,\beta,b} + \Omega_{\alpha,\beta,L_f,L_g,C_h} \|\mu - \bar{\mu}\| \\ \|\mu - \bar{\mu}\| &\leqslant \frac{\lambda_{\alpha,\beta,b}}{1 - \Omega_{\alpha,\beta,L_f,L_g,C_h}} \epsilon. \end{aligned}$$

Hence the solution of (2.1) is U-H Rassias stable.
Further there exist a solution $\psi : [0, b] \to R$ non-decreasing function such that $\psi(\epsilon) = \epsilon$, then

$$\|\mu - \bar{\mu}\| \leqslant \frac{\lambda_{\alpha,\beta,b}}{1 - \Omega_{\alpha,\beta,L_f,L_g,C_h}} \psi(\epsilon).$$

which implies that the solution of considered problem is generalized U-H Rassias stable. □

5. Application

Here, we present examples regarding illustration of essential main results.

Example 5.1. Consider

$$\begin{cases} {}^{CFF}D^{\frac{1}{2},\frac{1}{5}}[\mu(t) - f(t, \mu(t))] = g(t, \mu(t))], \quad t \in [0, 1], \\ \mu(0) = \int_0^1 h(\mu(s))ds, \end{cases} \tag{5.1}$$

where

$$f(t, \mu(t)) = \frac{|\mu(t)| + 1}{t^2 + 30}, \quad g(t, \mu(t)) = \frac{|\mu(t)| + 2}{t^2 + 20} \text{ and } h(\mu(t)) = \frac{e^{-|\mu(t)|}}{t^2 + 100}.$$

Clearly

$$|f(t, \mu(t)) - f(t, \bar{\mu}(t))| \leqslant \frac{1}{30}|\mu - \bar{\mu}| \quad \text{and} \quad |g(t, \mu(t)) - g(t, \bar{\mu}(t))| \leqslant \frac{1}{20}|\mu - \bar{\mu}|.$$

Also

$$|g(t, \mu(t))| \leqslant \frac{1}{20}|\mu(t)| + \frac{1}{10} \quad \text{and} \quad |h(\mu(t)) - h(\bar{\mu}(t))| \leqslant \frac{1}{100}|\mu - \bar{\mu}|.$$

So, we have

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{5}, \quad b = 1, \quad L_f = \frac{1}{30}, \quad L_g = \frac{1}{20} \text{ and } C_h = \frac{1}{100}.$$

Obviously hypothesis $(H_1 - H_3)$ holds. Thus by Theorem 3.2, problem (5.1) has at least one solution. And

$$\Omega_{\alpha,\beta,L_f,L_g,C_h} = C_h + \beta L_g B(\alpha, \beta) b^{\alpha+\beta-1} + L_f \approx 0.10602 < 1.$$

Hence, Theorem 3.1 implies that problem (5.1) has a unique solution. Also, assumptions of Theorem 4.2 holds, consequently the solution of (5.1) is U-H stable. In the same way, we can deduce the other kinds of U-H stability also.

Example 5.2. Consider

$$\begin{cases} {}^{CFF}D^{\frac{1}{3},\frac{1}{2}}[\mu(t)-f(t,\mu(t))]=g(t,\mu(t))], \quad t\in[0,1] \\ \mu(0)=\int_0^1 h(\mu(s))ds, \end{cases} \tag{5.2}$$

where

$$f(t,\mu(t))=\frac{\sin|\mu(t)|+10}{t^2+50}, \quad g(t,\mu(t))=\frac{\cos|\mu(t)|+3}{t^2+30} \text{ and } h(\mu(t))=\frac{|\mu(t)|}{t^3+200}.$$

Clearly

$$|f(t,\mu(t))-f(t,\bar{\mu}(t))|\leqslant\frac{1}{50}|\mu-\bar{\mu}| \quad \text{and} \quad |g(t,\mu(t))-g(t,\bar{\mu}(t))|\leqslant\frac{1}{30}|\mu-\bar{\mu}|.$$

Also

$$|g(t,\mu(t))|\leqslant\frac{1}{30}|\mu(t)|+\frac{1}{10} \quad \text{and} \quad |h(\mu(t))-h(\bar{\mu}(t))|\leqslant\frac{1}{200}|\mu-\bar{\mu}|.$$

So, we have

$$\alpha=\frac{1}{3}, \ \beta=\frac{1}{2}, \ b=1, \ L_f=\frac{1}{50}, \ L_g=\frac{1}{30} \text{ and } C_h=\frac{1}{200}.$$

Obviously hypothesis (H_1-H_3) holds. Thus by Theorem 3.2, problem (5.2) has at least one solution. And

$$\Omega_{\alpha,\beta,L_f,L_g,C_h}=C_h+\beta L_g B(\alpha,\beta)b^{\alpha+\beta-1}+L_f\approx 0.05833<1.$$

Hence, Theorem 3.1 implies that problem (5.2) has a unique solution. Also Theorem 4.2 holds, consequently the solution to (5.2) is U-H stable. In the same way, we can deduce the other kinds of U-H stability also.

6. Conclusion

This word was devoted to investigate a class of HDEs with CFFD. We have deduced sufficient results for the existence theory and different kinds of U-H stability using fixed point theory and functional analysis tools. Some pertinent examples were given to demonstrate the establish results. The CFFD is a powerful tool and can be used as another alternative in investigation of various dynamical problems. We concluded that the presented analysis will provide basis for deducing new results for more general nonlinear problem using CFFD. Therefore, in the future for useful results, various problems can be examine through this type of analysis.

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